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THE SMALLEST HARMONIC INDEX OF TREES WITH GIVEN MAXIMUM DEGREE

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Abstract

The harmonic index of a graph G, denoted by $H(G)$, is defined as the sum of weights $2/[d(u) + d(v)]$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u. In this paper we establish a lower bound on the harmonic index of a tree T.

Keywords: harmonic index, trees.

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1. Introduction

Let G be a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and the size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex $v \in V$ is $d_v = d(v) = d_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. An *leaf* of a tree T is a vertex of degree 1, a *stem* is a vertex adjacent to a leaf, whereas a *strong stem* is a stem adjacent to at least two leaves. An *end stem* is a stem whose all neighbors with exception at most one are leaves. For every two vertices x, y of a tree T, we denote the unique (x, y)-path by xTy. A path $P = u_0u_1 \cdots u_k$ $(k \geq 1)$ in G is called a *pendant path* if $d_{u_0} \geq 3$, $d_{u_k} = 1$ and the degree of any other vertex of the path is 2. To *contract* an edge e of a graph G, is to delete the edge and then identify its ends. The resulting graph is denoted by G/e. Let $\mathcal{T}_{n,\Delta}$ be the family of trees T of order n and maximum degree Δ.

A large variety of degree based topological indices has been defined in the mathematical and mathematico-chemical literature; for details we refer the reader to [4, 6]. Here, we focus on the harmonic index. For a simple graph G, the *harmonic index* of G, denoted $H(G)$, is defined in [3] as the sum of weights $2/[d(u)+d(v)]$ of all edges uv of G. That is, $H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}$. For some related works see [9, 17, 24–28, 30–33]. Wu *et al.* [20] established a lower bound on $H(G)$ of a graph with minimum degree two. Favaron *et al.* [5] investigated the relation between graph eigenvalues of graphs and the harmonic index. Deng *et al.* [1] considered the relation between $H(G)$ and the chromatic index $\chi(G)$, and proved that $\chi(G) \leq 2H(G)$. Liu [13] proposed a conjecture concerning the relation between the harmonic index and the diameter of a connected graph, and showed that the conjecture is true for trees. Relationships between the harmonic index and several other topological indices were established in [8, 22, 29]. For additional results on this index, see [11, 12, 14–17, 21].

In this paper we establish a lower bound for the harmonic index of a tree T in terms of its order and maximum degree. Our result is an extension of some well-known lower bound on the harmonic index of a tree T.

2. A Lower Bound on the Harmonic Index of Trees

In this section we prove the following lower bound for the harmonic index of a tree T of order n with maximum degree Δ.

Theorem 1. *Let $\Delta \geq 3$ and $T \in \mathcal{T}_{n,\Delta}$. If $n \equiv r \pmod{\Delta - 1}$, then*

$$H(T) \geq \begin{cases} 2\left(\frac{n(\Delta-2)}{\Delta^2-1} + \frac{\Delta-2}{2\Delta-2} + \frac{n-(\Delta-1)^2}{2\Delta(\Delta-1)}\right) & \text{if } r = 0 \text{ and } n > (\Delta-1)(\Delta-2),\\ 2\left(\frac{(\Delta-1)^2-n}{(\Delta-1)^2} + \frac{n-\Delta+1}{\Delta+1} + \frac{n-\Delta+1}{2(\Delta-1)^2}\right) & \text{if } r = 0 \text{ and } n \leq (\Delta-1)(\Delta-2),\\ 2\left(\frac{n(\Delta-2)+1}{\Delta^2-1} + \frac{\Delta-1}{2\Delta-1} + \frac{n-1-\Delta(\Delta-1)}{2\Delta(\Delta-1)}\right) & \text{if } r = 1 \text{ and } n > (\Delta-1)^2+1,\\ 2\left(\frac{\Delta(\Delta-1)-n+1}{\Delta(\Delta-1)} + \frac{n-\Delta}{\Delta+1} + \frac{n-\Delta}{(2\Delta-1)(\Delta-1)}\right) & \text{if } r = 1 \text{ and } n \leq (\Delta-1)^2+1,\\ 2\left(\frac{n(\Delta-2)+2}{\Delta^2-1} + \frac{n-\Delta-1}{2\Delta(\Delta-1)}\right) & \text{if } r = 2,\\ 2\left(\frac{n(\Delta-2)+r-\Delta+1}{\Delta^2-1} + \frac{r-1}{\Delta+r-1} + \frac{n-(r-1)\Delta-1}{2\Delta(\Delta-1)}\right) & \text{if } r \geq 3 \text{ and } n \geq \Delta(r-1)+1,\\ 2\left(\frac{(r-1)\Delta-n+1}{r(\Delta-1)} + \frac{n-r}{\Delta+1} + \frac{n-r}{(\Delta+r-1)(\Delta-1)}\right) & \text{if } r \geq 3 \text{ and } n < \Delta(r-1)+1. \end{cases}$$

For notational convenience, let $h_\omega : E(T) \to \mathbb{R}$ denote a function defined by $h_\omega(uv) = 1/[d(u) + d(v)]$. Hence $H(T) = 2\sum_{e\in E(G)} h_\omega(e)$. We begin with some lemmas.

