

## ON THE NUMBER OF DISJOINT 4-CYCLES IN REGULAR TOURNAMENTS <sup>1</sup>

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### Abstract

In this paper, we prove that for an integer  $r \geq 1$ , every regular tournament  $T$  of degree  $3r - 1$  contains at least  $\frac{21}{16}r - \frac{10}{3}$  disjoint directed 4-cycles. Our result is an improvement of Lichiardopol's theorem when taking  $q = 4$  [Discrete Math. **310** (2010) 2567–2570]: for given integers  $q \geq 3$  and  $r \geq 1$ , a tournament  $T$  with minimum out-degree and in-degree both at least  $(q - 1)r - 1$  contains at least  $r$  disjoint directed cycles of length  $q$ .

**Keywords:** regular tournament,  $C_4$ -free, disjoint cycles.

**2010 Mathematics Subject Classification:** 05C70, 05C38.

### 1. INTRODUCTION

This paper considers only digraphs. For a digraph  $D$ , we write  $V(D)$  for the vertex set of  $D$ , and the order of  $D$  is the cardinality of  $V(D)$ . We write  $A(D)$  for the set of the arcs of  $D$ . Two or several subgraphs are *independent* or *disjoint* if they are pairwise vertex-disjoint.

We say that a vertex  $y$  is an *out-neighbor* (*in-neighbor*) of a vertex  $x$  if  $(x, y)$  (respectively  $(y, x)$ ) is an arc of  $D$ . The number of out-neighbors of  $x$  is the *out-degree*  $d^+(x)$  of  $x$ , and the number of in-neighbors of  $x$  is the *in-degree*  $d^-(x)$  of  $x$ . The *minimum out-degree*  $\delta^+(D)$  of  $D$  is the smallest of the out-degrees of the vertices of  $D$ , and the *minimum in-degree*  $\delta^-(D)$  of  $D$  is the smallest of the in-degrees of the vertices of  $D$ .

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<sup>1</sup>The author's work is supported by NNSF of China (No. 11271230, 11671232).

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A *path* of length  $m$  of a digraph  $D$  is a sequence  $P$  with  $P = (x_1, \dots, x_{m+1})$  of distinct vertices of  $D$  such that  $(x_i, x_{i+1}) \in A(D)$  for  $1 \leq i \leq m$ . If  $\{x_1, \dots, x_{m+1}\} = V(D)$ , then  $P$  is a Hamiltonian path. A *cycle* of length  $m$  in  $D$  is a sequence  $C$  with  $C = (x_1, \dots, x_m, x_1)$  such that the vertices  $x_1, \dots, x_m$  are distinct,  $(x_i, x_{i+1}) \in A(D)$  for  $1 \leq i \leq m - 1$ , and  $(x_m, x_1) \in A(D)$ . If  $\{x_1, \dots, x_m\} = V(D)$ , then  $C$  is a Hamiltonian cycle. A cycle of length 3 is a *triangle*. A triangle  $(x, y, z, x)$  will often be denoted by  $(x, u, x)$ , where  $u$  is the arc  $(y, z)$ .

A *tournament* is a digraph  $T$  such that for any two distinct vertices  $x$  and  $y$ , exactly one of the ordered pairs  $(x, y)$  and  $(y, x)$  is an arc of  $T$ . A *regular tournament* of degree  $d$  is a tournament  $T$  such that  $d^+(x) = d^-(x) = d$  for every vertex  $x$ . Necessarily the order of  $T$  is  $2d + 1$ . For a subset  $S$  of  $V(T)$ ,  $T[S]$  denotes the *subtournament* induced by the vertices of  $S$ .

It is well-known (Redei's Theorem) that any tournament contains a Hamiltonian path, and (Camion's Theorem) a tournament is strong if and only if it contains a Hamiltonian cycle. It is also known (Moon's Theorem) that a strong tournament  $T$  of order  $|T|$  is pancyclic, i.e., it has cycles of all lengths  $3, \dots, |T|$ . In particular this means that if  $C$  is a  $q$ -cycle of  $T$ , then the tournament  $T[V(C)]$  has cycles of all lengths  $3, \dots, q$ . A  $C_q$ -free tournament is a tournament  $T$  without a  $q$ -cycle.

