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ON THE TOTAL k-DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph; a set $S \subseteq V$ is a total k-dominating set if every vertex $v \in V$ has at least k neighbors in S. The total k-domination number $\gamma_{kt}(G)$ is the minimum cardinality among all total k-dominating sets. In this paper we obtain several tight bounds for the total k-domination number of a graph. In particular, we investigate the relationship between the total k-domination number of a graph and the order, the size, the girth, the minimum and maximum degree, the diameter, and other domination parameters of the graph.

Keywords: k-domination, total k-domination, k-tuple domination, k-tuple total domination.

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1. Introduction

We begin by stating some notation and terminology. Let $G = (V, E)$ denote a simple graph of order $n = |V|$ and size $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V : u \sim v\}$, where $u \sim v$ means that u and v are adjacent vertices, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ will be denoted by $\deg(v) = |N(v)|$, and δ and Δ will be the minimum and maximum degree of the graph, respectively. The graph $G[S]$ is the subgraph induced by a set $S \subseteq V$, and, for any vertex $v \in V$, $N_S(v) = \{u \in S : u \sim v\}$ and $\deg_S(v) = |N_S(v)|$. The *complement* of the vertex-set S in V is denoted by $\overline{S}$, so that $N_{\overline{S}}(v)$ is the set of neighbors v has in $\overline{S} = V \setminus S$. Finally, for every $A, B \subseteq V$ we denote by $E(A, B)$ the number of edges from vertices in A to vertices in B.

Given a graph $G = (V, E)$, we are interested in finding the minimum cardinality of a set $S \subseteq V$ such that every vertex in V has at least k neighbors in S. This number has been studied by different authors using different names. For instance, in [5] it is called k-total k-domination number and denoted by $\gamma_{k,k}(G)$, in [4] it is called total k-tuple domination number and denoted by $\gamma_t^{(\times k)}(G)$, in [12] it is called k-tuple total domination number and denoted by $\gamma_{\times k,t}(G)$, and, more recently, in [6] and [15] it is called total k-domination number. We will follow the notation given in [6], and we will use $\gamma_{kt}(G)$ for the total k-domination number.

A set $S \subseteq V$ is a *k-dominating set* if every vertex $v \in V \setminus S$ satisfies $\deg_S(v) \geq k$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among all k-dominating sets (see [9, 10]). The *domination number* is the 1-domination number, denoted by $\gamma(G)$. A set $S \subseteq V$ is a *total k-dominating set* if every vertex $v \in V$ satisfies $\deg_S(v) \geq k$. In such a case, it is necessary to have $k \leq \delta$ and $|S| \geq k + 1$. The *total k-domination number* $\gamma_{kt}(G)$ is the minimum cardinality among all total k-dominating sets. A *total dominating set* is a total 1-dominating set, and the *total domination number*, denote by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets, that is, $\gamma_t(G) = \gamma_{1t}(G)$ (see [11, 14]). A set $S \subseteq V$ is a *k-tuple dominating set* if every vertex $v \in V$ satisfies $|N[v] \cap S| \geq k$. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality among all k-tuple dominating sets (see [3, 7, 8]). From the definitions we can directly obtain that $\gamma_k(G) \leq \gamma_{\times k}(G)$ and $\gamma_{(k-1)t}(G) \leq \gamma_{\times k}(G) \leq \gamma_{kt}(G)$.

2. Basic Results on the Total k-Dominating Set

The following lemmas will be very useful throughout this paper.

Lemma 1 ([12]). *If $k = \delta$ and v is a vertex such that $\deg(v) = \delta$, then $N(v)$ is included in every total k-dominating set.*

Corollary 2. *If $k = \delta$ and there exist two adjacent vertices v_1 and v_2 of minimum degree, then v_1 and v_2 belong to every total k-dominating set.*

Lemma 3. *Let G be a graph and S be a total k-dominating set in G. If $|S| = \gamma_{kt}(G)$, then for all $v \in S$ there exists $u \in N(v)$ such that $\deg_S(u) = k$.*

Proof. If there exists $v \in S$ such that every $u \in N(v)$ satisfies $\deg_S(u) \geq k+1$, then $S' = S \setminus \{v\}$ would be a total k-dominating set, a contradiction. ■

It is known (see [16]) that

$$\gamma_{1t}(P_n) = \gamma_{1t}(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if} \quad n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil & \text{otherwise.} \end{cases}$$

The following proposition shows closed formulas for the total k-domination numbers for well known graphs.

Proposition 4. *For the complete graph K_n, the cycle C_n and the wheel W_n with n vertices we have the following total k-domination numbers*

(a) $\gamma_{kt}(K_n) = k + 1$;
(b) $\gamma_{2t}(C_n) = n$;
(c) $\gamma_{1t}(W_n) = \gamma_t(W_n) = 2$, $\gamma_{3t}(W_n) = n$ *and*

$$\gamma_{2t}(W_n) = \begin{cases} \frac{n-1}{2} + 2 & \textit{if} \quad n \equiv 3 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil + 1 & \textit{otherwise.} \end{cases}$$

Proof. (a) and (b) were proved in [12] and [17], respectively. By Corollary 2 we have $\gamma_{3t}(W_n) = n$. In a wheel graph, any set of two vertices containing the vertex of degree $n-1$ is a total 1-dominating set. Hence $\gamma_{1t}(W_n) = 2$. If $V(W_n) = \{v_1, v_2, \ldots, v_n\}$ where $v_1 \sim v_i$ for every $i = 2, \ldots, n$, then v_1 belongs to any total 2-dominating set of cardinality at most $n-2$. Since v_1 is adjacent to every vertex in the graph,

$$\gamma_{2t}(W_n) = \gamma_{1t}(C_{n-1}) + 1 = \begin{cases} \frac{n-1}{2} + 2 & \text{if} \quad n \equiv 3 \pmod 4, \\ \lceil \frac{n-1}{2} \rceil + 1 & \text{otherwise.} \end{cases}$$

■

The wheel graph shows that it is not possible to find a relation $\gamma_{kt}(G) \leq g(k)\gamma_{(k-1)t}(G)$. It would be necessary to use another parameter in the function g.