Lemma 2. *Let $T \in \mathcal{T}_{n,\Delta}$. If u and v are two adjacent vertices each of degree at least two in T with $d_T(u) + d_T(v) \le \Delta + 1$, then there exists a tree T' of order n with maximum degree $\Delta(T)$ such that $H(T') < H(T)$.*

Proof. Let $T' := (T/e) + up$ be the tree obtained from T by contracting the edge $e = uv$ and adding a pendant edge up. Clearly, T' is a tree of order n with $\Delta(T') \le \Delta(T)$. By the assumptions and the constriction of T', we have $d_T(u) \le \Delta - 1$, $d_T(v) \le \Delta - 1$, and $d_{T'}(u) \le \Delta$. If $w \in V(T)$ is a vertex with maximum degree $\Delta(T)$, then we have $w \notin \{u, v\}$ and $d_T(w) = d_{T'}(w)$. Hence $\Delta(T') = \Delta(T)$. Assume that $d(u) = \alpha$, $d(v) = \beta$, $N(u) = \{x_1, \ldots, x_{\alpha-1}, v\}$, $N(v) = \{y_1, \ldots, y_{\beta-1}, u\}$ and $S = \{xu \,|\, x \in N(u)\} \cup \{yv|y \in N(v)\}$. Then we have

$$\frac{1}{2}H(T) = \sum_{e\in E(T)-S} h_\omega(e) + \frac{1}{\alpha+\beta} + \sum_{i=1}^{\alpha-1} \frac{1}{d(x_i)+\alpha} + \sum_{i=1}^{\beta-1} \frac{1}{d(y_i)+\beta}$$

and

$$\frac{1}{2}H(T') = \sum_{e\in E(T)-S} h_\omega(e) + \frac{1}{\alpha+\beta} + \sum_{i=1}^{\alpha-1} \frac{1}{d(x_i)+\alpha+\beta-1} + \sum_{i=1}^{\beta-1} \frac{1}{d(y_i)+\alpha+\beta-1}.$$

Clearly $H(T') < H(T)$ and the proof is complete. ■

Lemma 3. *Let $T \in \mathcal{T}_{n,\Delta}$, let u and v be two vertices of T with $d_T(u) = \alpha < \beta = d_T(v)$ and let $x \in N(u)$ and $y \in N(v)$ such that $x, y \notin uTv$ or $x, y \in uTv$. If $d_T(x) < d_T(y)$, then there exists a tree T' of order n with maximum degree $\Delta(T)$ such that $H(T') < H(T)$.*

Proof. Let T' be the tree obtained from T by removing the edges ux, vy and adding new edges vx, uy (see Figure 1). Clearly, T' is a connected graph of order n with $n-1$ edges and so T' is a tree. Also, we have $d_T(z) = d_{T'}(z)$ for each $z \in V(T)$ and hence $\Delta(T') = \Delta(T)$. Let $S = \{ux, vy\}$. Then we have

$$\frac{1}{2}H(T) = \sum_{e\in E(T)\setminus S} h_\omega(e) + \frac{1}{\alpha+d_T(x)} + \frac{1}{\beta+d_T(y)}$$

and

$$\frac{1}{2}H(T') = \sum_{e\in E(T)\setminus S} h_\omega(e) + \frac{1}{\beta+d_T(x)} + \frac{1}{\alpha+d_T(y)}.$$

It follows from $\alpha < \beta$ and $d_T(x) < d_T(y)$ that $H(T') < H(T)$. ■

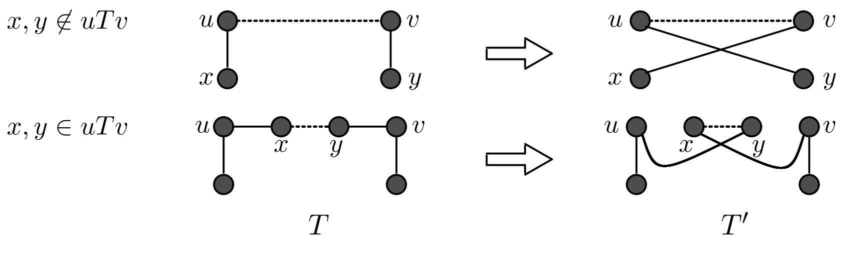

Figure 1. The switching process used in the proof of Lemma 3.

Lemma 4. *Let $T \in \mathcal{T}_{n,\Delta}$ be an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$. If u and v are two vertices of T of degree α with $2 \le \alpha \le \Delta - 1$, then there exists an extremal tree T^* with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$ such that $V(T^*) = V(T)$, $d_T(z) = d_{T^*}(z)$ for each $z \in V(T)$, and $d_{T^*}(x) \ge d_{T^*}(y)$ for each $x \in N_{T^*}(u) - V(uTv)$ and $y \in N_{T^*}(v) - V(uTv)$.*

Proof. If $d_T(x) \ge d_T(y)$ for each $x \in N_T(u) - V(uTv)$ and $y \in N_T(v) - V(uTv)$, then we are done. Let $d_T(x) < d_T(y)$ for some $x \in N_T(u) - V(uTv)$ and some $y \in N_T(v) - V(uTv)$. Assume T_1 to be the tree obtained from T by deleting the edges ux, vy and adding new edges uy, vx. Clearly, $V(T_1) = V(T)$ and $d_T(z) = d_{T_1}(z)$ for each $z \in V(T)$ and hence $T_1 \in \mathcal{T}_{n,\Delta}$. Since $d_{T_1}(u) = d_{T_1}(v) = \alpha$, it is easy to verify that $H(T) = H(T_1)$. Thus T_1 is a extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$. By repeating this process, we obtain a desired tree T^*. ■

Lemma 5. *If $T \in \mathcal{T}_{n,\Delta}$ is an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$, then T has at most one vertex of degree $1 < t < \Delta$.*

Proof. Assume, to the contrary, that T has two distinct vertices u and v such that $1 < d(u) = \alpha \le \beta = d(v) < \Delta$. Also, suppose that among two vertices with this property we choose two distinct vertices u, v such that $d(u, v)$ is as small as possible. Let $N(u) = \{x_1, \dots, x_\alpha\}$, $N(v) = \{y_1, \dots, y_\beta\}$, $S = \{xu | x \in N(u)\} \cup \{yv | y \in N(v)\}$ and $K = \sum_{e \in E(T)-S} h_\omega(e)$. Assume that $x_1, y_1 \in uTv$, $d_{x_\alpha} \ge \cdots \ge d_{x_2}$ and $d_{y_\beta} \ge \cdots \ge d_{y_2}$. By Lemmas 3 and 4, we may suppose that $d_{x_\alpha} \ge \cdots \ge d_{x_2} \ge d_{y_\beta} \ge \cdots \ge d_{y_2}$. Let $T' := T - ux_2 + vx_2$ be the tree obtained from T by removing the edge ux_2 and adding a new edge vx_2 (see Figure 2). We show that $H(T') < H(T)$. Consider four cases.

