# BOUNDS ON THE LOCATING-DOMINATION NUMBER AND DIFFERENTIATING-TOTAL DOMINATION NUMBER IN TREES 

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#### Abstract

A subset $S$ of vertices in a graph $G=(V, E)$ is a dominating set of $G$ if every vertex in $V-S$ has a neighbor in $S$, and is a total dominating set if every vertex in $V$ has a neighbor in $S$. A dominating set $S$ is a locating-dominating set of $G$ if every two vertices $x, y \in V-S$ satisfy $N(x) \cap S \neq N(y) \cap S$. The locating-domination number $\gamma_{L}(G)$ is the minimum cardinality of a locating-dominating set of $G$. A total dominating set $S$ is called a differentiating-total dominating set if for every pair of distinct vertices $u$ and $v$ of $G, N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a differentiating-total dominating set of $G$ is the differentiating-total domination number of $G$, denoted by $\gamma_{t}^{D}(G)$. We obtain new upper bounds for the locating-domination number, and the differentiating-total domination number in trees. Moreover, we characterize all trees achieving equality for the new bounds.


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## 1. Introduction

For notation and graph theory terminology in general we follow [9]. We consider finite, undirected, and simple graphs $G$ with vertex set $V=V(G)$ and edge set $E=E(G)$. The number of vertices $|V(G)|$ of a graph $G$ is called the order of $G$ and is denoted by $n=n(G)$. The open neighborhood of a vertex $v \in V$ is $N(v)=N_{G}(v)=\{u \in V \mid u v \in E\}$ and the closed neighborhood of $v$ is $N[v]=$
$N(v) \cup\{v\}$. The degree of $v$, denoted by $\operatorname{deg}(v)$ (or $\operatorname{deg}_{G}(v)$ to refer to $G$ ), is the cardinality of its open neighborhood. A leaf of a tree $T$ is a vertex of degree one, while a support vertex of $T$ is a vertex adjacent to a leaf. A strong support vertex is a support vertex adjacent to at least two leaves. We denote the set of all support vertices of a tree $T$ by $S(T)$ and the set of leaves by $L(T)$. We always denote $\ell=\ell(T)=|L(T)|$, and $s=s(T)=|S(T)|$. Whenever a tree $T^{\prime}$ (or $T^{\prime \prime}, \ldots$ ) is introduced, we let $n^{\prime}, \ell^{\prime}$ (or $n^{\prime \prime}, \ell^{\prime \prime}, \ldots$ ) be its order, and number of leaves, respectively. We denote by $\ell_{v}$ the number of leaves adjacent to a support vertex $v$, and by $L_{v}$ the set of leaves adjacent to $v$. We denote a path of order $n$ by $P_{n}$ (or $P_{n}: v_{1} v_{2} \cdots v_{n}$, where $V\left(P_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\}$ and $v_{i}$ is adjacent to $v_{i+1}$ for $i=1,2, \ldots, n-1)$. The distance $d(x, y)$ between two vertices $x$ and $y$ is the length of a shortest path from $x$ to $y$. The diameter $\operatorname{diam}(G)$ of a graph $G$ is the maximum distance over all pair of vertices of $G$. For a rooted tree $T$ and a vertex $v$, we denote by $T_{v}$ the sub-rooted tree, rooted at $v$.

A subset $S \subseteq V$ is a dominating set of $G$ if every vertex in $V-S$ has a neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set $S$ is a locating-dominating set (or just LDS) of $G$ if every two vertices $x, y \in V-S$ satisfy $N(x) \cap S \neq N(y) \cap S$. The locatingdomination number $\gamma_{L}(G)$ is the minimum cardinality of a locating-dominating set of $G$. A locating-dominating set of $G$ of cardinality $\gamma_{L}(G)$ is referred as a $\gamma_{L}(G)$-set. The concept of locating domination in graphs was pioneered by Slater $[15,16]$, and has been further studied in, for example, $[1,2,4,7,13,14]$.