Proposition 5. *If G is a graph of order n and minimum degree δ, then $\gamma_{kt}(G) \leq n - \delta + k$.*

Proof. We see that every set $S \subseteq V$ such that $|S| \geq n - \delta + k$ is a total k-dominating set. Since $|\overline{S}| \leq \delta - k$ for any vertex $v \in V$, we have

$$\delta \leq \deg_S(v) + \deg_{\overline{S}}(v) \leq \deg_S(v) + \delta - k.$$

Consequently, $\deg_S(v) \geq k$. ■

This upper bound is attained for any k in any complete graph. If $k = \delta$ the bound is attained in any graph satisfying the conditions given in Proposition 7.

Theorem 6. *Let G be a graph of order n and minimum degree δ, and let $A = \{v \in V : \deg(v) = \delta\}$. Then $\gamma_{kt}(G) = n$ if and only if $k = \delta$ and A is a total dominating set.*

Proof. If $\gamma_{kt}(G) = n$, by Proposition 5, we have $k = \delta$. Moreover, since V is the only total δ-dominating set, every vertex $u \in V$ has a neighbor of degree δ. Otherwise, $V \setminus \{u\}$ would be a total δ-dominating set, a contradiction. Finally, say $k = \delta$, A is a total dominating set and S is a minimum total δ-dominating set. Since every $u \in V \setminus S$ has a neighbor $v_i \in A$, it follows that $\deg_S(v_i) \leq \delta - 1$. A contradiction, so $S = V$. ■

As a consequence of Theorem 6 every δ-regular graph G of order n satisfies $\gamma_{\delta t}(G) = n$. Nevertheless, there exist many non-regular graphs satisfying the same property.

Proposition 7. *For every k and $n = 2jk$, there exists a non-regular graph G such that its minimum degree is k and $\gamma_{kt}(G) = n$.*

Proof. We consider j copies ($i = 1, \ldots, j$) of the graph presented in Figure 1 and we join $\left\{v^1_{1,1}, v^2_{1,1}, \ldots, v^{k-1}_{1,1}, v^1_{2,1}, \ldots, v^{k-1}_{2,1}, \ldots, v^{k-1}_{j,1}\right\}$ and $\left\{v^1_{1,2}, v^2_{1,2}, \ldots, v^{k-1}_{1,2}, v^1_{2,2}, \ldots, v^{k-1}_{2,2}, \ldots, v^{k-1}_{j,2}\right\}$ to obtain two complete graphs of order $j(k-1)$. Since $u_{i,1}$ and $u_{i,2}$ are adjacent two vertices of minimum degree, by Lemma 1 and Corollary 2, the graph obtained satisfies $\gamma_{kt}(G) = n$. ■

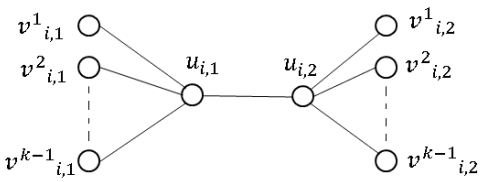

Figure 1.

Theorem 8. *Let S be a total k-dominating set in a graph G. If $\deg_S(v) = k$ for every $v \in S$ and $\sum_{v\in\overline{S}} \deg_S(v) \leq k|\overline{S}| + k - 1$, then $|S| = \gamma_{kt}(G)$.*

Proof. We assume that S' is a minimum total k-dominating set such that $|S'| < |S|$. If $S' \subset S$, there exist $u \in S \setminus S'$ and $v \in N_S(u)$ such that $\deg_S(v) = k$ and $\deg_{S'}(v) < k$, a contradiction. Hence $S' \nsubseteq S$. Since S is a total k-dominating set and $\sum_{v\in\overline{S}} \deg_S(v) \leq k|\overline{S}| + k - 1$, we have that $E(A, S) \leq |A|k + k - 1$ for every set $A \subseteq \overline{S}$. If $\{u_{i_1}, \ldots, u_{i_j}\} = S' \cap S$, then $|S' \cap \overline{S}| \leq |S| - j - 1$, thus $E\left(S' \cap \overline{S}, S\right) \leq (|S| - j - 1)k + k - 1$. Since every $u \in S$ must have at least k neighbors in S', we have $E(S, S') \geq |S|k$. On the other hand, every vertex in $\{u_{i_1}, \ldots, u_{i_j}\}$ has k neighbors in S, so $E(S \cap S', S) = jk$. Therefore,

$$\begin{aligned}(|S| - j)k - 1 &= (|S| - j - 1)k + k - 1 \geq E\left(S, S' \cap \overline{S}\right)\\ &= E(S, S') - E(S, S \cap S') = (S, S') - E(S \cap S', S)\\ &\geq |S|k - jk = (|S| - j)k,\end{aligned}$$

a contradiction. Consequently, $|S| = \gamma_{kt}(G)$. ■

As a consequence, we obtain the following corollary.

Corollary 9. *If S is a set such that $\deg_S(v) = k$ for every $v \in V$, then S is a total k-dominating and $|S| = \gamma_{kt}(G)$.*

Finding a total k-dominating set in a given graph is relatively easy. However, to determine whether such a set has minimum cardinality is more challenging. In this sense, these two results are very useful because, in many cases, they let us identify the minimum set without proving it. For example, we can find a set S in the Cartesian products $P_j \Box C_n$ when j is an odd number, or in $C_j \Box C_n$ when j is an even number, satisfying the conditions given in this corollary when $k = 2$ (see [1]). The last corollary is also very useful to construct an infinite number of graphs with a given total k-domination number. If we consider a k-regular graph $G' = (V', E')$ and another graph $G = (V, E)$ such that $V' \subseteq V$, $E' \subseteq E$, G' is an induced subgraph of G and every $v \in V \setminus V'$ satisfies $\deg_{V'}(v) = k$, then $\gamma_{kt}(G) = |V'|$. This is what happens in Figure 2 on the left side, where $k = 3$ and $V' = \{u_1, \ldots, u_6\}$.