Case 1. $uv \in E(T)$ and $d_u = d_v = \alpha$. Then $x_1 = v$ and $y_1 = u$. By definition we have

$$\frac{1}{2}H(T) = K + \frac{1}{2\alpha} + \frac{1}{d_{x_2} + \alpha} + \sum_{i=3}^{\alpha} \frac{1}{d_{x_i} + \alpha} + \sum_{i=2}^{\alpha} \frac{1}{d_{y_i} + \alpha}$$

and

$$\frac{1}{2}H(T') = K + \frac{1}{2\alpha} + \frac{1}{d_{x_2}+\alpha+1} + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha-1} + \sum_{i=2}^{\alpha}\frac{1}{d_{y_i}+\alpha+1}.$$

Now, we have

$$\begin{aligned}
&\frac{1}{2}\big(H(T')-H(T)\big)\\
&= \sum_{i=3}^{\alpha}\frac{1}{(d_{x_i}+\alpha)(d_{x_i}+\alpha-1)} + \sum_{i=2}^{\alpha}\frac{-1}{(d_{y_i}+\alpha)(d_{y_i}+\alpha+1)}\\
&+ \frac{-1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\\
&= \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_i}+\alpha)(d_{x_i}+\alpha-1)} - \frac{1}{(d_{y_i}+\alpha)(d_{y_i}+\alpha+1)}\right)\\
&+ \left(\frac{-1}{(d_{y_2}+\alpha)(d_{y_2}+\alpha+1)} + \frac{-1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\right)\\
&\le \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)} - \frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\right)\\
&+ \left(\frac{-1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)} + \frac{-1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\right)\\
&\le \sum_{i=3}^{\alpha}\left(\frac{2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)}\right) + \frac{-2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\\
&\le \frac{2(\alpha-2)}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)} + \frac{-2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\\
&= \frac{-2d_{x_2}-2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)} < 0.
\end{aligned}$$

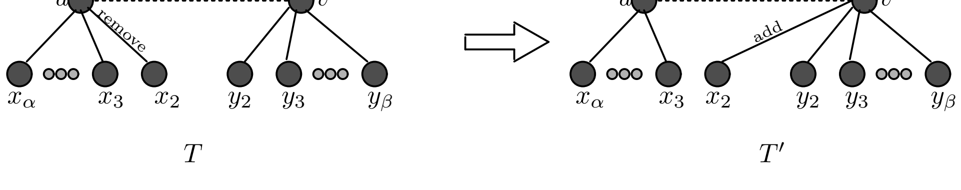

Figure 2. The switching process used in the proof of Lemma 5.

Case 2. $uv \in E(T)$, $d_u = \alpha < \beta = d_v$. As above $x_1 = v$ and $y_1 = u$. By definition we have

$$\frac{1}{2}H(T) = K + \frac{1}{\alpha+\beta} + \frac{1}{d_{x_2}+\alpha} + \frac{1}{d_{y_2}+\beta} + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\beta}$$

and

$$\frac{1}{2}H(T') = K + \frac{1}{\alpha+\beta} + \frac{1}{d_{x_2}+\beta+1} + \frac{1}{d_{y_2}+\beta+1} + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha-1} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\beta+1}.$$

Now, we have

$$\begin{aligned}
&\frac{1}{2}(H(T')-H(T))\\
&= \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_i}+\alpha)(d_{x_i}+\alpha-1)} - \frac{1}{(d_{y_i}+\beta)(d_{y_i}+\beta+1)}\right)\\
&+ \sum_{i=\alpha+1}^{\beta}\frac{-1}{(d_{y_i}+\beta)(d_{y_i}+\beta+1)} + \left(\frac{\alpha-\beta-1}{(d_{x_2}+\alpha)(d_{x_2}+\beta+1)} + \frac{-1}{(d_{y_2}+\beta)(d_{y_2}+\beta+1)}\right)\\
&\le \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)} - \frac{1}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\right)\\
&+ \sum_{i=\alpha+1}^{\beta}\frac{-1}{(d_{y_i}+\beta)(d_{y_i}+\beta+1)} + \left(\frac{\alpha-\beta-1}{(d_{x_2}+\alpha)(d_{x_2}+\beta+1)} + \frac{-1}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\right)\\
&= \sum_{i=3}^{\alpha}\left(\frac{(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\right)\\
&+ \sum_{i=\alpha+1}^{\beta}\frac{-1}{(d_{y_i}+\beta)(d_{y_i}+\beta+1)} + \frac{(\alpha-\beta)(d_{x_2}+\beta)-(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\\
&\le \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\\
&+ \frac{\alpha-\beta}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{(\alpha-\beta)(d_{x_2}+\beta)-(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\\
&= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\\
&+ \frac{(\alpha-\beta)(d_{x_2}+\alpha)+(\alpha-\beta)(d_{x_2}+\beta)-(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\\
&= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{(\alpha-\beta-1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}
\end{aligned}$$

$$= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})+(\alpha-\beta-1)(d_{x_2}+\alpha-1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)(d_{x_2}+\alpha-1)}$$
$$= \frac{(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})(-d_{x_2}-1)}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)(d_{x_2}+\alpha-1)} < 0$$

