Discussiones Mathematicae
Graph Theory 38 (2018) 455–462
doi:10.7151/dmgt.2012

BOUNDS ON THE LOCATING-DOMINATION NUMBER AND DIFFERENTIATING-TOTAL DOMINATION NUMBER IN TREES

Nader Jafari Rad and Hadi Rahbani

Department of Mathematics
Shahrood University of Technology
Shahrood, Iran

e-mail: n.jafarirad@gmail.com

Abstract

A subset S of vertices in a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S, and is a total dominating set if every vertex in V has a neighbor in S. A dominating set S is a *locating-dominating set* of G if every two vertices $x, y \in V - S$ satisfy $N(x) \cap S \neq N(y) \cap S$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set of G. A total dominating set S is called a *differentiating-total dominating set* if for every pair of distinct vertices u and v of G, $N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a differentiating-total dominating set of G is the *differentiating-total domination number* of G, denoted by $\gamma_t^D(G)$. We obtain new upper bounds for the locating-domination number, and the differentiating-total domination number in trees. Moreover, we characterize all trees achieving equality for the new bounds.

Keywords: locating-dominating set, differentiating-total dominating set, tree.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and graph theory terminology in general we follow [9]. We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] =$

$N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. We denote the set of all support vertices of a tree T by $S(T)$ and the set of leaves by $L(T)$. We always denote $\ell = \ell(T) = |L(T)|$, and $s = s(T) = |S(T)|$. Whenever a tree T' (or $T'', \ldots$) is introduced, we let n', ℓ' (or $n'', \ell'', \ldots$) be its order, and number of leaves, respectively. We denote by ℓ_v the number of leaves adjacent to a support vertex v, and by L_v the set of leaves adjacent to v. We denote a path of order n by P_n (or $P_n : v_1 v_2 \cdots v_n$, where $V(P_n) = \{v_1, \ldots, v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$). The *distance* $d(x, y)$ between two vertices x and y is the length of a shortest path from x to y. The *diameter* $diam(G)$ of a graph G is the maximum distance over all pair of vertices of G. For a rooted tree T and a vertex v, we denote by T_v the sub-rooted tree, rooted at v.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S is a *locating-dominating set* (or just LDS) of G if every two vertices $x, y \in V - S$ satisfy $N(x) \cap S \neq N(y) \cap S$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set of G. A locating-dominating set of G of cardinality $\gamma_L(G)$ is referred as a $\gamma_L(G)$-set. The concept of locating domination in graphs was pioneered by Slater [15, 16], and has been further studied in, for example, [1, 2, 4, 7, 13, 14].

A dominating set S is a *total dominating set* (or just TDS) of G if the induced subgraph $G[S]$ has no isolated vertex. A total dominating set S is called a *differentiating-total dominating set* (or just DTDS) if for every pair of distinct vertices u and v of G, $N[u] \cap S \neq N[v] \cap S$. The minimum cardinality of a differentiating-total dominating set of G is the *differentiating-total domination number* of G, denoted by $\gamma_t^D(G)$. The concept of differentiating-total domination was introduced by Haynes, Henning and Howard [8] and further studied in, for example, [3, 5, 6, 10, 11, 12].

Blidia *et al.* [1] obtained the following upper bound for the locating-domination number of a tree.

Theorem 1 (Blidia *et al.* [1]). *For any tree T of order $n \geq 2$, with ℓ leaves and s support vertices, $\gamma_L(T) \leq (n + \ell - s)/2$.*

Ning *et al.* [12] constructed the following family $\mathcal{F}$ of trees as follows. For each tree $T \in \mathcal{F}$, every vertex v in T has a label $s(v) \in \{A, B, C, D\}$, called its status. Let $\mathcal{F}$ be the family of labeled trees $T = T_k$ that can be obtained as follows. Let $T_0 = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ be a path of order 8 in which $s(x_1) = s(x_8) = C$, $s(x_2) = s(x_7) = A$, $s(x_3) = s(x_6) = B$ and $s(x_4) = s(x_5) = D$. If $k \geq 1$, then T_k can be obtained from T_{k-1} by one of the following operations.