In the graph on the right side in Figure 2, the set S containing the black vertices is a total 2-dominating set such that every vertex $u \in V \setminus \{v\}$ satisfies $\deg_S(u) = 2$. Although S does not satisfy the condition in Corollary 9, it does satisfy the conditions in Theorem 8. Hence S is a minimum total 2-dominating set.

Moreover, note that the conditions given in Theorem 8 are necessary. In the graph on the left showed in Figure 3, the black vertices form a total 2-dominating set. Since u and v satisfy $\deg_S(u) = \deg_S(v) = 3$, we can find a smaller total 2-dominating set represented by the grey vertices in the graph on the right side.

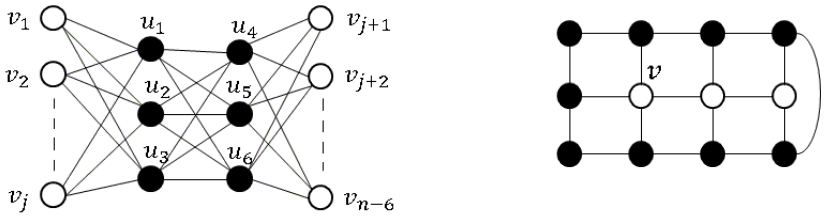

Figure 2.

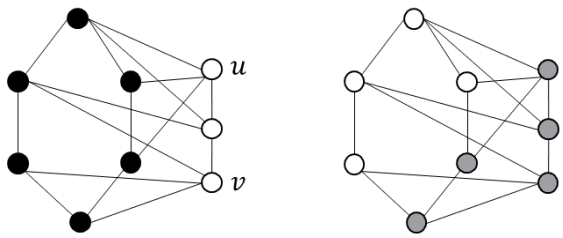

Figure 3.

3. Bounds for the Total k-Domination Number of a Graph

It was proved in [6] and [13] that for any graph $G = (V, E)$ of order n and maximum degree Δ it holds $\gamma_{kt}(G) \geq \frac{kn}{\Delta}$, and there are many graphs attaining this lower bound. In order to get a better lower bound using the same parameters, it is necessary to give additional conditions on the graph.

A *packing* of a graph G is a set of vertices in G that are pairwise at distance more than two. The *packing number* $\rho(G)$ of a graph G is the size of a largest packing in G. For every graph G, $\rho(G) \geq 1$. We have $\rho(G) \leq \gamma(G)$ because, for every vertex v in a packing set, $N[v]$ must contain at least one vertex of the dominating set. Therefore, since an upper bound for the domination number is a half of the order n, we have $\rho(G) \leq \frac{n}{2}$. This upper bound for the packing number will be needed in the proof of the next result.

Theorem 10. *Let $G = (V, E)$ be a graph of order n and maximum degree Δ and let $A = \{v \in V : \deg(v) = \Delta\}$. Then*

$$\gamma_{kt}(G) \geq \left\lceil \frac{kn - |A|}{\Delta - 1} \right\rceil.$$

Moreover, if A is a packing of G, then $\gamma_{kt}(G) \geq \left\lceil \frac{2kn}{2\Delta - 1} \right\rceil$.

Proof. Let S be a total k-dominating set and $d_1, \ldots, d_{|S|}$ the degrees of the vertices of S. Since

$$\frac{d_1 + \cdots + d_{|S|} - k(n - |S|)}{2} \geq \frac{d_1 + \cdots + d_{|S|} - E\left(S, \overline{S}\right)}{2} = m(G[S]) \geq \frac{k|S|}{2},$$

we have

$$kn \leq d_1 + \cdots + d_{|S|} \leq |A \cap S|\Delta + (|S| - |A \cap S|)(\Delta - 1) = |S|(\Delta - 1) + |A \cap S|.$$

It follows that $|S| \geq \frac{kn - |A \cap S|}{\Delta - 1} \geq \frac{kn - |A|}{\Delta - 1}$. Now, if A is a packing of G, then $A \cap S$ is a packing in $G[S]$. Therefore we have $|A \cap S| \leq \rho(G[S]) \leq \frac{|S|}{2}$. Using this in the above inequality we have

$$|S| \geq \frac{kn - |A \cap S|}{\Delta - 1} \geq \frac{2kn - |S|}{2(\Delta - 1)},$$

that is, $|S| \geq \frac{2kn}{2\Delta - 1}$. ■

Given a minimum total k-dominating (or k-tuple dominating) set S, we will say that a vertex $v \in S$ is a *helping vertex* of S if $\deg_{\overline{S}}(v) = 0$. We denote by $H(S)$ the set of all helping vertices of S.

Lemma 11. *For every $k < \delta$ and every minimum total k-dominating set S, it holds that $|H(S)| \leq \frac{k|S|}{k+1}$.*

Proof. If S is a minimum total k-dominating set, by Lemma 3, we know that for every vertex $v \in H(S)$ there exists $u \in N(v) \cap S$ such that $\deg_S(u) = k$. Hence $\deg_{\overline{S}}(u) \geq 1$, that is, $u \in S \setminus H(S)$. So, if we denote $S_k = \{w \in S : \deg_S(w) = k\}$ ($H(S) \cap S_k = \emptyset$ because $k < \delta$), we have $|H(S)| \leq E\left(H(S), S_k\right) \leq k|S_k| \leq k(|S| - |H(S)|)$ or equivalently $|H(S)| \leq \frac{k|S|}{k+1}$. ■

Notice that the inequality in this lemma is an equality if we consider the graph showed in Figure 4, where $k = 1$. In this graph, the set of black vertices is a minimum total dominating set. We show now that condition $k < \delta$ is necessary. If we consider the graphs formed from a cycle C_r given by $v_1 v_2 \cdots v_r v_1$ by adding t new vertices $u_1, \ldots, u_t$ such that $u_i \sim v_1$ and $u_i \sim v_2$ for every $i \in \{1, \ldots, t\}$, then the set $S = \{v_1, \ldots, v_r\}$ is a minimum total 2-dominating set and $|H(S)| = r - 2 > \frac{2|S|}{3} = \frac{2r}{3}$, for $r > 6$.