In 1981, Bermond and Thomassen [3] conjectured that for any positive integer  $r$ , any digraph of minimum out-degree at least  $2r - 1$  contains at least  $r$  disjoint directed cycles. It is trivially true when  $r = 1$ . It was proved by Thomassen [8] when  $r = 2$  in 1983. The case  $r = 3$  was proved by Lichiardopol *et al.* in [5]. It is still open for large values of  $r$ . In 2014, Bang-Jensen *et al.* proved the conjecture for tournaments in [2]. Lichiardopol proposed a conjecture for tournaments [7]: for given integers  $q \geq 3$  and  $r \geq 1$ , a tournament  $T$  with minimum out-degree at least  $(q - 1)r - 1$  contains at least  $r$  disjoint  $q$ -cycles. In 2012, Lichiardopol [6] proved that for an integer  $r \geq 1$ , every regular tournament  $T$  of degree  $2r - 1$  contains at least  $\frac{7}{6}r - \frac{7}{3}$  disjoint directed cycles. By pancyclic property of tournaments, the following is easy to see.

**Theorem 1.1.** *For an integer  $r \geq 1$ , every regular tournament  $T$  of degree  $2r - 1$  contains at least  $\frac{7}{6}r - \frac{7}{3}$  disjoint triangles.*

We consider the number of 4-cycles in a regular tournament and prove the following theorem.

**Theorem 1.2.** *For an integer  $r \geq 1$ , every regular tournament  $T$  with degree  $3r - 1$  contains at least  $\frac{21}{16}r - \frac{10}{3}$  disjoint 4-cycles.*

In 2012, Lichiardopol [7] proved the following theorem.

**Theorem 1.3** ([7]). *For given integers  $q \geq 3$  and  $r \geq 1$ , a tournament  $T$  with  $\min \{\delta^+(T), \delta^-(T)\} \geq (q - 1)r - 1$  contains at least  $r$  disjoint  $q$ -cycles.*

If we take  $q = 4$ , it is easy to see

**Theorem 1.4.** *For an integer  $r \geq 1$ , every regular tournament  $T$  with degree  $3r - 1$  contains at least  $r$  disjoint 4-cycles.*

Our result improves this lower bound to  $\frac{21}{16}r - \frac{10}{3}$ .

There are many analogous results on bipartite tournaments, for example, Bai *et al.* in [1] proved the following theorem.

**Theorem 1.5** ([1]). *Let  $BT$  be a bipartite tournament with minimum out-degree at least  $qr - 1$  and let  $t_1, \dots, t_r \in [4, 2q]$  be any  $r$  even integers. Then  $BT$  contains  $r$  disjoint cycles of length  $t'_1, \dots, t'_r$  such that  $t'_i = t_i$  for  $t_i \equiv 0 \pmod{4}$  and  $t'_i \in \{t_i, t_i + 2\}$  for  $t_i \equiv 2 \pmod{4}$ , where  $1 \leq i \leq r$ .*

## 2. LEMMA

In this section, we list a lemma to prove Theorem 1.2.

**Lemma 2.1.** *Let  $M$  be a proper subset of  $N$  with  $|N| = n$  and  $|M| = m$ . Suppose that  $T[N]$  is  $C_4$ -free and  $P = (x_1, x_2, \dots, x_{n-1}, x_n)$  is a Hamiltonian path of  $T[N]$ . If  $\{x_1, x_2, x_{n-1}, x_n\} \subseteq M$ , then there is a Hamiltonian path  $Q = (y_1, \dots, y_m)$  of  $T[M]$  such that  $y_1 = x_1, y_m = x_n$ .*

**Proof.** We construct  $Q$  from  $P$  by deleting vertices that are not contained in  $M$  by the following two steps.

**Step 1.** (1) If there exists  $x_i$  for  $i \geq 3$  such that none of  $x_i, x_{i+1}, \dots, x_j$  belongs to  $M$  ( $j \geq i + 1$ ), delete  $x_i, x_{i+1}, \dots, x_j$  from  $P$ .

