# GENERALIZED RAINBOW CONNECTION OF GRAPHS AND THEIR COMPLEMENTS 

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#### Abstract

Let $G$ be an edge-colored connected graph. A path $P$ in $G$ is called $\ell$-rainbow if each subpath of length at most $\ell+1$ is rainbow. The graph $G$ is called $(k, \ell)$-rainbow connected if there is an edge-coloring such that every pair of distinct vertices of $G$ is connected by $k$ pairwise internally vertex-disjoint $\ell$-rainbow paths in $G$. The minimum number of colors needed to make $G(k, \ell)$-rainbow connected is called the $(k, \ell)$-rainbow connection number of $G$ and denoted by $r c_{k, \ell}(G)$. In this paper, we first focus on the $(1,2)$-rainbow connection number of $G$ depending on some constraints of $\bar{G}$. Then, we characterize the graphs of order $n$ with ( 1,2 )-rainbow connection number $n-1$ or $n-2$. Using this result, we investigate the Nordhaus-Gaddum-Type problem of $(1,2)$-rainbow connection number and prove that $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq n+2$ for connected graphs $G$ and $\bar{G}$. The equality holds if and only if $G$ or $\bar{G}$ is isomorphic to a double star.


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## 1. Introduction

All graphs in this paper are finite, undirected, simple and connected. We follow the notation and terminology in the book [3].

When considering the transmission of information between agencies of the government, an immediate question is put forward as follows: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This question can be represented by a graph and studied by means of what is called rainbow colorings introduced by Chartrand et al. in [5]. An edgecoloring of a graph is a mapping from its edge set to the set of natural numbers (colors). A path in an edge-colored graph with no two edges sharing the same color is called a rainbow path. A graph $G$ with an edge-coloring $c$ is said to be rainbow connected if every pair of distinct vertices of $G$ is connected by at least one rainbow path in $G$. The coloring $c$ is called a rainbow coloring of the graph $G$. For a connected graph $G$, the minimum number of colors needed to make $G$ rainbow connected is defined as the rainbow connection number of $G$ and denoted by $r c(G)$. Many researchers have studied problems on rainbow connection. See $[9,12,14]$ for example. For more details we refer to the survey paper [13] and the book [14].

The following question provides a relaxation of this concept: What is the minimum number of passwords or firewalls that allows one or more secure paths between every two agencies such that as we progress from one agency to another along such a path, we are required to change passwords at each step? Inspired by this, Borozan et al. in [2] and Andrews et al. in [1] introduced the concept of proper-path coloring of graphs. Let $G$ be an edge-colored graph. A path $P$ in $G$ is called a proper path if no two adjacent edges of $P$ are colored with the same color. An edge-colored graph $G$ is $k$-proper connected if every pair of distinct vertices $u, v$ of $G$ is connected by $k$ pairwise internally vertex-disjoint proper $(u, v)$-paths in $G$. For a connected graph $G$, the minimum number of colors needed to make $G$ $k$-proper connected is called the $k$-proper connection number of $G$ and denoted by $p c_{k}(G)$. Particularly for $k=1$, we write $p c_{1}(G)$, the proper connection number of $G$, as $p c(G)$ for simplicity. Recently, many results have been obtained on the proper connection number. For details, we refer to the dynamic survey [10].

