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# THE SECOND NEIGHBOURHOOD FOR BIPARTITE TOURNAMENTS

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## Abstract

Let $T(X \cup Y, A)$ be a bipartite tournament with partite sets $X, Y$ and arc set $A$. For any vertex $x \in X \cup Y$, the second out-neighbourhood $N^{++}(x)$ of $x$ is the set of all vertices with distance 2 from $x$. In this paper, we prove that $T$ contains at least two vertices $x$ such that $|N^{++}(x)| \geq |N^{+}(x)|$ unless $T$ is in a special class $\mathcal{B}_1$ of bipartite tournaments; show that $T$ contains at least a vertex $x$ such that $|N^{++}(x)| \geq |N^{-}(x)|$ and characterize the class $\mathcal{B}_2$ of bipartite tournaments in which there exists exactly one vertex $x$ with this property; and prove that if $|X| = |Y|$ or $|X| \geq 4|Y|$, then the bipartite tournament $T$ contains a vertex $x$ such that $|N^{++}(x)| + |N^{+}(x)| \geq 2|N^{-}(x)|$.

**Keywords:** second out-neighbourhood, out-neighbourhood, in-neighbourhood, bipartite tournament.

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## 1. Terminology and Introduction

We will assume that the reader is familiar with the standard terminology on digraphs and refer to [1] for terminology not discussed here. In this paper, all digraphs have no multiple arcs and no loops.

We denote the vertex set and the arc set of a digraph $D$ by $V(D)$ and $A(D)$, respectively. For a vertex subset $X$, we denote by $D\langle X\rangle$ the subdigraph of $D$ induced by $X$, $D\langle V(D)-X\rangle$ by $D-X$. In addition, $D-x=D-\{x\}$ for a vertex $x$ of $D$.

Let $x, y$ be distinct vertices in $D$. If there is an arc from $x$ to $y$ then we say that $x$ *dominates* $y$, write $x \to y$ and call $y$ (respectively, $x$) an *out-neighbour* (respectively, an *in-neighbour*) of $x$ (respectively, $y$). For a subdigraph or simply a vertex subset $H$ of $D$ (possibly, $H=D$), we let $N_H^+(x)$ (respectively, $N_H^-(x)$) denote the set of out-neighbours (respectively, the set of in-neighbours) of $x$ in $H$ and call it *out-neighbourhood* (respectively, *in-neighbourhood*) of $x$ in $H$. Furthermore, $d_H^+(x)=|N_H^+(x)|$ (respectively, $d_H^-(x)=|N_H^-(x)|$) is called the *out-degree* (respectively, *in-degree*) of $x$ in $H$. Let

$$N_H^{++}(x)=\bigcup_{u\in N_H^+(x)} N_H^+(u)-N_H^+(x),$$

which is called the *second out-neighbourhood* of $x$ in $H$. Furthermore, $d_H^{++}(x)=|N_H^{++}(x)|$. We will omit the subscript if $H=D$ is known from the context.

Let $X, Y$ be two disjoint subsets of vertices of $D$. We let $E(X,Y)$ denote the set of all arcs with head in $Y$ and tail in $X$. If $E(Y,X)=\emptyset$ and $x\to y$ for all $x\in X$ and $y\in Y$, then we say that $X$ *completely dominates* $Y$ and denote this by $X\to Y$.

An *oriented graph* is a digraph with no cycle of length two. One of the most interesting and challenging open questions concerning digraphs is Seymour's Second Neighbourhood Conjecture (SSNC) (see [5] and Problem 325, page 804 in volume 197/198 (1999) of Discrete Mathematics), which asserts that one can always find, in an oriented graph $D$, a vertex $x$ whose second out-neighbourhood is at least as large as its out-neighbourhood.

**Conjecture 1** (Seymour's Second Neighbourhood Conjecture)**.** *In every oriented graph $D$, there exists a vertex $x$ such that $d^{++}(x)\ge d^+(x)$.*

Following [4], we will call such a vertex $x$ a *Seymour vertex*.