Operation ϕ_1. For $x \in V(T_{k-1})$, if $s(x) = C$ and $\deg_{T_{k-1}}(x) = 1$, then add a path $Q = yzuvw$ and the edge xy. Let $s(y) = D$, $s(z) = D$, $s(u) = B$, $s(v) = A$ and $s(w) = C$.

Operation ϕ_2. For $x \in V(T_{k-1})$, if $s(x) = D$, then add a path $Q = yzuv$ and the edge xy. Let $s(y) = D$, $s(z) = B$, $s(u) = A$ and $s(v) = C$.

Theorem 2 (Ning *et al.* [12]). *If T is a tree of order $n \geq 3$ with ℓ leaves, then $\gamma_t^D(T) \leq 3(n+\ell)/5$, with equality if and only if $T = P_3$, or $T \in \mathcal{F}$.*

In Section 2, we prove that for any tree T of order $n \geq 2$, with ℓ leaves, $\gamma_L(T) \leq (2n + 3\ell - 2)/5$, and characterize all trees achieving equality for this bound. We note that our bound is an improvement of the bound of Theorem 1 for trees T with $n > \ell + 5s - 4$. In Section 3, we prove that for any tree T of order $n \geq 4$ with ℓ leaves and s support vertices, $\gamma_t^D(T) \leq (3n + 2\ell + s)/5$, and characterize all trees achieving equality for this bound. We note that our bound is an improvement of the bound of Theorem 2 for trees T with $\ell > s$. We make use of the following.

Lemma 3 (Blidia *et al.* [1]). *In any tree T of order $n \geq 3$ there is a $\gamma_L(T)$-set S such that:*

(1) *If x is a support vertex and ℓ_x is the number of leaves adjacent to x, then S contains x and exactly $\ell_x - 1$ leaves adjacent to x.*
(2) *If $abcd$ is a path with $\deg(a) = 1$, $\deg(b) = \deg(c) = 2$ and $\deg(d) > 1$, then $S \cap \{a, b, c, d\} = \{b, d\}$.*

2. An Upper Bound for $\gamma_L(T)$

We begin with the following observation.

Observation 4. *If T is a tree obtained from a tree T' by adding a leaf or a path P_2 to T', then $\gamma_L(T) \leq \gamma_L(T') + 1$.*

Let $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ $(k \geq 1)$ of trees, where $T_1 = P_3$, and T_{i+1} can be obtained recursively from T_i by one of the following operations for $1 \leq i \leq k - 1$.

Operation $\mathcal{O}_1$. Assume that w is a support vertex of T_i. Then T_{i+1} is obtained from T_i by adding a leaf to w.

Operation $\mathcal{O}_2$. Assume that w is a leaf of T_i. Then T_{i+1} is obtained from T_i by adding a path P_5 and joining w to a leaf of P_5.

Lemma 5. *If $\gamma_L(T_i) = (2n(T_i) + 3\ell(T_i) - 2)/5$, and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$ or Operation $\mathcal{O}_2$, then $\gamma_L(T_{i+1}) = (2n(T_{i+1}) + 3\ell(T_{i+1}) - 2)/5$.*

Proof. Let $\gamma_L(T_i) = \frac{2n_i+3\ell_i-2}{5}$, where $n_i = n(T_i)$ and $\ell_i = \ell(T_i)$. Assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$. Let T_{i+1} be obtained from T_i by adding a leaf v to a support vertex w of T_i. By Observation 4, $\gamma_L(T_{i+1}) \leq \gamma_T(T_i) + 1$. Let S be a $\gamma_L(T_{i+1})$-set satisfying the conditions of Lemma 3. Thus $w \in S$, and without loss of generality, we may assume that $v \in S$. Then $S - \{v\}$ is an LDS for T_i, implying that $\gamma_L(T_i) \leq \gamma_L(T_{i+1}) - 1$. Thus $\gamma_L(T_{i+1}) = \gamma_T(T_i) + 1$. Now $\gamma_L(T_{i+1}) = (2n(T_i) + 3\ell(T_i) - 2)/5 + 1 = (2(n(T_i) + 1) + 3(\ell(T_i) + 1) - 2)/5 = (2n(T_{i+1}) + 3\ell(T_{i+1}) - 2)/5$.