Case 3. $uv \notin E(T)$ and $d_u = d_v = \alpha$. By the choice of u, v, we may assume that $d_{x_1} = d_{y_1} = \Delta$. We have

$$\frac{1}{2}H(T) = K + \frac{1}{\alpha+\Delta} + \frac{1}{\alpha+\Delta} + \frac{1}{d_{x_2}+\alpha} + \frac{1}{d_{y_2}+\alpha} + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\alpha}$$

and

$$\begin{aligned}\frac{1}{2}H(T') &= K + \frac{1}{\alpha+\Delta-1} + \frac{1}{\alpha+\Delta+1} + \frac{1}{d_{x_2}+\alpha+1} + \frac{1}{d_{y_2}+\alpha+1}\\ &\quad + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha-1} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\alpha+1}.\end{aligned}$$

Now, we have

$$\begin{aligned}
&\frac{1}{2}(H(T')-H(T))\\
&= \sum_{i=3}^{\alpha}\frac{1}{(d_{x_i}+\alpha)(d_{x_i}+\alpha-1)} + \sum_{i=3}^{\alpha}\frac{-1}{(d_{y_i}+\alpha)(d_{y_i}+\alpha+1)}\\
&+ \frac{2}{(\alpha+\Delta)(\alpha+\Delta-1)(\alpha+\Delta+1)} + \frac{-1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)} + \frac{-1}{(d_{y_2}+\alpha)(d_{y_2}+\alpha+1)}\\
&\le \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)} - \frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\right)\\
&+ \frac{2}{(\alpha+\Delta)(\alpha+\Delta-1)(\alpha+\Delta+1)} - \frac{2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\\
&= \frac{2(\alpha-2)}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)} + \frac{2}{(\alpha+\Delta)(\alpha+\Delta-1)(\alpha+\Delta+1)}\\
&- \frac{2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha+1)}\\
&= \frac{2(\alpha-2)-2(d_{x_2}+\alpha-1)}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)} + \frac{2}{(\alpha+\Delta)(\alpha+\Delta-1)(\alpha+\Delta+1)}\\
&= \frac{-2d_{x_2}-2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)} + \frac{2}{(\alpha+\Delta)(\alpha+\Delta-1)(\alpha+\Delta+1)}
\end{aligned}$$

$$\leq \frac{-2d_{x_2}-2}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)}+\frac{2}{(\alpha+d_{x_2})(\alpha+d_{x_2}-1)(\alpha+d_{x_2}+1)}$$
$$\leq \frac{-2d_{x_2}}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\alpha+1)}<0.$$

Case 4. $uv \notin E(T)$ and $d_u = \alpha < \beta = d_v$. As in Case 3, we may assume that $d_{x_1} = d_{y_1} = \Delta$. By definition we have

$$\frac{1}{2}H(T) = K + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\beta} + \frac{1}{\alpha+\Delta} + \frac{1}{\beta+\Delta} + \frac{1}{d_{x_2}+\alpha} + \frac{1}{d_{y_2}+\beta}$$

and

$$\frac{1}{2}H(T') = K + \sum_{i=3}^{\alpha}\frac{1}{d_{x_i}+\alpha-1} + \sum_{i=3}^{\beta}\frac{1}{d_{y_i}+\beta+1} + \frac{1}{\alpha+\Delta-1} + \frac{1}{\beta+\Delta+1}$$
$$+ \frac{1}{d_{x_2}+\beta+1} + \frac{1}{d_{y_2}+\beta+1}.$$

Then we have

$$\frac{1}{2}(H(T')-H(T))$$
$$= \sum_{i=3}^{\alpha}\frac{1}{(d_{x_i}+\alpha)(d_{x_i}+\alpha-1)} + \sum_{i=3}^{\beta}\frac{-1}{(d_{y_i}+\beta)(d_{y_i}+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)}$$
$$+ \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)} + \frac{\alpha-\beta-1}{(d_{x_2}+\alpha)(d_{x_2}+\beta+1)} + \frac{-1}{(d_{y_2}+\beta)(d_{y_2}+\beta+1)}$$
$$\leq \sum_{i=3}^{\alpha}\left(\frac{1}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)} - \frac{1}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)}\right)$$
$$+ \sum_{i=\alpha+1}^{\beta}\frac{-1}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)}$$
$$+ \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)} + \frac{\alpha-\beta-1}{(d_{x_2}+\alpha)(d_{x_2}+\beta+1)} + \frac{-1}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$
$$= \frac{(\alpha-2)((d_{x_2}+\beta)(d_{x_2}+\beta+1)-(d_{x_2}+\alpha)(d_{x_2}+\alpha-1))}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{\alpha-\beta}{(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$
$$+ \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)} + \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)} + \frac{(\alpha-\beta-1)(d_{x_2}+\beta)-(d_{x_2}+\alpha)}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$

$$= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)}$$

$$+ \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)} + \frac{(\alpha-\beta)(d_{x_2}+\beta)-(\alpha+\beta+2d_{x_2})+(\alpha-\beta)(d_{x_2}+\alpha)}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$

$$= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)}$$

$$+ \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)} + \frac{(\alpha-\beta-1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$

$$= \frac{(\alpha-2)(\beta-\alpha+1)(\alpha+\beta+2d_{x_2})+(d_{x_2}+\alpha-1)(\alpha-\beta-1)(\alpha+\beta+2d_{x_2})}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)}$$

$$+ \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)} + \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)}$$

$$= \frac{(\alpha+\beta+2d_{x_2})(\beta-\alpha+1)(-d_{x_2}-1)}{(d_{x_2}+\alpha)(d_{x_2}+\alpha-1)(d_{x_2}+\beta)(d_{x_2}+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)}$$