A dominating set $S$ is a total dominating set (or just TDS) of $G$ if the induced subgraph $G[S]$ has no isolated vertex. A total dominating set $S$ is called a differentiating-total dominating set (or just DTDS) if for every pair of distinct vertices $u$ and $v$ of $G, N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a differentiating-total dominating set of $G$ is the differentiating-total domination number of $G$, denoted by $\gamma_{t}^{D}(G)$. The concept of differentiating-total domination was introduced by Haynes, Henning and Howard [8] and further studied in, for example, $[3,5,6,10,11,12]$.

Blidia et al. [1] obtained the following upper bound for the locating-domination number of a tree.

Theorem 1 (Blidia et al. [1]). For any tree $T$ of order $n \geq 2$, with $\ell$ leaves and $s$ support vertices, $\gamma_{L}(T) \leq(n+\ell-s) / 2$.

Ning et al. [12] constructed the following family $\mathcal{F}$ of trees as follows. For each tree $T \in \mathcal{F}$, every vertex $v$ in $T$ has a label $s(v) \in\{A, B, C, D\}$, called its status. Let $\mathcal{F}$ be the family of labeled trees $T=T_{k}$ that can be obtained as follows. Let $T_{0}=x_{1} x_{2} x_{3} x_{4} x_{5} x_{6} x_{7} x_{8}$ be a path of order 8 in which $s\left(x_{1}\right)=$ $s\left(x_{8}\right)=C, s\left(x_{2}\right)=s\left(x_{7}\right)=A, s\left(x_{3}\right)=s\left(x_{6}\right)=B$ and $s\left(x_{4}\right)=s\left(x_{5}\right)=D$. If $k \geq 1$, then $T_{k}$ can be obtained from $T_{k-1}$ by one of the following operations.

Operation $\phi_{\mathbf{1}}$. For $x \in V\left(T_{k-1}\right)$, if $s(x)=C$ and $\operatorname{deg}_{T_{k-1}}(x)=1$, then add a path $Q=y z u v w$ and the edge $x y$. Let $s(y)=D, s(z)=D, s(u)=B, s(v)=A$ and $s(w)=C$.
Operation $\phi_{\mathbf{2}}$. For $x \in V\left(T_{k-1}\right)$, if $s(x)=D$, then add a path $Q=y z u v$ and the edge $x y$. Let $s(y)=D, s(z)=B, s(u)=A$ and $s(v)=C$.

Theorem 2 (Ning et al. [12]). If $T$ is a tree of order $n \geq 3$ with $\ell$ leaves, then $\gamma_{t}^{D}(T) \leq 3(n+\ell) / 5$, with equality if and only if $T=P_{3}$, or $T \in \mathcal{F}$.

In Section 2, we prove that for any tree $T$ of order $n \geq 2$, with $\ell$ leaves, $\gamma_{L}(T) \leq(2 n+3 \ell-2) / 5$, and characterize all trees achieving equality for this bound. We note that our bound is an improvement of the bound of Theorem 1 for trees $T$ with $n>\ell+5 s-4$. In Section 3, we prove that for any tree $T$ of order $n \geq 4$ with $\ell$ leaves and $s$ support vertices, $\gamma_{t}^{D}(T) \leq(3 n+2 \ell+s) / 5$, and characterize all trees achieving equality for this bound. We note that our bound is an improvement of the bound of Theorem 2 for trees $T$ with $\ell>s$. We make use of the following.

Lemma 3 (Blidia et al. [1]). In any tree $T$ of order $n \geq 3$ there is a $\gamma_{L}(T)$-set $S$ such that:
(1) If $x$ is a support vertex and $\ell_{x}$ is the number of leaves adjacent to $x$, then $S$ contains $x$ and exactly $\ell_{x}-1$ leaves adjacent to $x$.
(2) If abcd is a path with $\operatorname{deg}(a)=1, \operatorname{deg}(b)=\operatorname{deg}(c)=2$ and $\operatorname{deg}(d)>1$, then $S \cap\{a, b, c, d\}=\{b, d\}$.

## 2. An Upper Bound for $\gamma_{L}(T)$

We begin with the following observation.
Observation 4. If $T$ is a tree obtained from a tree $T^{\prime}$ by adding a leaf or a path $P_{2}$ to $T^{\prime}$, then $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1$.