Next assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$. Let T_{i+1} be obtained from T_i by joining a leaf v of T_i to the leaf a of a path $P_5 : abcde$. If S is a $\gamma_L(T_i)$-set, then $S \cup \{b, d\}$ is an LDS for T_{i+1}, and so $\gamma_L(T_{i+1}) \leq \gamma_L(T_i) + 2$. Let D be a $\gamma_L(T_{i+1})$-set satisfying the conditions of Lemma 3. Thus $S \cap \{b, c, d, e\} = \{b, d\}$. If $a \notin D$, then $D - \{b, d\}$ is an LDS for T_i, and if $a \in D$, then $(D - \{a, b, d\}) \cup \{v\}$ is an LDS for T_i, and so $\gamma_L(T_i) \leq \gamma_L(T_{i+1}) - 2$. Thus $\gamma_L(T_{i+1}) = \gamma_T(T_i) + 2$. Now $\gamma_L(T_{i+1}) = (2n(T_i) + 3\ell(T_i) - 2)/5 + 2 = (2(n(T_i) + 5) + 3\ell(T_i) - 2)/5 = (2n(T_{i+1}) + 3\ell(T_{i+1}) - 2)/5$. ■

By a simple induction on the operations performed to construct a tree $T \in \mathcal{T}$, and Lemma 5 we obtain the following.

Lemma 6. *For any tree $T \in \mathcal{T}$ of order $n \geq 3$ and with ℓ leaves, $\gamma_L(T) = (2n + 3\ell - 2)/5$.*

We are now ready to present the main result of this section.

Theorem 7. *For any tree T of order $n \geq 2$ with ℓ leaves, $\gamma_L(T) \leq (2n+3\ell-2)/5$, with equality if and only if $T \in \mathcal{T}$.*

Proof. We first use an induction on the order n of T to show that $\gamma_L(T) \leq (2n + 3\ell - 2)/5$. The base step is obvious for $n = 2$ and 3. Assume that for any nontrivial tree T' of order $n' < n$, with l' leaves, $\gamma_L(T') \leq (2n' + 3\ell' - 2)/5$. Now consider the tree T of order $n \geq 4$. Assume that T has a strong support vertex. Let v be a strong support vertex, and u be a leaf adjacent to v. Let $T' = T - u$. By Observation 4, $\gamma_L(T) \leq \gamma_L(T') + 1$. By the inductive hypothesis, $\gamma_L(T) \leq \gamma_L(T')+1 \leq (2n(T')+3\ell(T')-2)/5+1 = (2(n-1)+3(\ell-1)-2)/5+1 = (2n + 3\ell - 2)/5$. Next assume that T has an edge $e = uv$ with $\deg(u) \geq 3$ and $\deg(v) \geq 3$. Let T_1 and T_2 be the components of $T - e$, with $u \in V(T_1)$ and $v \in V(T_2)$. By the inductive hypothesis, $\gamma_L(T) \leq \gamma_L(T_1) + \gamma_L(T_2) \leq (2n(T_1) + 3\ell(T_1) - 2)/5 + (2n(T_2) + 3\ell(T_2) - 2)/5 = (2n + 3\ell - 4)/5 < (2n + 3\ell - 2)/5$. Thus for the next, we may assume that the following facts hold.

Fact 1. T has no strong support vertex.

Fact 2. For each edge $e = uv$, $\deg(u) \leq 2$ or $\deg(v) \leq 2$.