$$+ \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)}$$

$$\leq \frac{(\alpha+\beta+2d_{x_2})(\beta-\alpha+1)(-d_{x_2}-1)}{(\Delta+\alpha)(\Delta+\alpha-1)(\Delta+\beta)(\Delta+\beta+1)} + \frac{1}{(\alpha+\Delta)(\alpha+\Delta-1)} + \frac{-1}{(\beta+\Delta)(\beta+\Delta+1)}$$

$$= \frac{(\alpha+\beta+2d_{x_2})(\beta-\alpha+1)(-d_{x_2}-1)+(\alpha+\beta+2\Delta)(\beta-\alpha+1)}{(\Delta+\alpha)(\Delta+\alpha-1)(\Delta+\beta)(\Delta+\beta+1)}$$

$$= \frac{(\beta-\alpha+1)((\alpha+\beta+2d_{x_2})(-d_{x_2}-1)+(\alpha+\beta+2\Delta))}{(\Delta+\alpha)(\Delta+\alpha-1)(\Delta+\beta)(\Delta+\beta+1)}.$$

Since $\alpha+\beta+2d_{x_2} < 2\Delta+2d_{x_2}$ and $-d_{x_2}-1 \leq -2$, we deduce that

$$(\alpha+\beta+2d)(-d_{x_2}-1)+(\alpha+\beta+2\Delta) < -4\Delta-4d_{x_2}+(\alpha+\beta+2\Delta) < -4d_{x_2} < 0$$

and hence $\frac{1}{2}(H(T')-H(T)) < 0$.

Thus all cases lead to a contradiction since T has the minimum harmonic index. This completes the proof. ■

Lemma 6. *Let $T \in \mathcal{T}_{n,\Delta}$ be an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$ where $\Delta \geq 3$, $n = (\Delta-1)k + r$ and $0 \leq r \leq \Delta-2$. If n_i is the number of vertices of T of degree i for each $i = 1, 2, \ldots, \Delta$, then the following hold:*

1. *if $r = 0, 1$, then $n_\Delta = k-1$, $n_{\Delta-2+r} = 1$ and $n_1 = n-k$,*
2. *if $r = 2$, then $n_\Delta = k$ and $n_1 = n-k$,*
3. *if $r \geq 3$, then $n_\Delta = k$, $n_{r-1} = 1$ and $n_1 = n-k-1$.*

Proof. Let n_i be the number of vertices of T of degree i for each $i = 1, 2, \ldots, \Delta$. Then $n_1 + n_2 + \cdots + n_\Delta = n$ and $n_1 + 2n_2 + \ldots + \Delta n_\Delta = 2n - 2$ and hence

$$n_2 + 2n_3 + \cdots + (\Delta - 1)n_\Delta = n - 2. \tag{1}$$

By Lemma 5 we have $n_2 + n_3 + \cdots + n_{\Delta-1} \leq 1$ that yields

$$n_2 + 2n_3 + \cdots + (\Delta - 2)n_{\Delta-1} \leq \Delta - 2. \tag{2}$$

Assume $n_t = 1$ if $n_2 + n_3 + \cdots + n_{\Delta-1} = 1$.

(1) If $r = 0, 1$, then we deduce from (1) that $n_2 + n_3 + \cdots + n_{\Delta-1} = 1$ and so

$$(t - 1) + (\Delta - 1)n_\Delta = (\Delta - 1)k + r - 2 = (\Delta - 1)(k - 1) + (\Delta - 3 + r).$$

This implies that $n_\Delta = k - 1$, $n_t = n_{\Delta-2+r} = 1$ and $n_1 = n - k$.

(2) If $r = 2$, then we conclude from (1) and (2) that $n_2 + n_3 + \cdots + n_{\Delta-1} = 0$ and so $n_\Delta = k$ and $n_1 = n - k$.

(3) Let $r \geq 3$. Then we have

$$(t - 1) + (\Delta - 1)n_\Delta = (\Delta - 1)k + r - 2,$$

and this implies that $n_\Delta = k$, $n_t = n_{r-1} = 1$ and $n_1 = n - k - 1$. ■

Let $E_{i,j}$ denote the set of all edges having a vertex of degree i at one end and a vertex of degree j at the other end and let $\varepsilon_{i,j} = |E_{i,j}|$.

Lemma 7. *Let $T \in \mathcal{T}_{n,\Delta}$ be an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$ and let T have a vertex v of degree t with $1 < t < \Delta$. Then $\varepsilon_{1,t}$ is as small as possible.*

Proof. It follows from Lemma 5 that $\deg(u) = 1$ or Δ for each $u \in V(T) - \{v\}$ and hence $E(T) = E_{1,t} \cup E_{1,\Delta} \cup E_{t,\Delta} \cup E_{\Delta,\Delta}$. By definition we have

$$\begin{aligned}
\frac{1}{2}H(T) &= \frac{\varepsilon_{1,t}}{1+t} + \frac{\varepsilon_{1,\Delta}}{1+\Delta} + \frac{\varepsilon_{t,\Delta}}{t+\Delta} + \frac{\varepsilon_{\Delta,\Delta}}{2\Delta} \\
&= \frac{\varepsilon_{1,t}}{1+t} + \frac{n_1 - \varepsilon_{1,t}}{1+\Delta} + \frac{t - \varepsilon_{1,t}}{t+\Delta} + \frac{n - 1 - n_1 - \varepsilon_{t,\Delta}}{2\Delta} \\
&= \frac{\varepsilon_{1,t}}{1+t} + \frac{n_1 - \varepsilon_{1,t}}{1+\Delta} + \frac{t - \varepsilon_{1,t}}{t+\Delta} + \frac{n - 1 - n_1 - t + \varepsilon_{1,t}}{2\Delta} \\
&= \varepsilon_{1,t}\left(\frac{1}{1+t} + \frac{1}{2\Delta}\right) - \varepsilon_{1,t}\left(\frac{1}{1+\Delta} + \frac{1}{t+\Delta}\right) + \left(\frac{n_1}{1+\Delta} + \frac{t}{t+\Delta} + \frac{n-1-n_1-t}{2\Delta}\right)
\end{aligned}$$