Proposition 12. *If G is a graph of order n and S is a minimum total k-dominating set in G, then*

$$\gamma_{kt}(G) \geq \frac{nk + |H(S)|(\Delta - k)}{\Delta}.$$

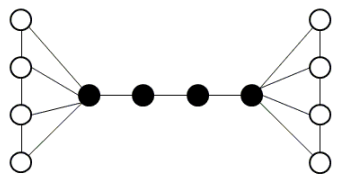

Figure 4.

Proof. If S is a minimum total k-dominating set, and $H(S)$ is the set of all helping vertices of S, then

$$(n-|S|)k \leq E\left(S,\overline{S}\right) \leq (|S|-|H(S)|)(\Delta-k).$$

Thus, we have that $|S| \geq \frac{nk+|H(S)|(\Delta-k)}{\Delta}$. ■

Theorem 13. *Let G be a graph of order n and maximum degree Δ. If $\gamma(G) > \frac{n(\Delta-k)}{\Delta}$, then*

$$\gamma_{kt}(G) \geq \frac{nk+\Delta-k}{\Delta}.$$

Proof. Let S be a minimum total k-dominating set of G. By Proposition 12, $H(S) \neq \emptyset$. Since

$$n = |\overline{S}| + |S| = |\overline{S}| + \gamma_{kt}(G) \geq |\overline{S}| + \frac{kn}{\Delta},$$

we have $|\overline{S}| \leq n - \frac{kn}{\Delta} = \frac{(\Delta-k)n}{\Delta}$. If we suppose that $H(S) = \emptyset$, then $\overline{S}$ is a dominating set and, as a consequence, $\gamma(G) \leq \frac{(\Delta-k)n}{\Delta}$ which is a contradiction. ■

For any graph $G=(V,E)$ of size m and any set $A \subseteq V$, it holds that

$$\begin{aligned} m &= m(G[A]) + E\left(A,\overline{A}\right) + m\left(G\left[\overline{A}\right]\right) \\ &= \frac{1}{2}\sum_{v\in A}\deg_A(v) + \sum_{v\in A}\deg_{\overline{A}}(v) + \frac{1}{2}\sum_{v\in \overline{A}}\deg_{\overline{A}}(v). \end{aligned}$$

This fact will be used throughout the paper.

Theorem 14. *Let G be a graph of size m, order n and minimum degree δ. Then*

$$\gamma_{kt}(G) \geq \min\left\{\delta, \frac{2(kn-m)+n}{k+1}\right\}.$$

Proof. Let S be a total k-dominating set. If there exists a vertex $v \in \overline{S}$ such that $\deg_{\overline{S}}(v) = 0$, then $\deg_S(v) = \deg(v)$ and $\gamma_{kt}(G) \geq \deg(v) \geq \delta$.

If every vertex $v \in \overline{S}$ satisfies $\deg_{\overline{S}}(v) \neq 0$, then

$$\begin{aligned} 2m &= \sum_{v\in S} \deg_S(v) + 2\sum_{v\in\overline{S}} \deg_S(v) + \sum_{v\in\overline{S}} \deg_{\overline{S}}(v) \\ &\geq |S|k + 2(n-|S|)k + n - |S| = -|S|(k+1) + n(2k+1). \end{aligned}$$

Thus, $|S| \geq \frac{2(kn-m)+n}{k+1}$. ■

Theorem 15. *Let G be a graph of size m and maximum degree Δ. If $\lceil \frac{\Delta}{2} \rceil \leq k$, then*

$$\gamma_{kt}(G) \geq \max\left\{ \left\lceil \frac{\sqrt{2m+1}+1}{2} \right\rceil, \left\lceil \frac{\sqrt{8m+(3(\Delta-k)-1)^2}-3(\Delta-k)+1}{2} \right\rceil \right\}.$$

Proof. Let S be a total k-dominating set. On one hand, since $\lceil \frac{\Delta}{2} \rceil \leq k$, we know that $\deg_S(v) \geq \deg_{\overline{S}}(v)$ for all $v \in V$, and therefore

$$\begin{aligned} 2m &= \sum_{v\in S} \deg_S(v) + 2\sum_{v\in\overline{S}} \deg_S(v) + \sum_{v\in\overline{S}} \deg_{\overline{S}}(v) \\ &\leq \sum_{v\in S} \deg_S(v) + 2\sum_{v\in\overline{S}} \deg_S(v) + \sum_{v\in\overline{S}} \deg_S(v) \\ &= \sum_{v\in S} \deg_S(v) + 3\sum_{v\in\overline{S}} \deg_S(v) = \sum_{v\in S} \deg_S(v) + 3\sum_{v\in S} \deg_{\overline{S}}(v) \leq 4\sum_{v\in S} \deg_S(v). \end{aligned}$$

Thus, $m \leq 2\sum_{v\in S} \deg_S(v) \leq 2|S|(|S|-1) = 2\left(\left(|S| - \frac{1}{2}\right)^2 - \frac{1}{4}\right)$. Hence $|S| \geq \frac{\sqrt{2m+1}+1}{2}$.

On the other hand,

$$2m \leq \sum_{v\in S} \deg_S(v) + 3\sum_{v\in S} \deg_{\overline{S}}(v) \leq |S|(|S|-1) + 3|S|(\Delta - k).$$

Therefore $|S| \geq \frac{\sqrt{8m+(3(\Delta-k)-1)^2}-3(\Delta-k)+1}{2}$. ■

Note that for every complete graph K_n, where $n = 2k+1$, since $0 \leq k + 1 - \left(\frac{\sqrt{4k^2+2k+1}+1}{2}\right) < 1$, we have

$$\gamma_{kt}(K_n) = k+1 = \left\lceil \frac{\sqrt{2m+1}+1}{2} \right\rceil = \left\lceil \frac{\sqrt{(2k+1)(2k)+1}+1}{2} \right\rceil.$$

Moreover, for a complete graph K_n, where $n = 2k+1$, we have

$$\begin{aligned}\frac{\sqrt{8m+(3(\Delta-k)-1)^2}-3(\Delta-k)+1}{2} &= \frac{\sqrt{4(2k+1)(2k)+(3k-1)^2}-3k+1}{2}\\ &= \frac{\sqrt{25k^2+2k+1}-3k+1}{2}.\end{aligned}$$

Since $0 \leq k+1-\left(\frac{\sqrt{25k^2+2k+1}-3k+1}{2}\right) < 1$, we also have

$$\gamma_{kt}(K_n) = k+1 = \left\lceil \frac{\sqrt{8m+(3(\Delta-k)-1)^2}-3(\Delta-k)+1}{2} \right\rceil.$$

The *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle contained in G.