Relaxing the notion of a rainbow path, the $(k, \ell)$-proper-path coloring was defined in [11] as a generalization of rainbow coloring and proper-path coloring. The notion of $\ell$-rainbow colorings was also independently defined and studied in [4, 6, 7]. A path $P$ in $G$ is called an $\ell$-rainbow path if each subpath of length at most $\ell+1$ is rainbow colored. The graph $G$ is called ( $k, \ell$ )-rainbow connected if there is an edge-coloring $c$ such that every pair of distinct vertices of $G$ is connected by $k$ pairwise internally vertex-disjoint $\ell$-rainbow paths in $G$. This
coloring is called a $(k, \ell)$-rainbow-path coloring of $G$. In addition, if $t$ colors are used, then $c$ is referred to as a ( $k, \ell$ )-rainbow-path $t$-coloring of $G$. For a connected graph $G$, the minimum number of colors needed to make $G(k, \ell)$-rainbow connected is called the ( $k, \ell$ )-rainbow connection number of $G$ and denoted by $r c_{k, \ell}(G)$. Particularly, for $k=1$ and $\ell=2$, there is an edge-coloring using $r c_{1,2}$ colors such that there exists a 2 -rainbow path between each pair of vertices of the graph $G$. Furthermore, if we ensure that every path in $G$ is a 2 -rainbow path, then such an edge-coloring is called a strong edge-coloring. In addition, the strong chromatic index $\chi_{s}^{\prime}(G)$, which was introduced by Fouquet and Jolivet [8], is the minimum number of colors needed in a strong edge-coloring of $G$. Immediately we get that $r c_{1,2}(G) \leq \chi_{s}^{\prime}(G)$. And this inspires us to pay our attention to the $(1,2)$-rainbow connection number of the connected graph $G$, i.e., $r c_{1,2}(G)$.

As an example of this concept, we consider the (2,3)-rainbow connection number of the cycle $C_{12}$. Since $\ell=3$, then each pair of edges with the same color must have at least 3 edges in between. Additionally, there are pairs of vertices at distance greater than 4 , we see that $r c_{2,3}\left(C_{12}\right) \geq 4$. On the other hand, if we color the edges of $C_{12}$ by alternating through the colors like $1,2,3,4,1, \ldots, 4$ in order around the cycle, then it is easy to see that this is a (2,3)-rainbow connected coloring using 4 colors, so $r c_{2,3}\left(C_{12}\right)=4$.

In this paper, we consider the $(k, \ell)$-rainbow connection number of graphs and their complements. This paper is organized as follows. In Section 2, we list some useful results about the $(k, \ell)$-rainbow connection number of a graph. In Section 3, we focus on $r c_{1,2}(G)$ depending on some constraints of $\bar{G}$. In Section 4, we first characterize the graphs of order $n$ with (1,2)-rainbow connection number $n-1$ or $n-2$. Using this result, we give the Nordhaus-Gaddum-Type result for the (1,2)-rainbow connection number, i.e., $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq n+2$ for connected graphs $G$ and $\bar{G}$, and the equality holds if and only if $G$ or $\bar{G}$ is isomorphic to a double star.

## 2. Preliminaries

In this section, we introduce some definitions and present several results which will be used later. Let $G$ be a connected graph. We denote by $n$ the number of its vertices and by $m$ the number of its edges. The distance between two vertices $u$ and $v$ in $G$, denoted by $d(u, v)$, is the length of a shortest path between them in $G$. The eccentricity of a vertex $v$ is $\operatorname{ecc}(v):=\max _{x \in V(G)} d(v, x)$. The radius of $G$ is $\operatorname{rad}(G):=\min _{x \in V(G)} \operatorname{ecc}(x)$. We also write $\sigma_{2}^{\prime}(G)$ as the largest sum of degrees of vertices $x$ and $y$, where $x$ and $y$ are taken over all couples of adjacent vertices in $G$. Additionally, we set $[n]=\{1,2, \ldots, n\}$ for any integer $n \geq 1$.

The following are some results that we will use in our proofs. The first
is a simple observation that the addition of edges cannot increase the rainbow connection number.

Proposition 2.1 [11]. If $G$ is a nontrivial connected graph and $H$ is a connected spanning subgraph of $G, \ell \geq 1$ is an integer, then $r c_{1, \ell}(G) \leq r c_{1, \ell}(H)$. Particularly, $r c_{1, \ell}(G) \leq r c_{1, \ell}(T)$ for every spanning tree $T$ of $G$.