$$= \varepsilon_{1,t}\left(\frac{2\Delta+t+1}{2(1+t)\Delta} - \frac{2\Delta+t+1}{(1+\Delta)(t+\Delta)}\right) + \left(\frac{n_1}{1+\Delta} + \frac{t}{t+\Delta} + \frac{n-1-n_1-t}{2\Delta}\right)$$
$$= \varepsilon_{1,t}(2\Delta+t+1)\left(\frac{1}{2\Delta(1+t)} - \frac{1}{(1+\Delta)(t+\Delta)}\right) + \left(\frac{n_1}{1+\Delta} + \frac{t}{t+\Delta} + \frac{n-1-n_1-t}{2\Delta}\right)$$
$$= \varepsilon_{1,t}(2\Delta+t+1)\left(\frac{t+\Delta+t\Delta+\Delta^2-2\Delta-2t\Delta}{2\Delta(1+t)(1+\Delta)(t+\Delta)}\right) + \left(\frac{n_1}{1+\Delta} + \frac{t}{t+\Delta} + \frac{n-1-n_1-t}{2\Delta}\right)$$
$$= \varepsilon_{1,t}(2\Delta+t+1)\cdot\frac{(\Delta-1)(\Delta-t)}{2\Delta(1+t)(1+\Delta)(t+\Delta)} + \left(\frac{n_1}{1+\Delta} + \frac{t}{t+\Delta} + \frac{n-1-n_1-t}{2\Delta}\right).$$

Since $\frac{(\Delta-1)(\Delta-t)}{2\Delta(1+t)(1+\Delta)(t+\Delta)} > 0$ and T is an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$, we conclude that $\varepsilon_{1,t}$ is as small as possible. ■

***Proof of Theorem* 1.** Let $T^* \in \mathcal{T}_{n,\Delta}$ be an extremal tree with the minimum harmonic index in $\mathcal{T}_{n,\Delta}$. We consider four cases.

Case 1. $r = 0$. Then $n_\Delta = k-1$, $n_t = n_{\Delta-2} = 1$ and $n_1 = n-k$ by Lemma 6. We have also $\varepsilon_{1,\Delta} = n-k-\varepsilon_{1,t}$, $\varepsilon_{t,\Delta} = \Delta-2-\varepsilon_{1,t}$ and $\varepsilon_{\Delta,\Delta} = k-\Delta+\varepsilon_{1,t}+1$. Consider two subcases.

Subcase 1.1. $k = \frac{n}{\Delta-1} > t = \Delta-2$, that is, $n > (\Delta-1)(\Delta-2)$. We conclude from Lemma 7 that $\varepsilon_{1,t} = 0$ and hence $\varepsilon_{1,\Delta} = n-k$, $\varepsilon_{t,\Delta} = \Delta-2$ and $\varepsilon_{\Delta,\Delta} = k-\Delta+1$. Therefore,

$$\frac{1}{2}H(T^*) = \frac{n-k}{1+\Delta} + \frac{\Delta-2}{t+\Delta} + \frac{k-\Delta+1}{2\Delta} = \frac{n-k}{\Delta+1} + \frac{\Delta-2}{2\Delta-2} + \frac{k-\Delta+1}{2\Delta}.$$

Subcase 1.2. $k = \frac{n}{\Delta-1} \leq t = \Delta-2$, that is, $n \leq (\Delta-1)(\Delta-2)$. Then we must have $\varepsilon_{1,t} = t-n_\Delta = t-k+1 = \Delta-k-1$ which implies that $\varepsilon_{1,\Delta} = n-\Delta+1$, $\varepsilon_{t,\Delta} = k-1$ and $\varepsilon_{\Delta,\Delta} = 0$. Therefore,

$$\frac{1}{2}H(T^*) = \frac{\Delta-k-1}{1+t} + \frac{n-\Delta+1}{1+\Delta} + \frac{k-1}{t+\Delta} = \frac{\Delta-k-1}{\Delta-1} + \frac{k-1}{2\Delta-2} + \frac{n-\Delta+1}{\Delta+1}.$$

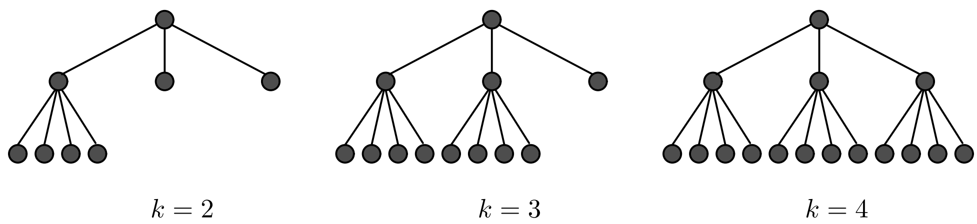

Figure 3. $\Delta = 5,\ r = 0,\ t = \Delta-2 = 3,\ n = (\Delta-1)k = 8, 12, 16.$

Case 2. $r = 1$. As in Case 1, we have $n_\Delta = k-1$, $n_t = n_{\Delta-1} = 1$, $n_1 = n-k$, $\varepsilon_{1,\Delta} = n-k-\varepsilon_{1,t}$, $\varepsilon_{t,\Delta} = \Delta-1-\varepsilon_{1,t}$ and $\varepsilon_{\Delta,\Delta} = k-\Delta+\varepsilon_{1,t}$. If $k = \frac{n-1}{\Delta-1} > t = \Delta-1$ that is $n > (\Delta-1)^2+1$, then as in Subcase 1.1. we have $\varepsilon_{1,t} = 0$, $\varepsilon_{1,\Delta} = n-k$, $\varepsilon_{t,\Delta} = \Delta-1$, $\varepsilon_{\Delta,\Delta} = k-\Delta$ and by definition we have

$$\frac{1}{2}H(T^*) = \frac{n-k}{\Delta+1} + \frac{\Delta-1}{2\Delta-1} + \frac{k-\Delta}{2\Delta}.$$