(2) If there exists  $x_i$  for  $i \geq 3$  such that  $x_i \notin M, x_{i-1}, x_{i+1} \in M$  and  $(x_{i-1}, x_{i+1}) \in A(T)$ , delete  $x_i$  from  $P$ . Do (1) and (2) until there are no such vertices.

We claim that after Step 1 the remaining vertices can still form a path as the prior order. It is obvious for Step 1(2). For Step 1(1), we can prove that  $(x_{i-1}, x_{j+1}) \in A(T)$ . Suppose on the contrary that  $(x_{j+1}, x_{i-1}) \in A(T)$ , then  $\{x_{i-1}, x_i, \dots, x_j, x_{j+1}, x_{i-1}\}$  is a cycle of length at least 4. By property of pancyclic, it has a 4-cycle, a contradiction (since  $T[N]$  is  $C_4$ -free). Denote this new path by  $Q' = (z_1, \dots, z_l)$ . Clearly,  $Q'$  has the following property: if  $x_i \notin M$  then  $x_{i-1}, x_{i+1} \in M$  and  $(x_{i+1}, x_{i-1}) \in A(T)$ . Since  $\{x_1, x_2, x_{n-1}, x_n\} \subseteq M$ , we have  $z_1 = x_1, z_2 = x_2, z_{l-1} = x_{n-1}, z_l = x_n$ .

**Step 2.** If none of  $z_j, z_{j+2}, \dots, z_{j+2i}$  belongs to  $M$  ( $i \geq 0$ ), but  $z_{j-1}, z_{j+2i+1} \in M$ , we delete  $z_j, z_{j+2}, \dots, z_{j+2i}$  from  $Q'$  and replace the segment  $(z_{j-2}, \dots, z_{j+2i+2})$

by  $(z_{j-2}, z_{j+2i+1}, z_{j+2i-1}, \dots, z_{j+1}, z_{j-1}, z_{j+2i+2})$ . Repeat the procedure until there are no such vertices.

Since  $z_1 = x_1, z_2 = x_2, z_{l-1} = x_{n-1}, z_l = x_n, j \geq 3$  and  $j + 2i \leq l - 2$ , we have  $j - 2 \geq 1$  and  $j + 2i + 2 \leq l$ . Denote the path after Step 2 by  $Q = (y_1, \dots, y_m)$ . Then it is the desired Hamiltonian path. ■

### 3. PROOF OF THEOREM 1.2

The proof of this theorem is inspired mainly by the proof of the main theorem in [6]. We begin with a preliminary result. Let  $(x, y)$  be an arc of a tournament  $T$  of order  $n$  with  $n \geq 3$ . We define:

$$B(x, y) = \{z \in V(T) : (x, z) \in A(T), (y, z) \in A(T)\},$$

$$E(x, y) = \{z \in V(T) : (z, x) \in A(T), (y, z) \in A(T)\},$$

$$F(x, y) = \{z \in V(T) : (x, z) \in A(T), (z, y) \in A(T)\}.$$

Observe that  $E(x, y)$  is the set of vertices  $z$  such that  $x, y$  and  $z$  form a triangle. We denote by  $b(x, y), e(x, y)$  and  $f(x, y)$  the respective cardinalities of these three sets. It is easy to see that  $d^+(x) = b(x, y) + f(x, y) + 1$  and  $d^+(y) = b(x, y) + e(x, y)$ . It follows that  $e(x, y) = f(x, y) + d^+(y) - d^+(x) + 1$ . Hence if  $T$  is regular, then we have

$$(1) \quad e(x, y) = f(x, y) + 1.$$

If  $u = (x, y)$ , then  $E(x, y), e(x, y), F(x, y)$  and  $f(x, y)$  will also be denoted by  $E(u), e(u), F(u)$  and  $f(u)$ , respectively.

The order of the regular tournament  $T$  of degree  $3r - 1$  is  $6r - 1$ . By Theorem 1.4,  $T$  contains at least  $r$  disjoint 4-cycles. When  $r \leq 10$ , it holds that  $r \geq \frac{21}{16}r - \frac{10}{3}$ , and so Theorem 1.2 holds in this case. So from now on, we suppose  $r \geq 11$ .