Note that if we allow 2-cycles, then SSNC is no longer true as can be seen by taking the complete digraph $\overleftrightarrow{K}_n$. Note also that SSNC trivially holds for digraphs $D$ which contain a vertex of out-degree zero, e.g. for acyclic digraphs.

A *tournament* is an oriented graph where every pair of distinct vertices are adjacent. SSNC in the case of tournaments was also stated by Dean and Latka [5].

This special case of the conjecture was proved by Fisher [7] using Farkas' Lemma and averaging arguments.

**Theorem 2** [7]**.** *In any tournament, there is a Seymour vertex.*

A more elementary proof of SSNC for tournaments was given by Havet and Thomassé [10] who introduced a median order approach. Their proof also yields the following stronger result.

**Theorem 3** [10]**.** *A tournament with no vertex of out-degree zero has at least two Seymour vertices.*

Kaneko and Locke [11] proved SSNC for oriented graphs with minimum out-degree at most 6. Fidler and Yuster [6] further developed the median order approach and proved that SSNC holds for oriented graphs $D$ with minimum degree $|V(D)|-2$, tournaments minus a star, and tournaments minus the arc set of a subtournament. The median order approach was also used by Ghazal [8] who proved a weighted version of SSNC for tournaments missing a generalized star. Cohn, Godbole, Wright Harkness, and Zhang [4] proved that the conjecture holds for random oriented graphs. Recently, Gutin and Li [9] proved SSNC for quasi-transitive oriented digraphs which is a superclass of tournaments and transitive acyclic digraphs. Another approach to SSNC is to determine the maximum value $\gamma$ such that in every oriented graph $D$, there exists a vertex $x$ such that $d^+(x) \leq \gamma d^{++}(x)$. SSNC asserts that $\gamma = 1$. Chen, Shen, and Yuster [3] proved that $\gamma \geq r$ where $r = 0.657298\ldots$ is the unique real root of $2x^3 + x^2 - 1 = 0$. They also claim a slight improvement to $r \geq 0.67815\ldots$.

Sullivan [13] stated the following "compromise conjectures" on SSNC, where $d^-(v)$ is used instead of or together with $d^+(v)$.

**Conjecture 4** [13]**.**

(1) *Every oriented graph $D$ has a vertex $x$ such that $d^{++}(x) \geq d^-(x)$.*

(2) *Every oriented graph $D$ has a vertex $x$ such that $d^{++}(x) + d^+(x) \geq 2d^-(x)$.*

For convenience, a vertex $x$ satisfying Conjecture 4(i) is called a *Sullivan-$i$ vertex* for $i = 1, 2$. Recently, we show that these conjectures hold for quasi-transitive oriented graphs. See [14].

A *bipartite tournament* is an oriented graph defined as an orientation of a complete bipartite graph. $T(X \cup Y, A)$ will denote a bipartite tournament with partite sets $X, Y$ and arc set $A$. When no confusion arises the short form $T$ will be used. In this paper, we consider Conjecture 1 and 4 for bipartite tournaments. It is not difficult to see that each vertex of minimum out-degree is a Seymour vertex in a bipartite tournament. In Section 2, we characterize the class of bipartite tournaments in which there exists exactly one Seymour vertex. In Section 3, we show that any bipartite tournament contains a Sullivan-1 vertex and characterize the class of bipartite tournaments in which there exists exactly one Sullivan-1 vertex. In Section 4, we prove that if $|X| = |Y|$ or $|X| \geq 4|Y|$, then the bipartite tournament $T$ contains a Sullivan-2 vertex.

## 2. SSNC for Bipartite Tournaments

We consider SSNC for bipartite tournaments. Let $T(X \cup Y, A)$ be a bipartite tournament. For any two vertices $x, y$ of a bipartite tournament $T$, if $x \to y$, then $N^+(y) \subseteq N^{++}(x)$. So we can obtain the following observation immediately.