Let $\mathcal{T}$ be the collection of trees $T$ that can be obtained from a sequence $T_{1}, T_{2}, \ldots, T_{k}=T(k \geq 1)$ of trees, where $T_{1}=P_{3}$, and $T_{i+1}$ can be obtained recursively from $T_{i}$ by one of the following operations for $1 \leq i \leq k-1$.
Operation $\mathcal{O}_{1}$. Assume that $w$ is a support vertex of $T_{i}$. Then $T_{i+1}$ is obtained from $T_{i}$ by adding a leaf to $w$.
Operation $\mathcal{O}_{2}$. Assume that $w$ is a leaf of $T_{i}$. Then $T_{i+1}$ is obtained from $T_{i}$ by adding a path $P_{5}$ and joining $w$ to a leaf of $P_{5}$.

Lemma 5. If $\gamma_{L}\left(T_{i}\right)=\left(2 n\left(T_{i}\right)+3 \ell\left(T_{i}\right)-2\right) / 5$, and $T_{i+1}$ is obtained from $T_{i}$ by Operation $\mathcal{O}_{1}$ or Operation $\mathcal{O}_{2}$, then $\gamma_{L}\left(T_{i+1}\right)=\left(2 n\left(T_{i+1}\right)+3 \ell\left(T_{i+1}\right)-2\right) / 5$.

Proof. Let $\gamma_{L}\left(T_{i}\right)=\frac{2 n_{i}+3 \ell_{i}-2}{5}$, where $n_{i}=n\left(T_{i}\right)$ and $\ell_{i}=\ell\left(T_{i}\right)$. Assume that $T_{i+1}$ is obtained from $T_{i}$ by Operation $\mathcal{O}_{1}$. Let $T_{i+1}$ be obtained from $T_{i}$ by adding a leaf $v$ to a support vertex $w$ of $T_{i}$. By Observation $4, \gamma_{L}\left(T_{i+1}\right) \leq \gamma_{T}\left(T_{i}\right)+1$. Let $S$ be a $\gamma_{L}\left(T_{i+1}\right)$-set satisfying the conditions of Lemma 3. Thus $w \in S$, and without loss of generality, we may assume that $v \in S$. Then $S-\{v\}$ is an LDS for $T_{i}$, implying that $\gamma_{L}\left(T_{i}\right) \leq \gamma_{L}\left(T_{i+1}\right)-1$. Thus $\gamma_{L}\left(T_{i+1}\right)=\gamma_{T}\left(T_{i}\right)+1$. Now $\gamma_{L}\left(T_{i+1}\right)=\left(2 n\left(T_{i}\right)+3 \ell\left(T_{i}\right)-2\right) / 5+1=\left(2\left(n\left(T_{i}\right)+1\right)+3\left(\ell\left(T_{i}\right)+1\right)-2\right) / 5=$ $\left(2 n\left(T_{i+1}\right)+3 \ell\left(T_{i+1}\right)-2\right) / 5$.

Next assume that $T_{i+1}$ is obtained from $T_{i}$ by Operation $\mathcal{O}_{2}$. Let $T_{i+1}$ be obtained from $T_{i}$ by joining a leaf $v$ of $T_{i}$ to the leaf $a$ of a path $P_{5}$ : abcde. If $S$ is a $\gamma_{L}\left(T_{i}\right)$-set, then $S \cup\{b, d\}$ is an LDS for $T_{i+1}$, and so $\gamma_{L}\left(T_{i+1}\right) \leq$ $\gamma_{L}\left(T_{i}\right)+2$. Let $D$ be a $\gamma_{L}\left(T_{i+1}\right)$-set satisfying the conditions of Lemma 3. Thus $S \cap\{b, c, d, e\}=\{b, d\}$. If $a \notin D$, then $D-\{b, d\}$ is an $\operatorname{LDS}$ for $T_{i}$, and if $a \in D$, then $(D-\{a, b, d\}) \cup\{v\}$ is an LDS for $T_{i}$, and so $\gamma_{L}\left(T_{i}\right) \leq \gamma_{L}\left(T_{i+1}\right)-2$. Thus $\gamma_{L}\left(T_{i+1}\right)=\gamma_{T}\left(T_{i}\right)+2$. Now $\gamma_{L}\left(T_{i+1}\right)=\left(2 n\left(T_{i}\right)+3 \ell\left(T_{i}\right)-2\right) / 5+2=$ $\left(2\left(n\left(T_{i}\right)+5\right)+3 \ell\left(T_{i}\right)-2\right) / 5=\left(2 n\left(T_{i+1}\right)+3 \ell\left(T_{i+1}\right)-2\right) / 5$.