Let $d = diam(T)$. By Fact 1, $d \geq 3$. If $d = 3$, then $T = P_4$, and $\gamma_L(T) = 2 < (2n + 3\ell - 2)/5$. Thus $d \geq 4$. We root T at a leaf x_0 of a diameterical path $x_0x_1 \cdots x_d$ from x_0 to a leaf x_d farthest from x_0. By Fact 1, $\deg(x_{d-1}) = \deg(x_1) = 2$. Assume that $d = 4$. If $\deg(x_2) = 2$ then $T = P_5$, and $\gamma_L(T) = 2 < (2n + 3\ell - 2)/5$. Thus assume that $\deg(x_2) > 2$. If x_2 is a support vertex, then T has $\deg(x_2) - 1$ support vertices of degree two. Then $N(x_2)$ is a LDS for T, implying that $\gamma_L(T) \leq \deg(x_2) < (2n + 3\ell - 2)/5$, since $n = 2\deg(x_2)$ and $\ell = \deg(x_2)$. Thus assume that x_2 is not a support vertex. Then T has $\deg(x_2)$ support vertices of degree two, and we can see that $N(x_2)$ is an LDS for T, implying that $\gamma_L(T) \leq \deg(x_2) < (2n+3\ell-2)/5$, since $n = 2\deg(x_2)+1$ and $\ell = \deg(x_2)$. Assume that $d = 5$. By Fact 2, we may assume that $\deg(x_3) = 2$. If $\deg(x_2) = 2$ then $T = P_6$, and $\gamma_L(T) = 3 < (2n + 3\ell - 2)/5$. Thus assume that $\deg(x_2) > 2$. Since $d = 5$, by Fact 1, any vertex of $N(x_2) - \{x_3\}$ is a leaf or a support vertex of degree two. Assume that x_2 is a support vertex. By Fact 1, there is a unique leaf adjacent to x_2. Then $S(T)$ is an LDS for T, implying that $\gamma_L(T) \leq |S(T)| = \deg(x_2) < (2n + 3\ell - 2)/5$, since $n = 2\deg(x_2) + 1$ and $\ell = \deg(x_2)$. Thus assume that x_2 is not a support vertex. Then $S(T) \cup \{x_2\}$ is an LDS for T, implying that $\gamma_L(T) \leq |S(T)|+1 = \deg(x_2)+1 < (2n+3\ell-2)/5$, since $n = 2\deg(x_2) + 2$ and $\ell = \deg(x_2)$. Thus assume that $d \geq 6$. Assume that $\deg(x_{d-2}) \geq 3$. Let $T' = T - \{x_d, x_{d-1}\}$. Using Observation 4 and the inductive hypothesis, we obtain $\gamma_L(T) \leq \gamma_L(T') + 1 \leq \frac{2(n-2)+3(\ell-1)-2}{5} + 1 < \frac{2n+3\ell-2}{5}$. We thus assume that $\deg(x_{d-2}) = 2$.

Assume that $\deg(x_{d-3}) \geq 3$. Suppose that x_{d-3} is a support vertex and u is the unique leaf adjacent to x_{d-3}. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}\}$. Note that x_{d-3} is a support vertex in T', and by Lemma 3 there is a $\gamma_L(T')$-set D containing x_{d-3}. Then $D \cup \{x_{d-1}\}$ is an LDS for T, implying that $\gamma_L(T) \leq \gamma_L(T') + 1$. By the inductive hypothesis, $\gamma_L(T) \leq \gamma_L(T') + 1 \leq \frac{2(n-3)+3(\ell-1)-2}{5} + 1 < \frac{2n+3\ell-2}{5}$. Thus assume that x_{d-3} is not a support vertex. Assume that there is a leaf y of $T_{x_{d-3}}$ with $d(y, x_{d-3}) = 2$. Let u be the father of y. By Fact 2, $\deg(u) = 2$. Let $T' = T - \{u, y\}$. By Observation 4 and the inductive hypothesis, $\gamma_L(T) \leq \gamma_L(T') + 1 \leq (2(n-2) + 3(\ell-1) - 2)/5 + 1 < (2n + 3\ell - 2)/5$. Thus assume for the next that any leaf of $T_{x_{d-3}}$ is at distance three from x_{d-3}. Since any such leaf plays the same role as x_d, any internal vertex in the shortest path from such leaf to x_{d-3} has degree two. Let $\deg(x_{d-3}) = k+1$ with $k \geq 2$. By Fact 2, $\deg(x_{d-4}) = 2$. Let $T' = T - T_{x_{d-4}}$. By the inductive hypothesis, $\gamma_L(T') \leq (2n' + 3\ell' - 2')/5$. But $l' \leq l - k + 1$, and $n' = n - 3k - 2$. Let S be a $\gamma_L(T')$-set. Then $S \cup \{x_{d-3}\} \cup U$ is an LDS for T, where U is the set of vertices of $T_{x_{d-4}}$ at distance two from x_{d-3}. Thus $\gamma_L(T) \leq \gamma_L(T') + k + 1 \leq (2n' + 3\ell' - 2)/5 + k + 1 \leq (2n + 3\ell - 2 - 4k + 4)/5 < (2n + 3\ell - 2)/5$ because $k \geq 2$. Thus assume that $\deg(x_{d-3}) = 2$.