**Lemma 5.** *Let $T$ be a bipartite tournament and $x, y$ two vertices of $T$. If $x \to y$ and $d^+(y) \geq d^+(x)$, then $x$ is a Seymour vertex of $T$.*

Moreover, SSNC is true for bipartite tournaments. In fact, in a bipartite tournament, each vertex of minimum out-degree is a Seymour vertex due to Lemma 5. Similarly to the Theorem 3 on tournaments, we have the following result on bipartite tournaments.

**Lemma 6.** *A bipartite tournament with no vertex of out-degree zero has at least two Seymour vertices.*

***Proof.*** Let $T = (X \cup Y, A)$ be a bipartite tournament with no vertex of out-degree zero. Without loss of generality, assume that $x \in X$ is a vertex of minimum out-degree in $T$. Then $x$ is a Seymour vertex of $T$, so we need to find another vertex with this property. Let $T_r = T - x$ and $y$ a vertex of minimum out-degree in $T_r$. Then $y$ is a Seymour vertex of the bipartite tournament $T_r$. We claim that

(1) If $y \in X$ or $y \in Y, x \to y$, then $y$ is also a Seymour vertex of $T$.

In fact, in both cases, $d^{++}(y) \geq d^{++}_{T_r}(y) \geq d^+_{T_r}(y) = d^+(y)$. So assume that $y \in Y$ and $y \to x$.

For the case when $N^+_{T_r}(y) = \emptyset$, we have $d^+_{T_r}(y) = 1$. Recall that the out-degree of $x$ is not zero. Hence $d^{++}(y) \geq d^+(x) = d^+(y)$ and $y$ is another Seymour vertex of $T$. For the case when $N^+_{T_r}(y) \neq \emptyset$, let $z \in N^+_{T_r}(y)$. Clearly, $z \in X$ and $d^+_{T_r}(z) \geq d^+_{T_r}(y)$. Note that $d^+_{T_r}(z) = d^+_{T_r}(y)$ implies that $z$ is also a vertex of minimum out-degree in $T_r$. By (1), $z$ is another Seymour vertex of $T$. So assume that $d^+_{T_r}(z) > d^+_{T_r}(y)$. Since $N^+_{T_r}(z) \subseteq N^{++}_{T_r}(y)$, we have

$$d^{++}(y) = d^{++}_{T_r}(y) \geq d^+_{T_r}(z) \geq d^+_{T_r}(y) + 1 = d^+(y).$$

$y$ is another Seymour vertex. The lemma holds. ■

Let $T = (X \cup Y, A)$ be a bipartite tournament. According to the out-degree of each vertex of $T$, we give a partition $V_1, \ldots, V_k$ of the vertex set $X \cup Y$ of $T$ such that

(a) $d^+(u) = d^+(v)$ for any $1 \le i \le k$ and any $u, v \in V_i$;

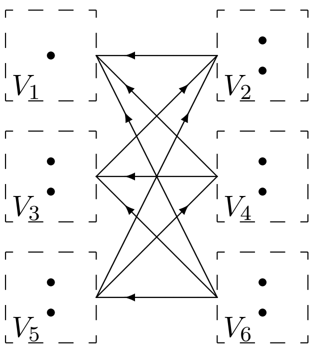
(b) $d^+(u_i) < d^+(u_j)$ for any $1 \le i < j \le k$ and any $u_i \in V_i$ and $u_j \in V_j$.

We call the unique sequence $V_1, \dots, V_k$ satisfying the statement (a) and (b) the *out-degree sequence* of $T$.