By a simple induction on the operations performed to construct a tree $T \in \mathcal{T}$, and Lemma 5 we obtain the following.

Lemma 6. For any tree $T \in \mathcal{T}$ of order $n \geq 3$ and with $\ell$ leaves, $\gamma_{L}(T)=$ $(2 n+3 \ell-2) / 5$.

We are now ready to present the main result of this section.
Theorem 7. For any tree $T$ of order $n \geq 2$ with $\ell$ leaves, $\gamma_{L}(T) \leq(2 n+3 \ell-2) / 5$, with equality if and only if $T \in \mathcal{T}$.

Proof. We first use an induction on the order $n$ of $T$ to show that $\gamma_{L}(T) \leq$ $(2 n+3 \ell-2) / 5$. The base step is obvious for $n=2$ and 3 . Assume that for any nontrivial tree $T^{\prime}$ of order $n^{\prime}<n$, with $l^{\prime}$ leaves, $\gamma_{L}\left(T^{\prime}\right) \leq\left(2 n^{\prime}+3 \ell^{\prime}-2\right) / 5$. Now consider the tree $T$ of order $n \geq 4$. Assume that $T$ has a strong support vertex. Let $v$ be a strong support vertex, and $u$ be a leaf adjacent to $v$. Let $T^{\prime}=T-u$. By Observation 4, $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1 \leq\left(2 n\left(T^{\prime}\right)+3 \ell\left(T^{\prime}\right)-2\right) / 5+1=(2(n-1)+3(\ell-1)-2) / 5+1=$ $(2 n+3 \ell-2) / 5$. Next assume that $T$ has an edge $e=u v$ with $\operatorname{deg}(u) \geq 3$ and $\operatorname{deg}(v) \geq 3$. Let $T_{1}$ and $T_{2}$ be the components of $T-e$, with $u \in V\left(T_{1}\right)$ and $v \in V\left(T_{2}\right)$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}\left(T_{1}\right)+\gamma_{L}\left(T_{2}\right) \leq\left(2 n\left(T_{1}\right)+\right.$ $\left.3 \ell\left(T_{1}\right)-2\right) / 5+\left(2 n\left(T_{2}\right)+3 \ell\left(T_{2}\right)-2\right) / 5=(2 n+3 \ell-4) / 5<(2 n+3 \ell-2) / 5$. Thus for the next, we may assume that the following facts hold.
Fact 1. $T$ has no strong support vertex.
Fact 2. For each edge $e=u v, \operatorname{deg}(u) \leq 2$ or $\operatorname{deg}(v) \leq 2$.