Let  $s$  be the maximum number of disjoint 4-cycles of  $T$ . In particular, let  $S = \{C_1, \dots, C_s\}$  be a set of  $s$  disjoint 4-cycles with  $C_i = (a_i, b_i, u_i, v_i, a_i)$  for  $1 \leq i \leq s$ . Let us define  $V_1 = \bigcup_{1 \leq i \leq s} V(C_i)$  and  $V_2 = V(T) \setminus V_1$ . Let  $T_s$  be the subtournament of  $T$  induced by the vertices of  $V_2$ . Its vertices can be ordered into a Hamiltonian path  $(x_1, \dots, x_t)$  where  $t = 6r - 1 - 4s$ . Note that  $T_s$  is a  $C_4$ -free tournament by the maximality of  $s$ .

Suppose first that  $t \leq 20$ . This means  $6r - 1 - 4s \leq 20$ , so  $s \geq \frac{3}{2}r - \frac{21}{4}$ . Since  $r \geq 11$  implies  $\frac{3}{2}r - \frac{21}{4} \geq \frac{21}{16}r - \frac{10}{3}$ , it follows that  $s \geq \frac{12}{16}r - \frac{10}{3}$  and Theorem 1.2 holds in this case.

So, from now on, we suppose that  $t \geq 21$  (and  $r \geq 11$ ).

Since  $T_s$  is  $C_4$ -free, it is easy to see the following.

**Claim 3.1.** For  $1 \leq i \leq t - 3$ ,  $j \geq i + 3$ ,  $(x_i, x_j) \in A(T)$ .

Since  $t \geq 21$ , by Claim 3.1, it is easy to see that  $\omega_i = (x_i, x_{t+1-i}) \in A(T)$  for each  $1 \leq i \leq 7$ . Denote by  $\Omega_s$  the set of the independent arcs  $\omega_1, \dots, \omega_7$ .

**Claim 3.2.** For  $1 \leq i \leq 7$ ,  $f(\omega_i) \geq t - 2i - 2$ ,  $e(\omega_i) \geq t - 2i - 1$ .

*Proof.* Since  $T_s$  is  $C_4$ -free, by Claim 3.1, there are at most two vertices (they are  $x_{i+2}, x_{t-i-1}$ ) between  $x_i$  and  $x_{t+1-i}$  that do not belong to  $F(\omega_i)$ . So we get  $f(\omega_i) \geq t - 2i - 2$ . By equation (1), we get  $e(\omega_i) \geq t - 2i - 1$ .  $\square$

Put  $e(\Omega_s) = \sum_{1 \leq i \leq 7} e(\omega_i)$ . Then we have

**Claim 3.3.**  $e(\Omega_s) \geq 7t - 63$ .

*Proof.* By Claim 3.2, we get  $e(\omega_i) \geq t - 2i - 1$ . It follows that  $e(\Omega_s) = \sum_{1 \leq i \leq 7} e(\omega_i) \geq \sum_{1 \leq i \leq 7} (t - 2i - 1)$ , so  $e(\Omega_s) \geq 7t - 63$ .  $\square$

Let  $W = \{x_8, \dots, x_{t-7}\}$  be the set of vertices between  $x_7$  and  $x_{t-6}$ ,  $F_W(\omega_i)$  denote the vertices in  $W$  that belong to  $F(\omega_i)$ , and  $f_W(\omega_i) = |F_W(\omega_i)|$ . Since  $t \geq 21$ , there are at least seven vertices between  $x_7$  and  $x_{t-6}$ . Similarly to the proof of Claim 3.2, there are at least five of these vertices in  $M$  belonging to  $F(\omega_i)$ , for each  $1 \leq i \leq 7$ , i.e.,  $f_W(\omega_i) \geq 5$ .