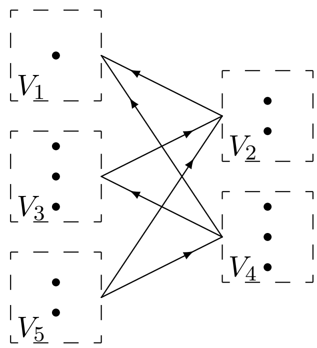
Now we consider a special class $\mathcal{B}_1$ of bipartite tournaments. $T \in \mathcal{B}_1$ if and only if $T$ is a bipartite tournament with the out-degree sequence $V_1, \dots, V_k$ satisfying that

- $|V_1| = 1$ and $|V_1| + |V_3| + \dots + |V_{2i-1}| < |V_2| + |V_4| + \dots + |V_{2i}| < |V_1| + |V_3| + \dots + |V_{2i+1}|$ for any $1 \le i \le \lceil \frac{k}{2} \rceil - 1$;
- all $V_i$'s for $i$ odd are contained in a common partite set and all $V_j$'s for $j$ even are contained in the other common partite set;
- $V_i \to V_2, V_4, \dots, V_{i-1}$ for any $i$ odd and $V_j \to V_1, V_3, \dots, V_{j-1}$ for any $j$ even.

It is not difficult to check that $v \in V_1$ is the only Seymour vertex of $T$. See two examples of the class $\mathcal{B}_1$ in Figure 1.




(a)




(b)

Figure 1. Two bipartite tournaments in $\mathcal{B}_1$. The dashed boxes indicate the partition of the vertex set of a bipartite tournament and an arc from a box $V_i$ to a box $V_j$ between two boxes indicates $V_i \to V_j$.

**Theorem 7.** *A bipartite tournament $T$ has at least two Seymour vertices unless $T \in \mathcal{B}_1$.*

***Proof.*** Let $T(X \cup Y, A)$ be a bipartite tournament. Suppose $T$ has exactly one Seymour vertex. We will show that $T \in \mathcal{B}_1$. Let $V_1, \dots, V_k$ be the out-degree sequence of $T$. Without loss of generality, assume that $k$ is even since the proof is very similar when $k$ is odd. Recall that a vertex of minimum out-degree is a Seymour vertex and each vertex of $V_1$ has the minimum out-degree in $T$. So $|V_1| = 1$. Lemma 6 shows that $V_2, V_4, \dots, V_k \to V_1$.

We claim that either $V_i \subseteq X$ or $V_i \subseteq Y$ for any $1 \le i \le k$. Suppose not. Let $u, v \in V_i$ but $u \in X, v \in Y$. Clearly, $i \ge 2$. By Lemma 5, $u \to v$ implies that $u$ is a Seymour vertex and $v \to u$ implies that $v$ is a Seymour vertex. In both cases, $T$ has two Seymour vertices. Hence $V_i \subseteq X$ or $V_i \subseteq Y$ for any $1 \le i \le k$.

We also claim that $V_i$ and $V_{i+1}$ are contained in different partite sets. Suppose to the contrary that $V_i, V_{i+1} \subseteq X$. For any $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$, there exists a vertex $y \in Y$ such that $v_{i+1} \to y \to v_i$ since $d^+(v_{i+1}) > d^+(v_i)$. Since neither $v_{i+1}$ nor $y$ is a Seymour vertex, we have $d^+(v_{i+1}) > d^+(y) > d^+(v_i)$ by Lemma 5. This contradicts the definition of $V_1, V_2, \ldots, V_k$. Hence $V_i$ and $V_{i+1}$ are contained in different partite sets.

For convenience, assume $V_1 \subseteq X$. The claims above show that $V_i \subseteq X$ for any $i$ odd and $V_j \subseteq Y$ for any $j$ even. Also for any $V_i, V_j$ with $i < j$, either $V_i, V_j$ are nonadjacent or $V_j \to V_i$ by Lemma 5 and the fact that $T$ has exactly one Seymour vertex. This means that $V_i \to V_2, V_4, \ldots, V_{i-1}$ for any $i$ odd and $V_j \to V_1, V_3, \ldots, V_{j-1}$ for any $j$ even.