Assume that $\deg(x_{d-4}) \geq 3$. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}$. Let D be $\gamma_L(T')$-set. Then $D \cup \{x_{d-2}, x_{d-1}\}$ is an LDS for T, and so $\gamma_L(T) \leq \gamma_L(T')$

$+\ 2$. By the inductive hypothesis, $\gamma_L(T) \leq \gamma_L(T') + 2 \leq \frac{2n(T')+3\ell(T')-2}{5} + 2 \leq \frac{2(n-4)+3(\ell-1)-2}{5} + 2 < \frac{2n+3\ell-2}{5}$. Thus assume that $\deg(x_{d-4}) = 2$.

Let $T' = T - T_{x_{d-4}}$. By the inductive hypothesis, $\gamma_L(T') \leq (2n' + 3\ell' - 2)/5$. Let S be a $\gamma_L(T')$-set. Then $S \cup \{x_{d-3}, x_{d-1}\}$ is an LDS for T. Thus $\gamma_L(T) \leq \gamma_L(T') + 2 \leq (2(n-5) + 3\ell - 2)/5 + 2 = (2n + 3\ell - 2)/5$.

We next prove the equality part. We use an induction on the order n of a tree T with ℓ leaves and $\gamma_L(T) = (2n + 3\ell - 2)/5$ to show that $T \in \mathcal{T}$. The basic step is obvious, since $P_3 \in \mathcal{T}$. Assume that any nontrivial tree T' of order $n' < n$, with ℓ' leaves and $\gamma_L(T') = (2n' + 3\ell' - 2)/5$ belongs to $\mathcal{T}$. Let $n = n(T) \geq 4$. Assume that T has a strong support vertex u, and v is a leaf adjacent to u. Let $T' = T - v$. Using Lemma 3, we can easily see that $\gamma_L(T) = \gamma_L(T') + 1$. Thus $\gamma_L(T') = \gamma_L(T) - 1 = (2n + 3\ell - 2)/5 - 1 = (2(n-1) + 3(\ell - 1) - 2)/5 = (2n(T') + 3\ell(T') - 2)/5$. By the inductive hypothesis, $T' \in \mathcal{T}$. Hence T is obtained from T' by Operation $\mathcal{O}_1$. Thus for the next assume that T has no strong support vertex.