When we focus on trees, the following holds.
Theorem 2.2 [11]. If $T$ is a nontrivial tree, then $r c_{1,2}(T)=\sigma_{2}^{\prime}(T)-1$.
For complete bipartite graphs, the situation is trickier.
Theorem 2.3 [11]. Let $\ell \geq 2$ be an integer and $m \leq n$. Then

$$
r c_{1, \ell}\left(K_{m, n}\right)= \begin{cases}n & \text { if } m=1, \\
2 & \text { if } m \geq 2 \text { and } m \leq n \leq 2^{m} \\
3 & \text { if } \ell=2, m \geq 2 \text { and } n>2^{m} \\
4 & \begin{array}{l}
\text { or } \ell \geq 3, m \geq 2 \text { and } 2^{m}<n \leq 3^{m} \\
4
\end{array} \\
\text { if } \ell \geq 3, m \geq 2 \text { and } n>3^{m} .\end{cases}
$$

For a general 2-connected graph, we gave in [11] an upper bound for the (1,2)-rainbow connection number.

Theorem 2.4 [11]. If a graph $G$ is 2 -connected, then $r c_{1,2}(G) \leq 5$.

## 3. (1,2)-Rainbow Connection Number for the Complement of a Graph

In this section, we investigate the (1,2)-rainbow connection number of $G$ depending on some properties of its complement $\bar{G}$.

Theorem 3.1. If $G$ is a graph with $\operatorname{diam}(\bar{G}) \geq 4$, then $r c_{1,2}(G) \leq 3$.
Proof. We first claim that $G$ must be connected. If not, $\bar{G}$ must contain a spanning complete bipartite graph which implies that $\operatorname{diam}(\bar{G}) \leq 2$, a contradiction. Choose a vertex $x$ with $\operatorname{ecc}_{\bar{G}}(x)=\operatorname{diam}(\bar{G})$. Let $N_{i}(x)=\left\{v: \operatorname{dist}_{\bar{G}}(x, v)=i\right\}$ for $0 \leq i \leq 3$ and $N_{4}(x)=\left\{v: \operatorname{dist}_{\bar{G}}(x, v) \geq 4\right\}$. Obviously $N_{0}(x)=\{x\}$. We write $N_{i}$ (for $0 \leq i \leq 4$ ) instead of $N_{i}(x)$ and $n_{i}$ instead of $\left|N_{i}\right|$ for convenience. It can be deduced that all edges are present in $G$ of the form $u v$, where $u \in N_{1}$ and $v \in N_{3} \cup N_{4}$ or $u \in N_{2}$ and $v \in N_{4}$ (see Figure 1).


Figure 1. The graph $G$ for the proof of Theorem 3.1.
We denote by $N_{i, j}(0 \leq i \neq j \leq 4)$ the edge set between $N_{i}$ and $N_{j}$ in $G$. We distinguish four cases and give in each of the cases a (1,2)-rainbow-path 3 -coloring, respectively. Again we use $f(e)(e \in E(\bar{G}))$ to represent the color assigned to $e$.

Case 1. $n_{4}>1$. We give all edges of $N_{1,3}$ the color 3 , edges of $N_{0,3}$ the color 3, edges of $N_{0,4}$ the color 2, edges of $N_{0,2}$ the color 3, edges of $N_{2,4}$ the color 1. Additionally, color the edges of $N_{1,4}$ such that for $v \in N_{1},\{f(v s): s \in$ $\left.N_{4}\right\}=\{1,2\}$. Then for any $u, v \in N_{1}\left(\right.$ if $\left.n_{1}>1\right)$, there must exist $s_{1}, s_{2} \in N_{4}$ (possibly with $s_{1}=s_{2}$ ) such that $f\left(u s_{1}\right)=1$ and $f\left(v s_{2}\right)=2$. Then one of $u s_{1} v$ or $u s_{1} x s s_{2} v$, where $s \in N_{2}$, is a 2 -rainbow $(u, v)$-path. Other situations can be checked similarly.