**Claim 3.4.** For each  $1 \leq i \leq 7$ ,  $E(\omega_i) \cap V_2 = \emptyset$ .

*Proof.* If  $E(\omega_i) \cap V_2 \neq \emptyset$ , then there exists a vertex  $x_j$  such that  $x_j \in E(\omega_i) \cap V_2$ . Since  $f_W(\omega_i) \geq 5$ , there is a vertex  $x_k$  with  $k \neq j$  such that  $x_k \in F_W(\omega_i)$ . Thus  $(x_i, x_k, x_{t+1-i}, x_j, x_i)$  is a 4-cycle of  $T_s$ , a contradiction.  $\square$

By Claim 3.4, the set  $E(\omega_i)$  does not contain any vertex of  $T_s$ .

For a vertex  $x \in V_1$ , let  $E_{\Omega_s}(x)$  denote the set of the arcs  $\omega_i \in \Omega_s$  such that  $x \in E(\omega_i)$ , and put  $e_{\Omega_s}(x) = |E_{\Omega_s}(x)|$ . For a 4-cycle  $C_i$  of  $S$ , let  $e_{\Omega_s}(C_i) = \sum_{x \in V(C_i)} e_{\Omega_s}(x)$ .

We then get  $e(\Omega_s) = \sum_{x \in V_1} e_{\Omega_s}(x) = \sum_{1 \leq i \leq s} e_{\Omega_s}(C_i)$ , by double-counting, and interchanging the order of summation. Then we get

**Claim 3.5.** If a vertex  $v$  of a 4-cycle  $C$  of  $S$  satisfies  $e_{\Omega_s}(v) \geq 2$ , then  $e_{\Omega_s}(w) = 0$  for every vertex  $w$  of  $C$  distinct from  $v$ .

*Proof.* If  $e_{\Omega_s}(w) > 0$ , then there exists an arc  $\omega_j$  of  $\Omega_s$  such that  $w \in E(\omega_j)$ . Since  $e_{\Omega_s}(v) \geq 2$ , there exists an arc  $\omega_k$  of  $\Omega_s$  with  $k \neq j$  such that  $v \in E(\omega_k)$ . Since  $f_W(\omega_j) \geq 5$  and  $f_W(\omega_k) \geq 5$ , there exist two distinct vertices  $x, y \in W$  such that  $x \in F_W(\omega_j), y \in F_W(\omega_k)$ . Clearly,  $C' = (w, x_j, x, x_{t+1-j}, w)$  and  $C'' = (v, x_k, y, x_{t+1-k}, v)$  are two disjoint 4-cycles. Now  $(S \setminus \{C\}) \cup \{C', C''\}$  is a collection of  $s + 1$  disjoint 4-cycles, which is impossible by the maximality of  $s$ . So the result is proved.  $\square$

Let  $U_s = \{x \in V_1 : e_{\Omega_s}(x) \geq 4\}$ , and let  $u_s = |U_s|$ . Clearly, this claim implies that every 4-cycle  $C$  of  $S$  which is disjoint from  $U_s$ , satisfies  $e_{\Omega_s}(C) \leq 4$ . It implies also that every 4-cycle of  $S$  contains at most one vertex of  $U_s$ .

Now, we choose  $S$  such that  $u_s$  is as large as possible. Suppose first that  $u_s = 0$ . Since  $e(\Omega_s) = \sum_{1 \leq i \leq s} e_{\Omega_s}(C_i)$ , from Claim 3.3 and Claim 3.5, we get  $7t - 63 \leq 4s$ . That is  $7(6r - 1 - 4s) - 63 \leq 4s$ , so  $32s \geq 42r - 70$ . Hence  $s \geq \frac{21}{16}r - \frac{35}{16} > \frac{21}{16}r - \frac{10}{3}$ . Therefore, Theorem 1.2 holds in this case.

Suppose now  $u_s > 0$ . By Claim 3.5, without loss of generality, we may suppose that the  $u_s$  vertices of  $U_s$  are  $a_1, \dots, a_{u_s}$ . We denote  $\Delta_s = \{C_1, \dots, C_{u_s}\}$ . Note that  $\Delta_s \subset S$  when  $u_s < s$ . For each 4-cycle  $C_i$  of  $\Delta_s$  we have  $e_{\Omega_s}(C_i) = e_{\Omega_s}(a_i) \leq 7$ .