If $k \le t = \Delta-1$ that is $n \le (\Delta-1)^2+1$, then we have $\varepsilon_{1,t} = \Delta-k$, $\varepsilon_{1,\Delta} = n-\Delta$, $\varepsilon_{t,\Delta} = k-1$ and $\varepsilon_{\Delta,\Delta} = 0$. Hence

$$\frac{1}{2}H(T^*) = \frac{\Delta-k}{1+t} + \frac{n-\Delta}{1+\Delta} + \frac{k-1}{t+\Delta} = \frac{\Delta-k}{\Delta} + \frac{n-\Delta}{\Delta+1} + \frac{k-1}{2\Delta-1}.$$

Case 3. $r = 2$. In this case we have $n_\Delta = k$, $n_1 = n-k$, $\varepsilon_{1,\Delta} = n_1 = n-k$ and $\varepsilon_{\Delta,\Delta} = (n-1)-(n-k) = k-1$. It follows from definition that

$$\frac{1}{2}H(T^*) = \frac{n-k}{\Delta+1} + \frac{k-1}{2\Delta}.$$

Case 4. $r \ge 3$. By Lemma 6 we have $n_\Delta = k$, $n_t = n_{r-1} = 1$ and $n_1 = n-k-1$. Also we have $\varepsilon_{1,\Delta} = n-k-1-\varepsilon_{1,t}$, $\varepsilon_{t,\Delta} = r-1-\varepsilon_{1,t}$ and $\varepsilon_{\Delta,\Delta} = k-r+\varepsilon_{1,t}+1$. An argument similar to that described in Case 1 shows that

$$\frac{1}{2}H(T^*) = \frac{n-k-1}{\Delta+1} + \frac{r-1}{\Delta+r-1} + \frac{k-r+1}{2\Delta}$$

if $k = \frac{n-r}{\Delta-1} \ge t = r-1$ that is $n \ge \Delta(r-1)+1$, and

$$\frac{1}{2}H(T^*) = \frac{r-k-1}{1+t} + \frac{n-r}{\Delta+1} + \frac{k}{\Delta+t} = \frac{r-k-1}{r} + \frac{n-r}{\Delta+1} + \frac{k}{\Delta+r-1}$$

when $k < t = r-1$ that is $n < \Delta(r-1)+1$.

Replacing k by $\frac{n-r}{\Delta-1}$ in all cases, we arrive at the bounds of Theorem 1. This completes the proof. ■

Applying Theorem 1, we can get two corollaries in the following.

Corollary 8. *Let T be a tree of order n and maximum degree Δ. If $\Delta \ge 3$ and $n = (\Delta-1)k+r$, $0 \le r \le \Delta-2$, then*

$$H(T) \ge 2\left(\frac{n(\Delta-2)+r}{\Delta^2-1} + \frac{n-\Delta-r+1}{2\Delta(\Delta-1)}\right)$$

with equality if and only if $n-2 = (\Delta-1)k$ and $n_\Delta = k$.

Corollary 9 ([10]). *For any tree T of order $n \geq 3$,*

$$H(T) \geq \frac{2(n-1)}{n}$$

with equality if and only if T is a star.

In Figure 4, we determine the harmonic index of all trees of order 6 and 7 with maximum degree at least 3.

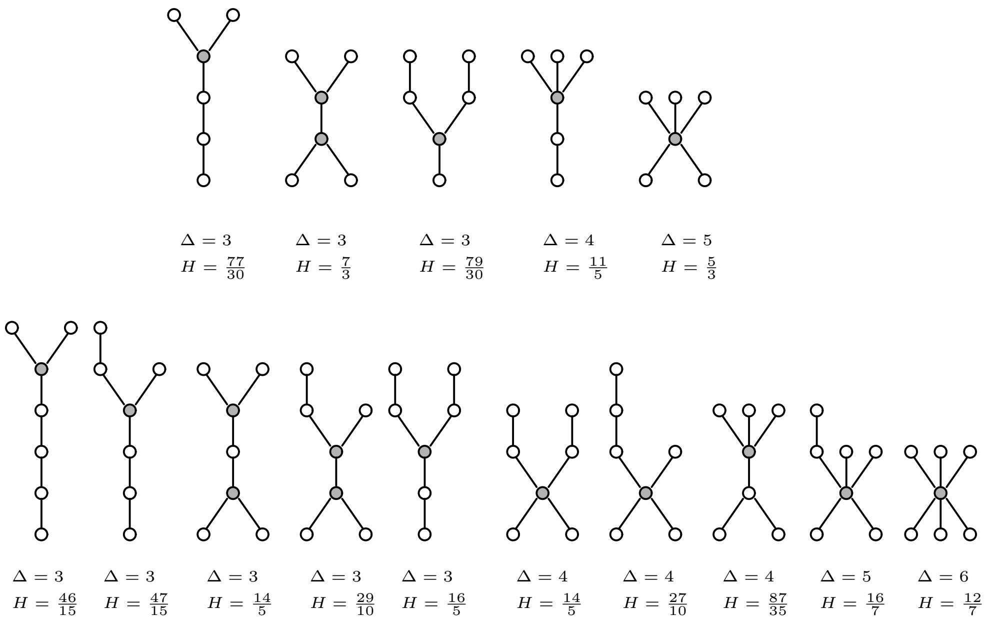

Figure 4. The harmonic index of all trees T of order 6 and 7 with $\Delta(T) \geq 3$.