Case 2. $n_{4}=1, n_{3}>1$ and $n_{1}=1$. Then we give all edges of $N_{1,3}$ the color 1 , the edge of $N_{1,4}$ the color 3, edges of $N_{0,3}$ the color 1, edges of $N_{0,4}$ the color 2 , edges of $N_{0,2}$ the color 1 and edges of $N_{2,4}$ the color 3. It is easy to verify this is indeed a $(1,2)$-rainbow-path 3 -coloring of $G$.

Case 3. $n_{4}=1, n_{3}>1$ and $n_{1}>1$. Let $G^{\prime}$ be the complete bipartite graph $G^{\prime}=G\left[N_{1} \cup N_{3}\right]$. By Theorem 2.3, we can use at most three colors to make $G^{\prime}$ (1,2)-rainbow connected. Then we give all edges of $N_{1,4}$ the color 1 , edges of $N_{0,3}$ the color 2 , the edge of $N_{0,4}$ the color 3 , edges of $N_{0,2}$ the color 1 and edges of $N_{2,4}$ the color 2 . One can easily check this is a (1,2)-rainbow-path 3 -coloring of $G$ and we omit the details here.

Case 4. $n_{4}=1$ and $n_{3}=1$. Then we give all edges of $N_{1,3}$ the color 1, edges of $N_{1,4}$ the color 1, the edge of $N_{0,3}$ the color 2, the edge of $N_{0,4}$ the color 3, edges of $N_{0,2}$ the color 2 and edges of $N_{2,4}$ the color 1 . We can again verify the correctness easily.

Thus, the proof is completed.
Theorem 3.2. For a graph $G$, if $\bar{G}$ is triangle-free and $\operatorname{diam}(\bar{G})=3$, then $r c_{1,2}(G) \leq 3$.
Proof. As in the proof of Theorem 3.1, it is easy to show that $G$ is connected. Choose a vertex $x$ such that $\operatorname{ecc}_{\bar{G}}(x)=\operatorname{diam}(\bar{G})=3$. In addition, $N_{i}, n_{i}$ and
$N_{i, j}$ for $0 \leq i \neq j \leq 3$ are defined as in the previous theorem. Again it can be deduced that there exist all edges of the form $u v$, where $u \in N_{0}$ and $v \in N_{2} \cup N_{3}$ or where $u \in N_{1}$ and $v \in N_{3}$. Since $\bar{G}$ is triangle-free and $x$ has all edges to $N_{1}$ in $\bar{G}$, we know that $N_{1}$ is a clique in $G$. We give a $(1,2)$-rainbow-path 3 -coloring for $G$ as follows.

We assign to the edges of $N_{0,2}$ the color 3 , edges of $N_{0,3}$ the color 1 , edges of $N_{1,3}$ the color 2, any edges of $N_{1,2}$ the color 3, any edges of $N_{2,3}$ the color 2 and the edges of the induced subgraph $G\left[N_{1}\right]$ the color 3.

It is obvious that for any $u \in N_{i}$ and $v \in N_{j}(i \neq j)$, there exists a 2 -rainbow path between them. Then it suffices to show that for any $u, v \in N_{2}$ or $N_{3}$, there is a 2 -rainbow path connecting them in $G$. First suppose $u, v \in N_{2}$ and there is no edge between them in $G$. Since $\bar{G}$ is triangle-free, there exists a vertex $w \in N_{1}$ such that $w v \in G$, then $u x t w v$ is a 2-rainbow path between $u$ and $v$, where $t \in N_{3}$. The situation for any vertices $u, v \in N_{3}$ can be dealt with similarly. Thus $r c_{1,2}(G) \leq 3$.

Theorem 3.3. Let $G$ be a connected graph. If $\bar{G}$ is triangle free and $\operatorname{diam}(\bar{G})$ $=2$, then $r c_{1,2}(G) \leq 3$.