Proposition 16. *If G is a graph with girth $g(G)$ and $k \geq 2$, then*

$$\gamma_{kt}(G) \geq (g(G)-2)(k-1)+2.$$

Proof. Firstly, if S is a total k-dominating set with $k \geq 2$, then $G[S]$ contains a cycle C_p. For any two vertices $u, v \in V(C_p)$ such that $d_{C_p}(u,v) \leq g(G)-3$ it holds $N(u) \cap N(v) \cap (V \setminus V(C_p)) = \emptyset$. If we take $g(G)-2$ consecutive vertices $\{u_1, \ldots, u_{g(G)-2}\}$ in $V(C_p)$ then, since $N(u_i) \cap (S \setminus \{u_1, \ldots, u_{g(G)-2}\}) \geq k-2$ for every $i \in \{2, \ldots, g(G)-3\}$, we get $N(u_1) \cap (S \setminus \{u_1, \ldots, u_{g(G)-2}\}) \geq k-1$ and $N(u_{g(G)-2}) \cap (S \setminus \{u_1, \ldots, u_{g(G)-2}\}) \geq k-1$. We obtain that $|S| \geq g(G)-2+2(k-1)+(g(G)-4)(k-2) = (g(G)-2)(k-1)+2$. ■

If $k=2$, note that the lower bound in the above proposition is attained for every cycle C_n. If $k=3$, this lower bound is attained in the graph G showed in Figure 2, where $g(G)=4$. The minimum total 3-dominating set is given by black vertices.

The *chromatic number* of a graph G is the smallest number of colors $\chi(G)$ needed to color the vertices of G so that no two adjacent vertices share the same color. Following the ideas showed in [12, Theorem 7] we obtain the next result.

Proposition 17. *Let G be a graph with chromatic number $\chi(G)$. Then*

$$\gamma_{kt}(G) \geq \frac{k\chi(G)}{\chi(G)-1}.$$

Proof. If $c = \chi(G)$, then $V(G)$ can be partitioned into c independent sets V_i. Let S be a total k-dominating set in G. If $S_i = S \cap V_i$, every vertex in V_i has at least k vertices in $S \setminus S_i$, that is, $|S| - |S_i| \geq k$. Consequently, $(c-1)|S| = c|S| - (|S_1| + \cdots + |S_c|) \geq ck$. ■

Note that the lower bound given in the above proposition is attained in the graph given in Figure 2 for $k = 3$, because we can partition the vertex set in two independent sets $\{v_1, \ldots, v_j, u_4, u_5, u_6\}$ and $\{u_1, u_2, u_3, v_{j+1}, \ldots, v_{n-6}\}$. So $\chi(G) = 2$.

Theorem 18. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. For every $k < \delta$ it holds that*

$$\gamma_{kt}(G) \leq \frac{\Delta(k+1)n}{\Delta(k+1)+1}.$$

Proof. Let S be a minimum total k-dominating set. As we have seen in the proof of Lemma 12, $V \setminus S$ is a dominating set in $G[V \setminus H(S)]$. Hence $n - |S| \geq \gamma(G[V \setminus H(S)]) \geq \frac{n-|H(S)|}{\Delta+1}$, and consequently

$$n\Delta \geq |S|(\Delta+1) - |H(S)| \geq |S|(\Delta+1) - \frac{k|S|}{k+1} = \left(\frac{k\Delta+\Delta+1}{k+1}\right)|S|. \qquad \blacksquare$$

The upper bound given in the theorem above is attained, for instance, in the cycle C_6 when $k = 1$, and in the complete graph K_n when $k = n - 2$.

Proposition 19. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. For every $k < \delta$ it holds that*

$$\gamma_{kt}(G) \leq n - \left\lceil \frac{\gamma(G)}{\Delta} \right\rceil.$$

Proof. In the proof of the above theorem we have seen that $n\Delta \geq |S|(\Delta + 1) - |H(S)|$. As we saw in the proof of Lemma 12, $S \setminus H(S)$ is a dominating set, thus $|S| - |H(S)| \geq \gamma(G)$. Using this in the inequality above we obtain $n\Delta \geq \Delta|S| + \gamma(G)$, thus $|S| \leq n - \frac{\gamma(G)}{\Delta}$. $\blacksquare$

The upper bound given in the last proposition is also attained in the complete graph K_n when $k = n - 2$.

We saw in Proposition 5, without any condition on the graph, that $\gamma_{kt}(G) \leq n - \delta + k$. If we want to improve this upper bound we will have to give some additional conditions on the graph.

Proposition 20. *Let G be a graph of order n and minimum degree δ. Suppose there exist a vertex $v \in V$ and $\{u_1, \ldots, u_r\} \subseteq N(v)$, where $r \geq \deg(v) - \delta + k$, such that $\deg(u_i) \geq \delta + 1$, for $1 \leq i \leq r$. Then $\gamma_{kt}(G) \leq n - \delta + k - 1$.*

Proof. If $N(v) = \left\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_{\deg(v)}\right\}$, we prove that $S = V \setminus \left\{v, u_{r+1}, \ldots, u_{\deg(v)}\right\}$ is a total k-dominating set. Firstly, for every $i = r+1, \ldots, \deg(v)$,

$\deg_{\overline{S}}(u_i) \leq \delta - k$; hence $\deg_S(u_i) \geq k$. Secondly, for every $i = 1, \ldots, r$, $\deg_{\overline{S}}(u_i) \leq \delta - k + 1$; hence $\deg_S(u_i) \geq \delta + 1 - (\delta - k + 1) = k$. Finally, for every vertex $w \in S \setminus \{u_1, \ldots, u_r\}$ it follows that $\deg_{\overline{S}}(w) \leq \delta - k$, and hence $\deg_S(w) \geq \delta - (\delta - k) = k$. Therefore, $\gamma_{kt}(G) \leq |S| \leq n - \delta + k - 1$. ■