Let $d=\operatorname{diam}(T)$. By Fact $1, d \geq 3$. If $d=3$, then $T=P_{4}$, and $\gamma_{L}(T)=$ $2<(2 n+3 \ell-2) / 5$. Thus $d \geq 4$. We root $T$ at a leaf $x_{0}$ of a diameterical path $x_{0} x_{1} \cdots x_{d}$ from $x_{0}$ to a leaf $x_{d}$ farthest from $x_{0}$. By Fact $1, \operatorname{deg}\left(x_{d-1}\right)=$ $\operatorname{deg}\left(x_{1}\right)=2$. Assume that $d=4$. If $\operatorname{deg}\left(x_{2}\right)=2$ then $T=P_{5}$, and $\gamma_{L}(T)=$ $2<(2 n+3 \ell-2) / 5$. Thus assume that $\operatorname{deg}\left(x_{2}\right)>2$. If $x_{2}$ is a support vertex, then $T$ has $\operatorname{deg}\left(x_{2}\right)-1$ support vertices of degree two. Then $N\left(x_{2}\right)$ is a LDS for $T$, implying that $\gamma_{L}(T) \leq \operatorname{deg}\left(x_{2}\right)<(2 n+3 \ell-2) / 5$, since $n=2 \operatorname{deg}\left(x_{2}\right)$ and $\ell=\operatorname{deg}\left(x_{2}\right)$. Thus assume that $x_{2}$ is not a support vertex. Then $T$ has $\operatorname{deg}\left(x_{2}\right)$ support vertices of degree two, and we can see that $N\left(x_{2}\right)$ is an LDS for $T$, implying that $\gamma_{L}(T) \leq \operatorname{deg}\left(x_{2}\right)<(2 n+3 \ell-2) / 5$, since $n=2 \operatorname{deg}\left(x_{2}\right)+1$ and $\ell=\operatorname{deg}\left(x_{2}\right)$. Assume that $d=5$. By Fact 2 , we may assume that $\operatorname{deg}\left(x_{3}\right)=2$. If $\operatorname{deg}\left(x_{2}\right)=2$ then $T=P_{6}$, and $\gamma_{L}(T)=3<(2 n+3 \ell-2) / 5$. Thus assume that $\operatorname{deg}\left(x_{2}\right)>2$. Since $d=5$, by Fact 1, any vertex of $N\left(x_{2}\right)-\left\{x_{3}\right\}$ is a leaf or a support vertex of degree two. Assume that $x_{2}$ is a support vertex. By Fact 1 , there is a unique leaf adjacent to $x_{2}$. Then $S(T)$ is an LDS for $T$, implying that $\gamma_{L}(T) \leq|S(T)|=\operatorname{deg}\left(x_{2}\right)<(2 n+3 \ell-2) / 5$, since $n=2 \operatorname{deg}\left(x_{2}\right)+1$ and $\ell=\operatorname{deg}\left(x_{2}\right)$. Thus assume that $x_{2}$ is not a support vertex. Then $S(T) \cup\left\{x_{2}\right\}$ is an LDS for $T$, implying that $\gamma_{L}(T) \leq|S(T)|+1=\operatorname{deg}\left(x_{2}\right)+1<(2 n+3 \ell-2) / 5$, since $n=2 \operatorname{deg}\left(x_{2}\right)+2$ and $\ell=\operatorname{deg}\left(x_{2}\right)$. Thus assume that $d \geq 6$. Assume that $\operatorname{deg}\left(x_{d-2}\right) \geq 3$. Let $T^{\prime}=T-\left\{x_{d}, x_{d-1}\right\}$. Using Observation 4 and the inductive hypothesis, we obtain $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1 \leq \frac{2(n-2)+3(\ell-1)-2}{5}+1<\frac{2 n+3 \ell-2}{5}$. We thus assume that $\operatorname{deg}\left(x_{d-2}\right)=2$.

Assume that $\operatorname{deg}\left(x_{d-3}\right) \geq 3$. Suppose that $x_{d-3}$ is a support vertex and $u$ is the unique leaf adjacent to $x_{d-3}$. Let $T^{\prime}=T-\left\{x_{d}, x_{d-1}, x_{d-2}\right\}$. Note that $x_{d-3}$ is a support vertex in $T^{\prime}$, and by Lemma 3 there is a $\gamma_{L}\left(T^{\prime}\right)$-set $D$ containing $x_{d-3}$. Then $D \cup\left\{x_{d-1}\right\}$ is an LDS for $T$, implying that $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1$. By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+1 \leq \frac{2(n-3)+3(\ell-1)-2}{5}+1<\frac{2 n+3 \ell-2}{5}$. Thus assume that $x_{d-3}$ is not a support vertex. Assume that there is a leaf $y$ of $T_{x_{d-3}}$ with $d\left(y, x_{d-3}\right)=2$. Let $u$ be the father of $y$. By Fact $2, \operatorname{deg}(u)=2$. Let $T^{\prime}=T-\{u, y\}$. By Observation 4 and the inductive hypothesis, $\gamma_{L}(T) \leq$ $\gamma_{L}\left(T^{\prime}\right)+1 \leq(2(n-2)+3(\ell-1)-2) / 5+1<(2 n+3 \ell-2) / 5$. Thus assume for the next that any leaf of $T_{x_{d-3}}$ is at distance three from $x_{d-3}$. Since any such leaf plays the same role as $x_{d}$, any internal vertex in the shortest path from such leaf to $x_{d-3}$ has degree two. Let $\operatorname{deg}\left(x_{d-3}\right)=k+1$ with $k \geq 2$. By Fact $2, \operatorname{deg}\left(x_{d-4}\right)=2$. Let $T^{\prime}=T-T_{x_{d-4}}$. By the inductive hypothesis, $\gamma_{L}\left(T^{\prime}\right) \leq\left(2 n^{\prime}+3 \ell^{\prime}-2^{\prime}\right) / 5$. But $l^{\prime} \leq l-k+1$, and $n^{\prime}=n-3 k-2$. Let $S$ be a $\gamma_{L}\left(T^{\prime}\right)$-set. Then $S \cup\left\{x_{d-3}\right\} \cup U$ is an LDS for $T$, where $U$ is the set of vertices of $T_{x_{d-4}}$ at distance two from $x_{d-3}$. Thus $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+k+1 \leq\left(2 n^{\prime}+3 \ell^{\prime}-2\right) / 5+k+1 \leq(2 n+3 \ell-2-4 k+4) / 5<$ $(2 n+3 \ell-2) / 5$ because $k \geq 2$. Thus assume that $\operatorname{deg}\left(x_{d-3}\right)=2$.