References

[1] H. Deng, S. Balachandran, S.K. Ayyaswamy and Y.B. Venkatakrishnan, *On the harmonic index and the chromatic number of a graph*, Discrete Appl. Math. **161** (2013) 2740–2744.
doi:10.1016/j.dam.2013.04.003

[2] H. Deng, S. Balachandran, S.K. Ayyaswamy and V.B. Venkatakrishnan, *On harmonic indices of trees, unicyclic graphs and bicyclic graphs*, Ars Combin. **130** (2017) 239–248.

[3] S. Fajtlowicz, *On conjectures of Graffiti*-II, Congr. Numer. **60** (1987) 187–197.

[4] B. Furtula, I. Gutman and M. Dehmer, *On structure-sensitivity of degree-based topological indices*, Appl. Math. Comput. **219** (2013) 8973–8978.
doi:10.1016/j.amc.2013.03.072

[5] O. Favaron, M. Mahio and J.F. Sacle, *Some eigenvalue properties in graphs* (*Conjectures of Graffiti*-II), Discrete Math. **111** (1993) 197–220. doi:10.1016/0012-365X(93)90156-N

[6] I. Gutman, *Degree-based topological indices*, Croat. Chem. Acta **86** (2013) 351–361. doi:10.5562/cca2294

[7] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals,Total-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972) 535–538. doi:10.1016/0009-2614(72)85099-1

[8] I. Gutman, L. Zhong and K. Xu, *Relating ABC and harmonic indices*, J. Serb. Chem. Soc. **79** (2014) 557–563. doi:10.2298/JSC130930001G

[9] Y. Hu and X. Zhou, *On the harmonic index of the unicyclic and bicyclic graphs*, WSEAS Trans. Math. **12** (2013) 716–726.

[10] A. Ilic, *Note on the harmonic index of a graph*, Appl. Math. Lett. **25** (2012) 561–566. doi:10.1016/j.aml.2011.09.059

[11] M.A. Iranmanesh and M. Saheli, *On the harmonic index and harmonic polynomial of caterpillars with diameter four*, Iranian J. Math. Chem. **6** (2015) 41–49.

[12] J. Li and W.C. Shiu, *The harmonic index of a graph*, Rocky Mountain J. Math. **44** (2014) 1607–1620. doi:0.1216/RMJ-2014-44-5-1607

[13] J. Liu, *On harmonic index and diameter of graphs*, J. Appl. Math. Phys. **1** (2013) 5–6. doi:10.4236/jamp.2013.13002

[14] J. Liu, *On the harmonic index of triangle-free graphs*, Appl. Math. **4** (2013) 1204–1206. doi:10.4236/am.2013.48161

[15] J. Liu, *Harmonic index of dense graphs*, Ars Combin. **120** (2015) 293–304.

[16] J. Liu and Q. Zhang, *Remarks on harmonic index of graphs*, Util. Math. **88** (2012) 281–285.

[17] J.B. Lv, J. Li and W.C. Shiu, *The harmonic index of unicyclic graphs with given matching number*, Kragujevac J. Math. **38** (2014) 173–183. doi:doi.org/10.5937/KgJMath1401173J

[18] S. Liu and J. Liu, *Some properties on the harmonic index of molecular trees*, ISRN Appl. Math. (2014) 1–8.

[19] R. Rasi, S.M. Sheikholeslami and I. Gutman, *On harmonic index of trees*, MATCH Commun. Math. Comput. Chem. **78** (2017) 405–416.

[20] R.Wu, Z. Tang and H. Deng, *A lower bound for the harmonic index of a graph with minimum degree at least two*, Filomat **27** (2013) 51–55. doi:10.2298/FIL1301051W

[21] R. Wu, Z. Tang and H. Deng, *On the harmonic index and the girth of a graph*, Util. Math. **91** (2013) 65–69.

[22] X. Xu, *Relationships between harmonic index and other topological indices*, Appl. Math. Sci. **6** (2012) 2013–2018.

[23] L. Yang and H. Hua, *The harmonic index of general graphs, nanocones and triangular benzenoid graphs*, Optoelectron. Adv. Mater. - Rapid Commun. **6** (2012) 660–663.

[24] L. Zhong, *The harmonic index for graphs*, Appl. Math. Lett. **25** (2012) 561–566. doi:10.1016/j.aml.2011.09.059

[25] L. Zhong, *The harmonic index of unicyclic graphs*, Ars Combin. **104** (2012) 261–269.

[26] L. Zhong, *The harmonic index for unicyclic and bicyclic graphs with given matching number*, Miskolc Math. Notes **16** (2015) 587–605.

[27] L. Zhong and Q. Cui, *The harmonic index for unicyclic graphs with given girth*, Filomat **29** (2015) 673–686. doi:10.2298/FIL1504673Z

[28] L. Zhong and K. Xu, *The harmonic index for bicyclic graphs*, Util. Math. **90** (2013) 23–32.

[29] L. Zhong and K. Xu, *Inequalities between vertex-degree-based topological indices*, MATCH Commun. Math. Comput. Chem. **71** (2014) 627–642.

[30] Y. Zhu, R. Chang and X. Wei, *The harmonic index on bicyclic graphs*, Ars Combin. **110** (2013) 97–104.

[31] Y. Zhu and R. Chang, *On the harmonic index of bicyclic conjugated molecular graphs*, Filomat **28** (2014) 421–428. doi:10.2298/FIL1402421Z

[32] Y. Zhu and R. Chang, *Minimum harmonic index of trees and unicyclic graphs with given number of pendant vertices and diameter*, Util. Math. **93** (2014) 345–374.

[33] A. Zolfi, A.R. Ashrafi and S. Moradi, *The top ten values of harmonic index in chemical trees*, Kragujevac J. Sci. **37** (2015) 91–98.

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