Proof. First we choose a vertex $x$ with $\operatorname{ecc}_{\bar{G}}(x)=\operatorname{diam}(\bar{G})=2$. In addition, $N_{i}, n_{i}$ and $N_{i, j}$ are defined as above. Clearly, all edges of the form $x v$ for $v \in N_{2}$ are present in $G$. Again $N_{1}$ is a clique in $G$, since all edges of the form $x u$ are in $\bar{G}$ for $u \in N_{1}$ and $\bar{G}$ is triangle free.

Suppose there exists a vertex $v_{0} \in N_{2}$ such that no edge $v w\left(w \in N_{1}\right)$ exists in $G$. Then $v_{0}$ is adjacent to every vertex of $N_{1}$ in $\bar{G}$. Thus, since every vertex of $N_{2}$ has at least one edge to $N_{1}$ in $\bar{G}$, the vertex $v_{0}$ must be adjacent to every other vertex of $N_{2}$ in $G$, since otherwise a triangle will appear in $\bar{G}$. Next we give an edge coloring $f$ for $G$. We set $f\left(x v_{0}\right)=3, f(x w)=2$ and $f\left(v_{0} w\right)=1$ ( $w \in N_{2}, w \neq v_{0}$ ), and we give any edges of $N_{1,2}$ the color 2, the edges of the induced subgraph $G\left[N_{1}\right]$ the color 3 . We only need to consider the 2 -rainbow path for $w_{1}, w_{2} \in N_{2}$ and $w_{1} v_{0} x w_{2}$ clearly suffices.

Next suppose there exists no such vertex $v_{0}$. Since $G$ and $\bar{G}$ are connected, we know that $n_{1} \geq 2$. We denote by $E_{G}(v)$ (for $v \in N_{2}$ ) the set of edges between $v$ and vertices of $N_{1}$ in $G$ and set $e_{G}(v)=\left|E_{G}(v)\right|$. Also $e_{\bar{G}}(v)$ (for $v \in N_{2}$ ) is defined similarly. Again we distinguish two cases to analyze.

If $\left|N_{1}\right| \geq 3$, then for each $u \in N_{2}$ with $e_{G}(u)=1$, we color this edge incident to $u$ with color 1 . And for $u \in N_{2}$ with $e_{G}(u) \geq 2$, we arbitrarily color these edges but confirm that $\left\{f(e): e \in E_{G}(u)\right\}=\{1,2\}$. Then we set $f(x u)=2\left(u \in N_{2}\right)$ and give the edges of the induced subgraph $G\left[N_{1}\right]$ the color 3 . The rest edges are colored arbitrarily with colors from [3]. Again we only need to consider the 2-rainbow path between the two non-adjacent vertices $v, w \in N_{2}$. Since $\left|N_{1}\right| \geq 3$ and $v$ and $w$ are non-adjacent in $G$, so $e_{\bar{G}}(v)+e_{\bar{G}}(w) \leq\left|N_{1}\right|$. Thus
$e_{G}(v)+e_{G}(w) \geq\left|N_{1}\right| \geq 3$, which implies that one of the vertices $v, w$, say $v$, must have $e_{G}(v) \geq 2$. So there exists one vertex $s \in N_{1}$ or two vertices $s, t \in N_{1}$ such that $v s w$ or $v s t w$ is a 2 -rainbow $(v, w)$-path in $G$.

If $\left|N_{1}\right|=2$ and $N_{1}=\{s, t\}$, then each vertex $u \in N_{2}$ is adjacent to only one vertex of $N_{1}$ in $G$, either $s$ or $t$ since otherwise $\operatorname{diam}(\bar{G}) \geq 3$. We denote by $V_{1}$ the set of vertices of $N_{2}$ adjacent to $s$ in $G$, that is, the set of vertices adjacent to $t$ in $\bar{G}$. And we write $V_{2}$ for the rest of the vertices of $N_{2}$. It is easy to see that $V_{1}$ and $V_{2}$ both induce cliques in $G$. We then set $f(x u)=1\left(u \in V_{1}\right), f(u s)=2\left(u \in V_{1}\right)$, $f(x u)=2\left(u \in V_{2}\right), f(u t)=1\left(u \in V_{2}\right), f(s t)=3$ and color any remaining edges with color 1 . It is easy to check that this is a ( 1,2 -rainbow-path 3 -coloring of $G$. Thus the proof is completed.