We denote  $U'_s = \bigcup_{1 \leq i \leq u_s} \{b_i, u_i, v_i\}$  (where  $V(C_i) = \{a_i, b_i, u_i, v_i\}$ ) and  $V'_s = V_2 \cup U'_s$ . Clearly,  $|V'_s| = 3u_s + t$ .

**Claim 3.6.** *The subtournament induced by the set  $V'_s$  is  $C_4$ -free.*

**Proof.** On the contrary, let  $C'$  be a 4-cycle of  $T[V'_s]$  with  $C' = (w, x, y, z, w)$ . Since  $T[V_2]$  is  $C_4$ -free, two cases are possible.

*Case 1.*  $C'$  contains exactly one vertex of  $U'_s$ . Let  $w$  be this vertex; there exists  $i$  with  $1 \leq i \leq u_s$  such that  $w \in V(C_i)$ , and  $w \neq a_i$ . Since  $e_{\Omega_s}(a_i) \geq 4$ , there exists an arc  $\omega_j$  of  $E_{\Omega_s}(a_i)$  disjoint from  $x, y, z$ . Since  $f_W(\omega_j) \geq 5$ , there exists a vertex  $a \in W$  distinct from  $x, y, z$  such that  $a \in F_W(\omega_j)$ . Clearly,  $C'$  and  $C''$ , where  $C'' = (a_i, x_j, a, x_{t+1-j}, a_i)$ , are disjoint 4-cycles. Now  $(S \setminus \{C_i\}) \cup \{C', C''\}$  is a collection of  $s + 1$  disjoint 4-cycles, a contraction.

*Case 2.*  $C'$  contains at least two vertices of  $U'_s$ . Denote the set of these vertices by  $\Gamma$ . Then  $2 \leq |\Gamma| \leq 4$ . Let  $m$  be the number of the 4-cycles of  $\Delta_s$  containing at least one vertex of  $\Gamma$ . Then  $1 \leq m \leq |\Gamma| \leq 4$ . Without loss of generality, we may suppose that  $C_1, \dots, C_m$  with  $C_i = (a_i, b_i, u_i, v_i, a_i)$  for  $1 \leq i \leq m$  are these 4-cycles. Note that  $a_i \in U_s$ . Since  $e_{\Omega_s}(a_i) \geq 4$ , there exist  $m$  independent arcs, say  $\omega_1, \dots, \omega_m$ , of  $\Omega_s$  which are disjoint with  $V(C') \setminus \Gamma$ , such that  $\omega_i \in e_{\Omega_s}(a_i)$  for each  $1 \leq i \leq m$ . Since  $f_W(\omega_i) \geq 5$  (for each  $1 \leq i \leq m$ ), there exist  $m$  vertices  $\gamma_1, \dots, \gamma_m$  of  $W$  distinct from the vertices of  $V(C') \setminus \Gamma$  such that  $\gamma_i \in F_W(\omega_i)$ . Clearly,  $C^i = (a_i, x_i, \gamma_i, x_{t+1-i}, a_i)$ ,  $1 \leq i \leq m$ , and  $C^i$  are  $m + 1$  disjoint 4-cycles. Now  $(S \setminus \{C_1, \dots, C_m\}) \cup \{C', C^1, \dots, C^m\}$  is a collection of  $s + 1$  disjoint 4-cycles, a contraction. □

Since the subtournament  $T[V'_s]$  is  $C_4$ -free, let  $(\alpha_1, \dots, \alpha_{\gamma_s})$  be a Hamiltonian path of  $T[V'_s]$ , where  $\gamma_s = 3u_s + t = |V'_s|$ .