Assume that $\operatorname{deg}\left(x_{d-4}\right) \geq 3$. Let $T^{\prime}=T-\left\{x_{d}, x_{d-1}, x_{d-2}, x_{d-3}\right\}$. Let $D$ be $\gamma_{L}\left(T^{\prime}\right)$-set. Then $D \cup\left\{x_{d-2}, x_{d-1}\right\}$ is an LDS for $T$, and so $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)$
+2 . By the inductive hypothesis, $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+2 \leq \frac{2 n\left(T^{\prime}\right)+3 \ell\left(T^{\prime}\right)-2}{5}+2 \leq$ $\frac{2(n-4)+3(\ell-1)-2}{5}+2<\frac{2 n+3 \ell-2}{5}$. Thus assume that $\operatorname{deg}\left(x_{d-4}\right)=2$.

Let $T^{\prime}=T-T_{x_{d-4}}$. By the inductive hypothesis, $\gamma_{L}\left(T^{\prime}\right) \leq\left(2 n^{\prime}+3 \ell^{\prime}-2\right) / 5$. Let $S$ be a $\gamma_{L}\left(T^{\prime}\right)$-set. Then $S \cup\left\{x_{d-3}, x_{d-1}\right\}$ is an LDS for $T$. Thus $\gamma_{L}(T) \leq$ $\gamma_{L}\left(T^{\prime}\right)+2 \leq(2(n-5)+3 \ell-2) / 5+2=(2 n+3 \ell-2) / 5$.

We next prove the equality part. We use an induction on the order $n$ of a tree $T$ with $\ell$ leaves and $\gamma_{L}(T)=(2 n+3 \ell-2) / 5$ to show that $T \in \mathcal{T}$. The basic step is obvious, since $P_{3} \in \mathcal{T}$. Assume that any nontrivial tree $T^{\prime}$ of order $n^{\prime}<n$, with $\ell^{\prime}$ leaves and $\gamma_{L}\left(T^{\prime}\right)=\left(2 n^{\prime}+3 \ell^{\prime}-2\right) / 5$ belongs to $\mathcal{T}$. Let $n=n(T) \geq 4$. Assume that $T$ has a strong support vertex $u$, and $v$ is a leaf adjacent to $u$. Let $T^{\prime}=T-v$. Using Lemma 3, we can easily see that $\gamma_{L}(T)=\gamma_{L}\left(T^{\prime}\right)+1$. Thus $\gamma_{L}\left(T^{\prime}\right)=\gamma_{L}(T)-1=(2 n+3 \ell-2) / 5-1=(2(n-1)+3(\ell-1)-2) / 5=$ $\left(2 n\left(T^{\prime}\right)+3 \ell\left(T^{\prime}\right)-2\right) / 5$. By the inductive hypothesis, $T^{\prime} \in \mathcal{T}$. Hence $T$ is obtained from $T^{\prime}$ by Operation $\mathcal{O}_{1}$. Thus for the next assume that $T$ has no strong support vertex.