We root T at a leaf x_0 of a diameterical path $x_0x_1 \cdots x_d$ from x_0 to a leaf x_d farthest from x_0. By the first part of the proof, we find that $d \geq 6$, and $\deg(x_{d-2}) = \deg(x_{d-3}) = \deg(x_{d-4}) = 2$. Since $\gamma_L(P_7) = 3 < (2n(P_7) + 3\ell(P_7) - 2)/5$, we have $d \geq 7$. We next show that $\deg(x_{d-5}) = 2$. Assume that $\deg(x_{d-5}) \geq 3$. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}\}$. Let D be a $\gamma_L(T')$-set. Then $D \cup \{x_{d-3}, x_{d-1}\}$ is an LDS for T, and so $\gamma_L(T) \leq \gamma_L(T') + 2$. By the first part of the theorem we have $\gamma_L(T) \leq \gamma_L(T') + 2 \leq (2n(T') + 3\ell(T') - 2)/5 + 2 \leq (2(n-5) + 3(\ell - 1) - 2)/5 + 2 < (2n + 3\ell - 2)/5$, a contradiction. Thus $\deg(x_{d-5}) = 2$. Let $T^* = T - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}\}$. If D^* is a $\gamma_L(T^*)$-set, then $D^* \cup \{x_{d-3}, x_{d-1}\}$ is an LDS for T, and so $\gamma_L(T) \leq \gamma_L(T^*) + 2$. By Lemma 3 there is a $\gamma_L(T)$-set D such that $D \cap \{x_{d-3}, x_{d-2}, x_{d-1}, x_d\} = \{x_{d-3}, x_{d-1}\}$. If $x_{d-4} \in D$, then $(D - \{x_{d-3}, x_{d-1}, x_{d-4}\}) \cup \{x_{d-5}\}$ is an LDS for T^*, and if $x_{d-4} \notin D$, then $D - \{x_{d-3}, x_{d-1}\}$ is an LDS for T^*. Thus $\gamma_L(T^*) \leq \gamma_L(T) - 2$. We deduce that $\gamma_L(T) = \gamma_L(T^*) + 2$. Now $\gamma_L(T^*) = \gamma_L(T) - 2 = (2n + 3\ell - 2)/5 - 2 = (2(n-5) + 3\ell - 2)/5 = (2n(T^*) + 3\ell(T^*) - 2)/5$. By the inductive hypothesis, $T^* \in \mathcal{T}$. Hence T is obtained from T^* by Operation $\mathcal{O}_2$.

The converse follows by Lemma 6. ■

We note that the bound of Theorem 7 is an improvement of the bound of Theorem 1 for trees T with $n > \ell + 5s - 4$.

3. An Upper Bound for $\gamma_t^D(T)$

We prove that for any tree T of order $n \geq 3$ with ℓ leaves and s support vertices, $\gamma_t^D(T) \leq (3n + 2\ell + s)/5$. We begin with the following observation of [12], since for any tree $T \in \mathcal{F}$, $\{v \in V(T) \mid s(v) \in \{A \cup B \cup C\}\}$ is a $\gamma_t^D(T)$-set.

Observation 8 (Ning *et al.* [12]). *Any tree $T \in \mathcal{F}$ has a $\gamma_t^D(T)$-set containing all leaves and all support vertices.*

Theorem 9. *If T is a tree of order $n \geq 4$ with ℓ leaves and s support vertices, then $\gamma_t^D(T) \leq (3n + 2\ell + s)/5$, with equality if and only if $T \in \mathcal{F}$.*

Proof. Let T be a tree of order $n \geq 4$ with ℓ leaves and s support vertices. We prove by induction on the order n of T that $\gamma_t^D(T) \leq (3n + 2\ell + s)/5$. For the base step of the induction, if $n = 4$, then $T \in \{P_4, K_{1,3}\}$, and it is obvious that $\gamma_t^D(T) < (3n + 2\ell + s)/5$.

Assume that for any tree T' of order $n' < n$ with ℓ' leaves and s' support vertices, $\gamma_t^D(T') \leq (3n' + 2\ell' + s')/5$. Now consider the tree T of order $n > 4$ with ℓ leaves and s support vertices. If T has no strong support vertices, then $\ell = s$, and the result follows from Theorem 2. Thus assume that T has some strong support vertex. Let u be a strong support vertex of T, and v be a leaf adjacent to u. Let $T' = T - v$. Then $n' \geq 4$. Clearly $\gamma_t^D(T) \leq \gamma_t^D(T') + 1$. By the inductive hypothesis, $\gamma_t^D(T) \leq \gamma_t^D(T') + 1 \leq (3n(T') + 2\ell(T') + s(T'))/5 + 1 = (3(n-1) + 2(\ell - 1) + s)/5 + 1 = (3n + 2\ell + s)/5$.