Now for any $1 \le i \le \lceil \frac{k}{2} \rceil - 1$ and for any $u \in V_{2i+1}$ and $v \in V_{2i+2}$, we see that

$$\begin{aligned} N^+(u) &= V_2 \cup V_4 \cup \cdots \cup V_{2i}, & N^{++}(u) &= V_1 \cup V_3 \cup \cdots \cup V_{2i-1}, \\ N^+(v) &= V_1 \cup V_3 \cup \cdots \cup V_{2i+1}, & N^{++}(v) &= V_2 \cup V_4 \cup \cdots \cup V_{2i}. \end{aligned}$$

Since $T$ has exactly one Seymour vertex, we have $d^{++}(u) < d^+(u)$ and $d^{++}(v) < d^+(v)$. This means that

$$|V_1| + |V_3| + \cdots + |V_{2i-1}| < |V_2| + |V_4| + \cdots + |V_{2i}| < |V_1| + |V_3| + \cdots + |V_{2i+1}|.$$

Thus $T \in \mathcal{B}_1$ and the theorem follows. ∎

## 3. Sullivan's Conjecture (1) for Bipartite Tournaments

We consider Conjecture 4(1) for bipartite tournaments. We begin with two observations.

**Lemma 8.** *Let $T$ be a bipartite tournament and $x, y$ two vertices of $T$. If $x \to y$ and $d^+(y) \ge d^-(x)$, then $x$ is a Sullivan-1 vertex.*

***Proof.*** Note that $N^+(y) \subseteq N^{++}(x)$. Then $d^{++}(x) \ge d^+(y) \ge d^-(x)$. ∎

**Lemma 9.** *Let $T = (X \cup Y, A)$ be a bipartite tournament. If $|E(Y, X)| \ge |E(X, Y)|$, then there exists a vertex $y \in Y$ such that $d^+(y) \ge d^-(y)$.*

***Proof.*** Suppose $d^+(y) < d^-(y)$ for any $y \in Y$. Then

$$E(Y,X) = \sum_{y\in Y} d^+(y) < \sum_{y\in Y} d^-(y) = |E(X,Y)|,$$

a contradiction. Thus there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$. ■

Now we show that Conjecture 4(1) is true in the case of bipartite tournaments.

**Theorem 10.** *Any bipartite tournament has a Sullivan-*1 *vertex.*

***Proof.*** Let $T = (X \cup Y, A)$ be a bipartite tournament. Without loss of generality, assume $|E(Y,X)| \geq |E(X,Y)|$. Then by Lemma 9, there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$. Let $y_0 \in Y$ such that $y_0$ has maximum out-degree among the vertices of $Y$. Clearly, $d^+(y_0) \geq d^-(y_0)$. We give a partition of the vertex set $X \cup Y$ of $T$. Set

$$V_1 = N^-(y_0), \quad V_2 = N^+(y_0), \quad V_3 = N^{++}(y_0), \quad V_4 = Y - V_3$$

and $t_i = |V_i|$ for $i = 1,2,3,4$. We claim that $V_1 \to V_4 \to V_2$. In fact, $V_3 = N^{++}(y_0) = \bigcup_{x\in V_2} N^+(x)$ implies $V_4 \to V_2$. Moreover, since $y_0$ has maximum out-degree in $Y$, we have $d^+(y) \leq d^+(y_0)$ for any $y \in V_4$. Note that $y \to V_2 = N^+(y_0)$. We have $N^+(y_0) \subseteq N^+(y)$. So $N^+(y) = N^+(y_0)$ and hence $N^-(y) = N^-(y_0)$ for any $y \in V_4$. Thus $V_1 \to V_4$. See Figure 2(a).

Now we will prove the following claim which directly implies the result.