**Claim 3.7.** *There exists a set  $S'$  of  $s$  disjoint 4-cycles such that  $\{\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}\} \subseteq V(T_{S'})$ .*

**Proof.** Let  $p$  be the number of the vertices of  $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$  which are in  $U'_s$ . When  $p = 0$ , we take  $S' = S$  and clearly the result is proved. Now suppose

that  $p \geq 1$  and let  $m$  be the number of the 4-cycles of  $\Delta_s$  containing at least one vertex of  $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$ . Without loss of generality, we may suppose that  $C_1, C_2, \dots, C_m$  (with  $C_i = (a_i, b_i, u_i, v_i, a_i)$ ,  $1 \leq i \leq m$ ) are these 4-cycles. Note that  $a_i \in U_s$  for each  $1 \leq i \leq m$ . We have  $1 \leq m \leq p \leq 4$  with  $m \geq 2$  when  $p = 4$ . Since  $e_{\Omega_s}(a_i) \geq 4$  for each  $1 \leq i \leq m$ , there exist  $m$  independent arcs, without loss of generality, say  $\omega_1, \dots, \omega_m$ , of  $\Omega_s$  with  $\omega_i \in E_{\Omega_s}(a_i)$  for each  $1 \leq i \leq m$ . Since  $f_W(\omega_i) \geq 5$ , there exist  $m$  distinct vertices  $y_i \in W$  for each  $1 \leq i \leq m$ . This yields  $m$  disjoint 4-cycles  $C'_i = (a_i, x_i, y_i, x_{t+1-i}, a_i)$  for each  $1 \leq i \leq m$ , and these 4-cycles do not contain any vertex of  $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$ . Then  $S' = (S \setminus \{C_1, \dots, C_m\}) \cup \{C'_1, \dots, C'_m\}$  is a set of  $s$  disjoint 4-cycles. The vertices  $\alpha_1, \alpha_2, \alpha_{\gamma_s-1}, \alpha_{\gamma_s}$  are in  $T_{s'}$ , and the vertices of  $V(T_{s'})$  are vertices of  $T[V'_s]$ .  $\square$

Recall that  $T_s$  is the  $C_4$ -free subtournament induced by the vertices of  $T$  not contained in a 4-cycle of  $S$ , and that the vertices of  $T_s$  can be ordered into a Hamiltonian path which we denote here by  $(x_1^S, \dots, x_t^S)$ . Clearly, this notation (and the other using  $S$  as subscript or superscript) is valid for every set of  $s$  disjoint 4-cycles.

Let  $N = V'_s, M = V(T_{s'}), P = (\alpha_1, \dots, \alpha_{\gamma_s})$ , by Claim 3.7 and Lemma 2.1, it is easy to see that

**Claim 3.8.** *There exists a set  $S'$  of  $s$  disjoint 4-cycles such that  $x_1^{S'} = \alpha_1, x_t^{S'} = \alpha_{\gamma_s}$ .*

Now we can achieve the proof of Theorem 1.2. We work on the set  $S'$  of  $s$  disjoint 4-cycles constructed in Claim 3.7. Here  $\Omega_{s'}$  is the set of the independent arcs  $\omega_i^{S'}$  with  $\omega_i^{S'} = (x_i^{S'}, x_{t+1-i}^{S'})$  for each  $1 \leq i \leq t$ .

First, since  $e(\omega_1^{S'}) \geq t + 3u_s - 3$ , we have  $e(\Omega_{s'}) \geq 7t - 63 + 3u_s$ .

On the other hand, since  $e_{\Omega_{s'}}(C) \leq 7$  when  $C$  is a 4-cycle of  $\Delta_{s'}$ , and  $e_{\Omega_{s'}} \leq 4$  when  $C$  is not a 4-cycle of  $\Delta_{s'}$  (by Claim 3.5), we deduce  $e(\Omega_{s'}) \leq 7u_{s'} + 4(s - u_{s'})$ . It follows that  $7t - 63 + 3u_s \leq 3u_{s'} + 4s$ .

As  $u_{s'} \leq u_s$  (by the maximality of  $u_s$ ), it follows that  $7t - 63 + 3u_s \leq 3u_s + 4s$ . Hence  $7t - 63 \leq 4s$ , which gives  $s \geq \frac{21}{16}r - \frac{35}{16} > \frac{21}{16}r - \frac{10}{3}$ . So Theorem 1.2 is proved.  $\blacksquare$

### Acknowledgment

The authors are indebted to anonymous referees for their valuable comments and suggestions.

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Received 30 June 2016

Revised 2 January 2017

Accepted 3 January 2017