Theorem 21. *If G is a graph of order n and minimum degree δ, then*

(a) *for every $k \leq \delta - 1$ we have $\gamma_{kt}(G) \leq n - \rho(G)$,*
(b) *for every $k \leq \delta$ we have $k\rho(G) \leq \gamma_{kt}(G)$.*

Proof. If $A = \{u_1, \ldots, u_s\}$ is a packing in G such that $s = \rho(G)$ and we consider $S = V \setminus A$, then $\deg_S(u_i) \geq \delta \geq k + 1$. Since no vertex in S can have two neighbors in A, $G[S]$ is a graph with minimum degree greater than or equal to k. Thus S is a total k-dominating set, and hence $\gamma_{kt}(G) \leq |S| = n - \rho(G)$.

Now, let S be a minimum total k-dominating set of G. Notice that $|N(u_i) \cap S| \geq k$ for all $1 \leq i \leq s$. Since $N(u_i) \cap N(u_j) = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq s$, we deduce that $|S| \geq \sum_{i=1}^{s} |N(u_i) \cap S| \geq k\rho(G)$. Therefore, $\gamma_{kt}(G) \geq k\rho(G)$. ■

The upper bound given in Theorem 21(a) is attained for $k = 2$ in the following family of graphs. We consider a cycle C_{3r} whose vertices are $\{u_1, \ldots, u_{3r}\}$, and a set of vertices $\{v_1, \ldots, v_r\}$ such that $N(v_i) = \{u_{3i-2}, u_{3i-1}, u_{3i}\}$ (see Figure 5). In such a graph, $\{v_1, \ldots, v_r\}$ and $\{u_1, \ldots, u_{3r}\}$ are a maximum packing set and a minimum 2-dominating set, respectively.

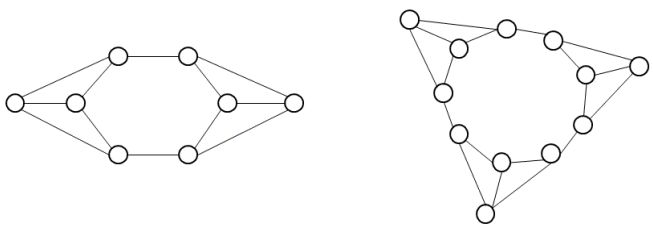

Figure 5. Graphs where $\gamma_{2t}(G) = n - \rho(G)$.

The lower bound given in Theorem 21(b) is attained for $k = 3$ in the following family of graphs. We consider a complete graph K_{3r} whose vertices are $\{u_1, \ldots, u_{3r}\}$, and a set of vertices $\{v_1, \ldots, v_r\}$ such that $N(v_i) = \{u_{3i-2}, u_{3i-1}, u_{3i}\}$ (see Figure 6). We have that $\{v_1, \ldots, v_r\}$ and $\{u_1, \ldots, u_{3r}\}$ are a maximum packing set and a minimum 3-dominating set, respectively.

Corollary 22. *If G is a graph of order n, minimum degree δ and diameter $D(G)$, then for every $k \leq \delta - 1$ we have $\gamma_{kt}(G) \leq n - \left\lceil \frac{D(G)}{3} \right\rceil$.*

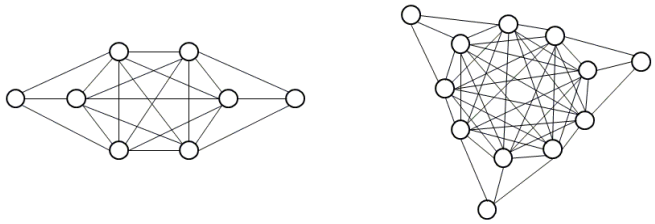

Figure 6. Graphs with $\gamma_{3t}(G) = 3\,\rho(G)$.

4. Total k-Domination Number and Other Domination Parameters

We show in this section some relations between the total k-domination number and the total domination number, the k-domination number and the k-tuple domination number.

Notice that if S is a total k-dominating set and $v \in S$, then $S \setminus \{v\}$ is a total $(k-1)$-dominating set, so $\gamma_{kt}(G) \geq \gamma_{(k-1)t}(G) + 1$. Consequently,

$$\gamma_{kt}(G) \geq \gamma_{(k-1)t}(G) + 1 \geq \gamma_{(k-2)t}(G) + 2 \geq \cdots \geq \gamma_{1t}(G) + k - 1 = \gamma_t(G) + k - 1.$$

All these inequalities become equalities when we consider a complete graph.

Lemma 23. *Let G be a graph of order n and let $v_1, \ldots, v_p$ be vertices of degree $n-1$.*

(a) *If $k \leq p-1$, then $\{v_1, \ldots, v_{k+1}\}$ is a minimum total k-dominating set.*
(b) *If $k \geq p$, then there exists a minimum total k-dominating set S such that $\{v_1, \ldots, v_p\} \subseteq S$.*

Proof. (a) Since $\gamma_{kt}(G) \geq k+1$ and $k+1 \leq p$, it follows that $\{v_1, \ldots, v_{k+1}\}$ is a minimum total k-dominating set.

(b) If $k \geq p$ and there exist a minimum total k-dominating set S and two vertices $u \in S$ and $v \in V \setminus S$ such that $\deg(u) < n-1 = \deg(v)$, then $(S \setminus \{u\}) \cup \{v\}$ is also a minimum total k-dominating set S. ■

The above lemma and the following theorem are very useful in order to find a minimum total k-dominating set in some particular graphs. If the graph has order n and contains a vertex v whose degree is equal to $n-1$, this vertex can be taken into the set we are looking for and we can continue looking for a minimum total $(k-1)$-dominating set in the graph induced by $V \setminus \{v\}$. This idea was also used in the proof of Proposition 4 for the wheel graph.