Now we prove the equality part. Assume that $\gamma_t^D(T) = (3n + 2\ell + s)/5$. We show that $\ell = s$. Suppose that $\ell > s$. Let T' be a tree obtained from T by removing $\ell_x - 1$ leaves of any support vertex x. Thus $n' = n(T') = n - \ell + s$ and $\ell(T') = s(T') = s(T)$. By the theorem, $\gamma_t^D(T') \leq (3n(T') + 2\ell(T') + s(T'))/5 = (3(n-\ell+s)+3s)/5$. If $\gamma_t^D(T') < (3(n-\ell+s)+3s)/5$, then $\gamma_t^D(T) \leq \gamma_t^D(T')+\ell-s < (3n+2\ell+s)/5$, a contradiction. Thus $\gamma_t^D(T') = (3(n-\ell+s)+3s)/5$. By Theorem 2, $T' \in \mathcal{F}$. By Observation 8, T' has a $\gamma_t^D(T')$-set S containing all leaves and all support vertices. Let $y \in L(T) - L(T')$. Then $(S \cup (L(T) - L(T'))) - \{y\}$ is a DTDS for T, and thus $\gamma_t^D(T) \leq \gamma_t^D(T')+\ell-s-1 = (3(n-\ell+s)+3s)/5+\ell-s-1 < (3n + 2\ell + s)/5$, a contradiction. Thus $\ell = s$, and so $\gamma_t^D(T) = (3n + 3l)/5$. Now the result follows from Theorem 2. The converse is obvious. ∎

We note that the bound of Theorem 9 is an improvement of the bound of Theorem 2 for trees T with $\ell > s$.

References

[1] M. Blidia, M. Chellali, F. Maffray, J. Moncel and A. Semri, *Locating-domination and identifying codes in trees*, Australas. J. Combin. **39** (2007) 219–232.

[2] M. Blidia and W. Dali, *A characterization of locating-domination edge critical graphs*, Australas. J. Combin. **44** (2009) 297–300.

[3] M. Blidia and W. Dali, *A characterization of locating-total domination edge critical graphs*, Discuss. Math. Graph Theory **31** (2011) 197–202. doi:10.7151/dmgt.1538

[4] M. Blidia, F. Favaron and R. Lounes, *Locating-domination, 2-domination and independence in trees*, Australas. J. Combin. **42** (2008) 309–319.

[5] M. Chellali, *On locating and differentiating-total domination in trees*, Discuss. Math. Graph Theory **28** (2008) 383–392.
doi:10.7151/dmgt.1414

[6] X.-G. Chen and M.Y. Sohn, *Bounds on the locating-total domination number of a tree*, Discrete Appl. Math. **159** (2011) 769–773.
doi:10.1016/j.dam.2010.12.025

[7] F. Foucaud, M.A. Henning, C. Löwenstein and T. Sasse, *Locating-dominating sets in twin-free graphs*, Discrete Appl. Math. **200** (2016) 52–58.
doi:10.1016/j.dam.2015.06.038

[8] T.W. Haynes, M.A. Henning and J. Howard, *Locating and total dominating sets in trees*, Discrete Appl. Math. **154** (2006) 1293–1300.
doi:10.1016/j.dam.2006.01.002

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[10] M.A. Henning and N. Jafari Rad, *Locating-total domination in graphs*, Discrete Appl. Math. **160** (2012) 1986–1993.
doi:10.1016/j.dam.2012.04.004

[11] M.A. Henning and C. Löwenstein, *Locating-total domination in claw-free cubic graphs*, Discrete Math. **312** (2012) 3107–3116.
doi:10.1016/j.disc.2012.06.024

[12] W. Ning, M. Lu and J. Guo, *Bounds on the differentiating-total domination number of a tree*, Discrete Appl. Math. **200** (2016) 153–160.
doi:10.1016/j.dam.2015.06.029

[13] J.L. Sewell and P.J. Slater, *A sharp lower bound for locating-dominating sets in trees*, Australas. J. Combin. **60** (2014) 136–149.

[14] S.J. Seo and P.J. Slater, *Open neighborhood locating dominating sets*, Australas. J. Combin. **46** (2010) 109–119.

[15] P.J. Slater, *Dominating and location in acyclic graphs*, Networks **17** (1987) 55–64.
doi:10.1002/net.3230170105

[16] P.J. Slater, *Dominating and reference sets in graphs*, J. Math. Phys. Sci. **22** (1988) 445–455.

Received 7 September 2016
Revised 15 December 2016
Accepted 15 December 2016