**Claim A.** *Either $y_0$ or $w \in N^-(y_0)$ is a Sullivan-*1 *vertex. Moreover, if $y_0$ is not a Sullivan-*1 *vertex, then $d^{++}(w) > d^-(w)$.*

If $t_3 \geq t_1$, then $d^{++}(y_0) \geq d^-(y_0)$ and $y_0$ is a Sullivan-1 vertex. We are done. So assume $t_3 < t_1$. Since $d^+(y_0) \geq d^-(y_0)$, we have $t_1 \leq t_2$. For any $w \in V_1$, $N^-(w) \subseteq V_3$ and $V_2 \subseteq N^{++}(w)$. Now

$$d^{++}(w) \geq t_2 \geq t_1 > t_3 \geq d^-(w).$$

$w$ is a Sullivan-1 vertex in $T$. The theorem follows. ■

We consider a special class $\mathcal{B}_2$ of bipartite tournaments. $T \in \mathcal{B}_2$ if and only if $T$ is a bipartite tournament with two partite sets $X$ and $Y$ such that $x \to Y \to X - x$ (possibly, $X - x = \emptyset$) for some $x \in X$. See Figure 2(b). It is not difficult to check that $x$ is the only Sullivan-1 vertex of $T$.

**Theorem 11.** *Any bipartite tournament has at least two Sullivan-*1 *vertices unless $T \in \mathcal{B}_2$.*

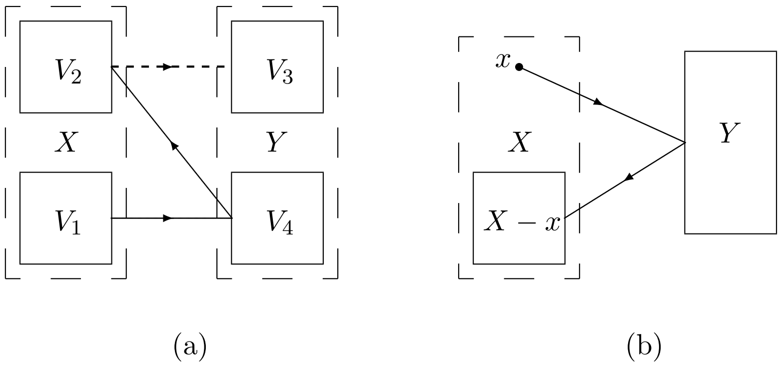



Figure 2. (a) A partition of the vertex set of a bipartite tournament $T = (X \cup Y, A)$. For any vertex $y \in V_1$, $d^+(y) \geq d^+(y)$ and $y$ has the maximum out-degree among all vertices in $Y$. $V_1 = N^-(y), V_2 = N^+(y), V_3 = N^{++}(y), V_4 = Y - V_3$. An dotted arc from a box $V_2$ to a box $V_3$ indicates $N^+(V_2) = V_3$. $V_1 \to V_4 \to V_2$. (b) A bipartite tournament in $\mathcal{B}_2$. $x \to Y \to X - x$.

***Proof.*** Let $T = (X \cup Y, A)$ be a bipartite tournament. Suppose $T$ has exactly one Sullivan-1 vertex. It is sufficient to show that $T \in \mathcal{B}_2$. Without loss of generality, assume $|E(Y,X)| \geq |E(X,Y)|$. Let $y_0$, $V_i$ and $t_i$ be defined as in the proof of Theorem 10. Then $d^+(y_0) \geq d^-(y_0)$ and $V_1 \to V_4 \to V_2$. We consider the following two cases.

*Case* 1. $t_3 \geq t_1$. Clearly, each vertex of $V_4$ is a Sullivan-1 vertex. So $|V_4| = 1$ and $V_4 = \{y_0\}$. Let $T_r = T - y_0$.

*Subcase* 1.1. There is a vertex $y \in Y - y_0$ such that $d^+(y) \geq d^-(y)$. Let $y_1$ be the vertex of maximum out-degree in $Y - y_0$. Then $d^+(y_1) \geq d^-(y_1)$. Clearly, $y_1 \in V_3$ and $d^+_{T_r}(y_1) = d^+(y_1) \geq d^-(y_1) = d^-_{T_r}(y_1)$. Applying Claim A of the proof of Theorem 10 to the bipartite tournament $T_r$, either $y_1$ is a Sullivan-1 vertex or $w \in N^-_{T_r}(y_1)$ is a Sullivan-1 vertex of $T_r$. And if $y_1$ is not a Sullivan-1 vertex of $T_r$, then $d^{++}_{T_r}(w) > d^-_{T_r}(w)$.