Theorem 24. *Let G be a graph of order n, $v_1, \ldots, v_p$ be vertices of degree $n-1$, and $G' = G[V \setminus \{v_1, \ldots, v_p\}]$. It follows that*

(a) *If* $k \leq p$, *then* $\gamma_{kt}(G) = k+1$.

(b) *If* $k \geq p+1$, *then* $\gamma_{kt}(G) = \gamma_{(k-p)t}(G') + p$.

Proof. (a) If $k \leq p-1$ then, by Lemma 23, we the result follows. If $k = p$ for every $u \in V \setminus \{v_1, \ldots, v_p\}$, we have that $\{v_1, \ldots, v_p, u\}$ is a minimum total k-dominating set.

(b) If we consider the minimum total $(k-p)$-dominating set S in the induced subgraph $G' = G[V \setminus \{v_1, \ldots, v_p\}]$, we see that the set $S' = S \cup \{v_1, \ldots, v_p\}$ is a total k-dominating set in G. Every vertex $u \in V \setminus \{v_1, \ldots, v_p\}$ has $k-p$ neighbors in S, so $\deg_{S'}(u) \geq |\{v_1, \ldots, v_p\}| + k - p = k$. For every $v_i \in \{v_1, \ldots, v_p\}$ we have $\deg_{S'}(v_i) = |S| + p - 1 \geq k - p + 1 + p - 1 = k$. Therefore, $\gamma_{kt}(G) \leq \gamma_{(k-p)t}(G') + p$. Finally, by Lemma 23, if S is a minimum total k-dominating set in G, then we can suppose $\{v_1, \ldots, v_p\} \subseteq S$ and denote $S_0 = S \setminus \{v_1, \ldots, v_p\}$. If there exists a vertex $u \in V(G')$ such that $\deg_{S_0}(u) < k - p$, then $\deg_S(u) = p + \deg_{S_0}(u) < k$, a contradiction. Consequently, S_0 is a minimum total $(k-p)$-dominating set in G', and therefore,

$$\gamma_{kt}(G) = |S_0| + p \geq \gamma_{(k-p)t}(G') + p. \qquad \blacksquare$$

Theorem 25. *For every graph* G *it holds that*

$$\gamma_{kt}(G) \leq 2\gamma_{\times k}(G) - k + 1.$$

Moreover, if n *and* Δ *are the order and maximum degree of* G *respectively, and* $n > \frac{k\Delta^2}{k-1}$, *then*

$$\gamma_{kt}(G) \leq 2\gamma_{\times k}(G) - k.$$

Proof. Let S be a k-tuple dominating set. If there exists a helping vertex v in S, then $\deg_S(v) \geq \delta \geq k$. Let $u \in V \setminus S$ be a vertex such that $N_S(u) = \{v_1, \ldots, v_j\}$ with $j \geq k$. If we denote $A = \{v_1, \ldots, v_j, u\}$, then $\deg_{S \cup \{u\}}(w) \geq k$ for every $w \in A$. Now we need to adapt $S \setminus (H(S) \cup \{v_1, \ldots, v_j\})$ to obtain a total k-dominating set. Since every vertex $v \in S \setminus (H(S) \cup \{v_1, \ldots, v_j\})$ has a neighbor $u_v \in V \setminus S$, if we take the union of set $B = \{u_{v_{j+1}}, \ldots, u_{v_{|S \setminus H(S)|}}\}$ and $S \cup \{u\}$, we obtain a new set S' which is a total k-dominating with cardinality

$$\begin{aligned} |S'| &= |S| + 1 + |B| \leq |S| + 1 + |S \setminus H(S)| - j = 2|S| + 1 - |H(S)| - j \\ &\leq 2|S| - k + 1. \end{aligned}$$

Consequently, $\gamma_{kt}(G) \leq 2\gamma_{\times k}(G) - k + 1$.

When two vertices in $S \setminus (H(S) \cup \{v_1, \ldots, v_j\})$ have a common neighbor in $V \setminus S$ or $j > k$, it follows that $|B| < |S \setminus H(S)| - j$. Then $|S'| \leq 2|S| - k$ and, consequently, $\gamma_{kt}(G) \leq 2\gamma_{\times k}(G) - k$. Therefore, we can suppose that $|B| = |S \setminus H(S)| - j$ and $j = k$. Now, if all vertices in $\{v_1, \ldots, v_k\}$ are adjacent to some vertex of B, it is not necessary to include u in this new set S'. Then its

cardinality satisfies $|S'| = |S|+|B| = |S|+(|S|-|H(S)|-k) = 2|S|-|H(S)|-k \leq 2|S|-k$. Otherwise, if a vertex in $\{v_1,\ldots,v_k\}$, for instance v_k, is not adjacent to the vertices of B, the number of edges from B to $\{v_1,\ldots,v_{k-1}\}$ is at least $|B|(k-1) = (|S|-|H(S)|-k)(k-1)$, but the maximum number of neighbors that every v_i, with $i = 1,\ldots,k-1$, can have in B is $(\Delta-1)-(k-1)$. So $(|S|-|H(S)|-k)(k-1) \leq (k-1)(\Delta-k)$ or, equivalently, $|S| \leq \Delta + |H(S)|$. Similarly to Lemma 12, it can be proved that for every $k \leq \delta$ and every minimum k-tuple dominating set S, it holds $|H(S)| \leq \frac{(k-1)|S|}{k}$. Therefore, we obtain $|S| \leq k\Delta$. Using the fact that $\gamma_{(k-1)t}(G) \leq \gamma_{\times k}(G)$ and $\frac{(k-1)n}{\Delta} \leq \gamma_{(k-1)t}(G)$ (see [6]), we conclude that $\frac{(k-1)\,n}{\Delta} \leq k\Delta$, which is a contradiction. ■

There exists an infinite family of graphs G_s satisfying $\gamma_{kt}(G_s) = 2\gamma_{\times k}(G_s) - k+1$. In Figure 7 we have a graph with $n = 4s+3$ and $\delta = 2$. The black vertices form a minimum 2-tuple dominating set with cardinality $2s+2$, and, by Lemma 1, $\gamma_{2t}(G_s) = n$.