We root $T$ at a leaf $x_{0}$ of a diameterical path $x_{0} x_{1} \cdots x_{d}$ from $x_{0}$ to a leaf $x_{d}$ farthest from $x_{0}$. By the first part of the proof, we find that $d \geq 6$, and $\operatorname{deg}\left(x_{d-2}\right)=\operatorname{deg}\left(x_{d-3}\right)=\operatorname{deg}\left(x_{d-4}\right)=2$. Since $\gamma_{L}\left(P_{7}\right)=3<\left(2 n\left(P_{7}\right)+3 \ell\left(P_{7}\right)-\right.$ $2) / 5$, we have $d \geq 7$. We next show that $\operatorname{deg}\left(x_{d-5}\right)=2$. Assume that $\operatorname{deg}\left(x_{d-5}\right) \geq$ 3. Let $T^{\prime}=T-\left\{x_{d}, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}\right\}$. Let $D$ be a $\gamma_{L}\left(T^{\prime}\right)$-set. Then $D \cup\left\{x_{d-3}, x_{d-1}\right\}$ is an LDS for $T$, and so $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+2$. By the first part of the theorem we have $\gamma_{L}(T) \leq \gamma_{L}\left(T^{\prime}\right)+2 \leq\left(2 n\left(T^{\prime}\right)+3 \ell\left(T^{\prime}\right)-2\right) / 5+$ $2 \leq(2(n-5)+3(\ell-1)-2) / 5+2<(2 n+3 \ell-2) / 5$, a contradiction. Thus $\operatorname{deg}\left(x_{d-5}\right)=2$. Let $T^{*}=T-\left\{x_{d}, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}\right\}$. If $D^{*}$ is a $\gamma_{L}\left(T^{*}\right)$-set, then $D^{*} \cup\left\{x_{d-3}, x_{d-1}\right\}$ is an LDS for $T$, and so $\gamma_{L}(T) \leq \gamma_{L}\left(T^{*}\right)+2$. By Lemma 3 there is a $\gamma_{L}(T)$-set $D$ such that $D \cap\left\{x_{d-3}, x_{d-2}, x_{d-1}, x_{d}\right\}=\left\{x_{d-3}, x_{d-1}\right\}$. If $x_{d-4} \in D$, then $\left(D-\left\{x_{d-3}, x_{d-1}, x_{d-4}\right\}\right) \cup\left\{x_{d-5}\right\}$ is an LDS for $T^{*}$, and if $x_{d-4} \notin D$, then $D-\left\{x_{d-3}, x_{d-1}\right\}$ is an LDS for $T^{*}$. Thus $\gamma_{L}\left(T^{*}\right) \leq \gamma_{L}(T)-2$. We deduce that $\gamma_{L}(T)=\gamma_{L}\left(T^{*}\right)+2$. Now $\gamma_{L}\left(T^{*}\right)=\gamma_{L}(T)-2=(2 n+3 \ell-2) / 5-2=$ $(2(n-5)+3 \ell-2) / 5=\left(2 n\left(T^{*}\right)+3 \ell\left(T^{*}\right)-2\right) / 5$. By the inductive hypothesis, $T^{*} \in \mathcal{T}$. Hence $T$ is obtained from $T^{*}$ by Operation $\mathcal{O}_{2}$.

The converse follows by Lemma 6.
We note that the bound of Theorem 7 is an improvement of the bound of Theorem 1 for trees $T$ with $n>\ell+5 s-4$.

## 3. An Upper Bound for $\gamma_{t}^{D}(T)$

We prove that for any tree $T$ of order $n \geq 3$ with $\ell$ leaves and $s$ support vertices, $\gamma_{t}^{D}(T) \leq(3 n+2 \ell+s) / 5$. We begin with the following observation of [12], since for any tree $T \in \mathcal{F},\{v \in V(T) \mid s(v) \in\{A \cup B \cup C\}\}$ is a $\gamma_{t}^{D}(T)$-set.

Observation 8 (Ning et al. [12]). Any tree $T \in \mathcal{F}$ has a $\gamma_{t}^{D}(T)$-set containing all leaves and all support vertices.