For the case when $y_1$ is a Sullivan-1 vertex of $T_r$, we have $d^{++}(y_1) \geq d^{++}_{T_r}(y_1) \geq d^-_{T_r}(y_1) = d^-(y_1)$. So $y_1$ is also Sullivan-1 vertex of $T$. For the case when $y_1$ is not a Sullivan-1 vertex of $T_r$, we have $w \in N^-_{T_r}(y_1)$ is a Sullivan-1 vertex and $d^{++}_{T_r}(w) > d^-_{T_r}(w)$. Now $d^{++}(w) \geq d^{++}_{T_r}(w) \geq d^-_{T_r}(w) + 1 \geq d^-(w)$. So $w$ is also Sullivan-1 vertex of $T$.

*Subcase* 1.2. For any vertex $y \in Y - y_0$, $d^+(y) < d^-(y)$. In the bipartite tournament $T_r$, we see that

$$|E(X, Y - y_0)| = \sum_{y \in Y - y_0} d^-(y) > \sum_{y \in Y - y_0} d^+(y) = |E(Y - y_0, X)|.$$

By Lemma 9, there exists a vertex $x \in X$ such that $d^+_{T_r}(x) \geq d^-_{T_r}(x)$. Let $x_0 \in X$ be the vertex of maximum out-degree among the vertices of $X$ in $T_r$. Clearly, $d^+_{T_r}(x_0) \geq d^-_{T_r}(x_0)$. Similarly to the proof of Theorem 10, set

$$V'_1 = N^-_{T_r}(x_0), \quad V'_2 = N^+_{T_r}(x_0), \quad V'_3 = N^{++}_{T_r}(x_0), \quad V'_4 = Y - y_0 - V'_3.$$

Let $t'_i = |V'_i|$ for $i = 1, 2, 3, 4$. By Claim A of the proof of Theorem 10, either $x_0$ is a Sullivan-1 vertex in $T_r$ or $z \in N^-_{T_r}(x_0)$ is a Sullivan-1 vertex of $T_r$. For the case when $z \in N^-_{T_r}(x_0)$ is a Sullivan-1 vertex of $T_r$, we have $d^{++}(z) \geq d^{++}_{T_r}(z) \geq d^-_{T_r}(z) = d^-(z)$. Then $z$ is also a Sullivan-1 vertex of $T$. For the case when $x_0$ is a Sullivan-1 vertex in $T_r$, we have $t'_3 \geq t'_1$. Note that $t'_3 > t'_1$ implies that $d^{++}(x_0) \geq d^{++}_{T_r}(x_0) \geq d^-_{T_r}(x_0) + 1 \geq d^-(x_0)$ Then $x_0$ is also a Sullivan-2 vertex of $T_r$. So assume $t'_3 = t'_1$. Recall that $t'_1 \leq t'_2$. So $t'_3 \leq t'_2$. On the other hand, $z$ is not a Sullivan-1 vertex of $T_r$ implies that $t'_2 \leq d^{++}_{T_r}(z) < d^-_{T_r}(z) \leq t'_3$, a contradiction.

In any case, we get a contradiction. Thus Case 1 is impossible.

*Case* 2. $t_3 < t_1$. Clearly, any vertex $y \in V_4$ is not a Sullivan-1 vertex. So any vertex $w \in V_1$ is a Sullivan-1 vertex and $d^{++}(w) > d^-(w)$ by Claim A of the proof of Theorem 10. Since $T$ has exactly one Sullivan-1 vertex, we have $t_1 = 1$. So $t_3 = 0$ and $V_3$ is an empty set. Thus $w \to Y \to X - w$ (possibly, $X - w = \emptyset$) and $T \in \mathcal{B}_2$. The theorem follows. ■

## 4. SUPPORT FOR SULLIVAN'S CONJECTURE (2) ON BIPARTITE TOURNAMENTS

The results in Section 4 provide support for Conjecture 4(2) on bipartite tournaments.