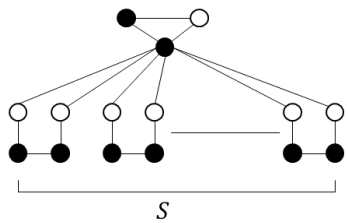

Figure 7. Family of graphs G_s satisfying $\gamma_{2t}(G_s) = 2\gamma_{\times 2}(G_s) - 1$.

Proposition 26. *Let G be a graph of order n and minimum degree δ. If $k < \delta$, then*

$$\gamma_{\times(k+1)}(G) \leq \frac{n+\gamma_{kt}(G)}{2}.$$

Proof. Let S be a minimum total k-dominating set. If $u_1 \in \overline{S}$ satisfies $\deg_S(u_1) = k$, then $\deg_{\overline{S}}(u_1) \geq 1$. We consider $S_1 = S \cup \{w_{u_1}\}$, where $w_{u_1} \in \overline{S} \cap N(u_1)$. Now, if $u_2 \in \overline{S_1}$ satisfies $\deg_{S_1}(u_2) = k$, then $\deg_{\overline{S_1}}(u_2) \geq 1$. We consider $S_2 = S_1 \cup \{w_{u_2}\}$, where $w_{u_2} \in \overline{S_1} \cap N(u_2)$ and continue this process. At the end we will get a $(k+1)$-tuple dominating set of size at most $|S| + \frac{n-|S|}{2} = \frac{n+|S|}{2}$. Consequently $\gamma_{\times(k+1)}(G) \leq \frac{n+\gamma_{kt}(G)}{2}$. ■

The inequality in the proposition above is attained for instance in the Cartesian product $G = P_2 \Box C_3$ with $k = 2$, where $\gamma_{\times 3}(G) = 5$ and $\gamma_{2t}(G) = 4$.

In the next theorem we compare the total k-domination number with the k-domination number, similarly as we did with the total k-domination number and the k-tuple domination number in Theorem 25. Given a k-dominating set

S in a graph G, if for any vertex $u \in S$ we take k adjacent vertices in $V \setminus S$, we obtain a total k-dominating set of cardinality $(k+1)|S|$. Therefore, we have $\gamma_{kt}(G) \leq (k+1)\,\gamma_k(G)$. A better result though is the following.

Theorem 27. *For every graph G it holds that*

$$\gamma_{kt}(G) \leq (k+1)\gamma_k(G) - k(k-1).$$

Proof. Let S be a minimum k-dominating set. We are going to take the union of some vertices from $V \setminus S$ and the vertices of S in order to obtain a total k-dominating set. It is clear that the worst case is when S is an independent set. If $S = \{v_1, \ldots, v_r\}$ and we add to S any k vertices from $V \setminus S$, we obtain a new set S' with cardinality $|S| + k$ such that $\deg_{S'}(v_1) + \cdots + \deg_{S'}(v_r) \geq k^2$ and $0 \leq \deg_{S'}(v_i) \leq k$ for every $i \in \{1, \ldots, r\}$. As we want to have $\deg_{S'}(v_i) \geq k$ for every $i \in \{1, \ldots, r\}$, in the worst case we need to add $k|S| - k^2$ vertices from $V \setminus S'$ to S'. This new set will be a total k-dominating set with cardinality at most $|S| + k + k|S| - k^2 = (k+1)|S| - k(k-1)$, and the result follows. ■

The upper bound given in the theorem above is attained in any complete bipartite graph $K_{k,r}$ with $r \geq k$, where $\gamma_{kt}(G) = 2k$ and $\gamma_k(G) = k$.

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References

[1] S. Bermudo, J.L. Sánchez and J.M. Sigarreta, *Total k-domination in Cartesian product graphs*, Period. Math. Hungar. (2016), to appear.

[2] S. Bermudo, D.L. Jalemskaya and J.M. Sigarreta, *Total 2-domination in grid graphs*, Util. Math. (2017), to appear.

[3] E.J. Cockayne and A.G. Thomason, *An upper bound for the k-tuple domination number*, J. Combin. Math. Combin. Comput. **64** (2008) 251–254.

[4] P. Dorbec, S. Gravier, S. Klavžar and S. Špacapan, *Some results on total domination in direct products of graphs*, Discuss. Math. Graph Theory **26** (2006) 103–112. doi:10.7151/dmgt.1305

[5] O. Favaron, M.A. Henning, J. Puech and D. Rautenbach, *On domination and annihilation in graphs with claw-free blocks*, Discrete Math. **231** (2001) 143–151. doi:10.1016/S0012-365X(00)00313-7

[6] H. Fernau, J.A. Rodríguez-Velázquez and J.M. Sigarreta, *Global powerful r-alliances and total k-domination in graphs*, Util. Math. **98** (2015) 127–147.

[7] J. Harant and M.A. Henning, *On double domination in graphs*, Discuss. Math. Graph Theory **25** (2005) 29–34.
doi:10.7151/dmgt.1256

[8] F. Harary and T.W. Haynes, *Double domination in graphs*, Ars Combin. **55** (2000) 201–213.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[11] M.A. Henning, *A survey of selected recent results on total domination in graphs*, Discrete Math. **309** (2009) 32–63.
doi:10.1016/j.disc.2007.12.044

[12] M.A. Henning and A.P. Kazemi, *k-tuple total domination in graphs*, Discrete Appl. Math. **158** (2010) 1006–1011.
doi:10.1016/j.dam.2010.01.009

[13] M.A. Henning and A.P. Kazemi, *k-tuple total domination in cross products of graphs*, J. Comb. Optim. **24** (2012) 339–346.
doi:10.1007/s10878-011-9389-z

[14] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer Monographs in Mathematics, 2013).

[15] A.P. Kazemi, *On the total k-domination number of graphs*, Discuss. Math. Graph Theory **32** (2012) 419–426.
doi:10.7151/dmgt.1616

[16] A. Klobucar, *Total domination numbers of Cartesian products*, Math. Commun. **9** (2004) 35–44.

[17] V.R. Kulli, *On n-total domination number in graphs*, Graph theory, Combinatorics, Algorithms, and Applications (San Francisco, CA, 1989), (SIAM, Philadelphia, PA, 1991) 319–324.

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