Theorem 9. If $T$ is a tree of order $n \geq 4$ with $\ell$ leaves and support vertices, then $\gamma_{t}^{D}(T) \leq(3 n+2 \ell+s) / 5$, with equality if and only if $T \in \mathcal{F}$.

Proof. Let $T$ be a tree of order $n \geq 4$ with $\ell$ leaves and $s$ support vertices. We prove by induction on the order $n$ of $T$ that $\gamma_{t}^{D}(T) \leq(3 n+2 \ell+s) / 5$. For the base step of the induction, if $n=4$, then $T \in\left\{P_{4}, K_{1,3}\right\}$, and it is obvious that $\gamma_{t}^{D}(T)<(3 n+2 \ell+s) / 5$.

Assume that for any tree $T^{\prime}$ of order $n^{\prime}<n$ with $\ell^{\prime}$ leaves and $s^{\prime}$ support vertices, $\gamma_{t}^{D}\left(T^{\prime}\right) \leq\left(3 n^{\prime}+2 \ell^{\prime}+s^{\prime}\right) / 5$. Now consider the tree $T$ of order $n>4$ with $\ell$ leaves and $s$ support vertices. If $T$ has no strong support vertices, then $\ell=s$, and the result follows from Theorem 2. Thus assume that $T$ has some strong support vertex. Let $u$ be a strong support vertex of $T$, and $v$ be a leaf adjacent to $u$. Let $T^{\prime}=T-v$. Then $n^{\prime} \geq 4$. Clearly $\gamma_{t}^{D}(T) \leq \gamma_{t}^{D}\left(T^{\prime}\right)+1$. By the inductive hypothesis, $\gamma_{t}^{D}(T) \leq \gamma_{t}^{D}\left(T^{\prime}\right)+1 \leq\left(3 n\left(T^{\prime}\right)+2 \ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)\right) / 5+1=$ $(3(n-1)+2(\ell-1)+s) / 5+1=(3 n+2 \ell+s) / 5$.

Now we prove the equality part. Assume that $\gamma_{t}^{D}(T)=(3 n+2 \ell+s) / 5$. We show that $\ell=s$. Suppose that $\ell>s$. Let $T^{\prime}$ be a tree obtained from $T$ by removing $\ell_{x}-1$ leaves of any support vertex $x$. Thus $n^{\prime}=n\left(T^{\prime}\right)=n-\ell+s$ and $\ell\left(T^{\prime}\right)=s\left(T^{\prime}\right)=s(T)$. By the theorem, $\gamma_{t}^{D}\left(T^{\prime}\right) \leq\left(3 n\left(T^{\prime}\right)+2 \ell\left(T^{\prime}\right)+s\left(T^{\prime}\right)\right) / 5=$ $(3(n-\ell+s)+3 s) / 5$. If $\gamma_{t}^{D}\left(T^{\prime}\right)<(3(n-\ell+s)+3 s) / 5$, then $\gamma_{t}^{D}(T) \leq \gamma_{t}^{D}\left(T^{\prime}\right)+\ell-s<$ $(3 n+2 \ell+s) / 5$, a contradiction. Thus $\gamma_{t}^{D}\left(T^{\prime}\right)=(3(n-\ell+s)+3 s) / 5$. By Theorem $2, T^{\prime} \in \mathcal{F}$. By Observation $8, T^{\prime}$ has a $\gamma_{t}^{D}\left(T^{\prime}\right)$-set $S$ containing all leaves and all support vertices. Let $y \in L(T)-L\left(T^{\prime}\right)$. Then $\left(S \cup\left(L(T)-L\left(T^{\prime}\right)\right)\right)-\{y\}$ is a DTDS for $T$, and thus $\gamma_{t}^{D}(T) \leq \gamma_{t}^{D}\left(T^{\prime}\right)+\ell-s-1=(3(n-\ell+s)+3 s) / 5+\ell-s-1<$ $(3 n+2 \ell+s) / 5$, a contradiction. Thus $\ell=s$, and so $\gamma_{t}^{D}(T)=(3 n+3 l) / 5$. Now the result follows from Theorem 2. The converse is obvious.

We note that the bound of Theorem 9 is an improvement of the bound of Theorem 2 for trees $T$ with $\ell>s$.

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