**Lemma 12.** *Let $T = (X \cup Y, A)$ be a bipartite tournament with $|X| \leq |Y|$. If there exists a vertex $y \in Y$ such that $d^+(y) \geq d^-(y)$, then $T$ has a Sullivan-*2 *vertex.*

***Proof.*** Choose $y_0 \in Y$ such that $y_0$ has maximum out-degree among the vertices of $Y$. By the assumption, $d^+(y_0) \geq d^-(y_0)$. Let $V_i$ and $t_i$ be defined as in the proof of Theorem 10. Then $|X| \leq |Y|$ implies that $t_1 + t_2 \leq t_3 + t_4$. Recall that $t_2 \geq t_1$. If $y_0$ is a Sullivan-2 vertex of $T$, we are done. So assume that $d^{++}(y_0) + d^+(y_0) < 2d^-(y_0)$, i.e., $t_2 + t_3 < 2t_1$. So $t_3 < t_1 \leq t_2$. For any $w \in N^-(y_0)$, suppose that $w$ is also not a Sullivan-2 vertex of $T$. We have $d^{++}(w) + d^+(w) < 2d^-(w)$, which means $t_2 + t_4 < 2t_3$. So $t_4 < t_3$. Now $t_3 + t_4 < 2t_3 < t_1 + t_2$, a contradiction. Thus either $y_0$ or $w \in N^-(y_0)$ is a Sullivan-2 vertex. The lemma follows. ■

**Corollary 13.** *Any balance bipartite tournament has a Sullivan-2 vertex.*

***Proof.*** Let $T = (X \cup Y, A)$ be a balance bipartite tournament. Then $|X| = |Y|$. By Lemma 9, there exists a vertex $u \in X \cup Y$ such that $d^+(u) \geq d^-(u)$. Now Lemma 12 yields the result. ■

**Lemma 14.** *Let $T = (X \cup Y, A)$ be a bipartite tournament. If there exists a vertex $x \in X$ such that $d^+(x) \geq 2|X| - 3$, then any $y \in N^-(x)$ is a Sullivan-2 vertex.*

***Proof.*** Note that $N^+(x) \subseteq N^{++}(y)$. So $d^{++}(y) \geq 2|X| - 3$. Thus $d^{++}(y) + d^+(y) \geq 2|X| - 3 + 1 \geq 2d^-(y)$ and $y$ is a Sullivan-2 vertex of $T$. ■

**Corollary 15.** *Let $T = (X \cup Y, A)$ be a bipartite tournament. If $|E(X,Y)| \geq 2|X|^2$, then there is a vertex $x \in X$ such that any $y \in N^-(x)$ is a Sullivan-2 vertex.*

***Proof.*** Since $|E(X,Y)| = \sum_{x \in X} d^+(x) \geq 2|X|^2$, there is a vertex $x \in X$ such that $d^+(x) \geq 2|X|$. By Lemma 14, any $y \in N^-(x)$ is a Sullivan-2 vertex. ■

**Corollary 16.** *A bipartite tournament $T = (X \cup Y, A)$ with $|Y| \geq 4|X|$ has a Sullivan-2 vertex.*

***Proof.*** By Lemma 9, there exists a vertex $u \in X \cup Y$ such that $d^+(u) \geq d^-(u)$. If $u \in Y$, by Lemma 12, $T$ has a Sullivan-2 vertex and we are done. So assume $u \in X$. Now $d^+(u) \geq \frac{|Y|}{2} \geq 2|X|$. By Lemma 14, any $y \in N^-(u)$ is a Sullivan-2 vertex of $T$. ■

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