## 4. Nordhaus-Gaddum-Type Theorem for (1,2)-Rainbow Connection Number

In this section, we first characterize the graphs on $n$ vertices with (1,2)-rainbow connection number $n-1$ or $n-2$, which is crucial to investigate the Nordhaus-Gaddum-Type result for the $(1,2)$-rainbow connection number of the graph $G$. We use $C_{n}, S_{n}$ to denote the cycle and the star graph on $n$ vertices, respectively. Denote by $T\left(n_{1}, n_{2}\right)$ the double star in which the degrees of its (adjacent) center vertices are $n_{1}+1$ and $n_{2}+1$ respectively. Additionally, we write $T^{1}\left(n_{1}, n_{2}\right)$ as the graph obtained by replacing one pendent edge with $P_{3}$ in the double star $T\left(n_{1}, n_{2}\right)$ and denote the new pendent vertex by $u_{0}$ (see Figure 2). Also define graphs $G_{1}, \ldots, G_{8}$ as in Figure 2.










Figure 2. Graphs $G_{i}(1 \leq i \leq 8)$ and $T^{1}\left(n_{1}, n_{2}\right)$ in $\mathcal{G}_{2}$.
Theorem 4.1. Let $G$ be a nontrivial connected graph on $n \geq 2$ vertices. Then
(i) $r c_{1,2}(G)=n-1$ if and only if $G \in \mathcal{G}_{1}=\left\{S_{n}(n \geq 2), T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)\right\}$;
(ii) $r c_{1,2}(G)=n-2$ if and only if $G \in \mathcal{G}_{2}=\left\{C_{3}, C_{4}, C_{5}, G_{1}, G_{2}, G_{3}, G_{4}, G_{5}, G_{6}\right.$, $\left.G_{7}, G_{8}, T^{1}\left(n_{1}, n_{2}\right)\right\}$.

Proof. Let $G$ be a connected graph of order $n \geq 2$ and $T$ be a spanning tree of $G$. Proposition 2.1 shows that $r c_{1,2}(G) \leq r c_{1,2}(T)$. Now we give proofs for (i) and (ii) separately.

Proof of (i). For any graph $G \in \mathcal{G}_{1}$, we can easily check that $r c_{1,2}(G)=n-1$. So it remains to verify the converse. Since $r c_{1,2}(G)=n-1$, we see that $n-1=$ $r c_{1,2}(G) \leq r c_{1,2}(T) \leq n-1$, i.e., $r c_{1,2}(T)=n-1$. Thus, by Theorem 2.2, we know that any spanning tree $T$ of $G$ must be a star or a double star, i.e., $T \in \mathcal{G}_{1}$. Without loss of generality, we can assume that $n_{2} \geq n_{1}$.


Figure 3. Graphs obtained by adding an edge to $S_{n}(n \geq 2)$.
If $G$ is a tree, then $G \in \mathcal{G}_{1}$. Now we suppose that $G$ is not a tree. Then since $T \in \mathcal{G}_{1}, G$ can be constructed from $S_{n}(n \geq 2)$ or $T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ by adding edges. Adding an edge to $S_{n}(n \geq 2)$, we will obtain one of the graphs depicted in Figure 3. However, all the graphs in Figure 3 have ( 1,2 )-rainbow connection number no more than $n-2$, which implies that any spanning tree $T$ of $G$ cannot be a star. Next, we will consider the graphs obtained by adding edges to $T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$.

If $n_{1}=n_{2}=1$, then $T(1,1)=P_{4}$. If an edge is added, then we will obtain either the cycle $C_{4}$ or the graph $G_{1}$ depicted in Figure 2. Obviously, both $C_{4}$ and $G_{1}$ have ( 1,2 )-rainbow connection number $2=n-2<n-1$. For the cases $n_{1}=1, n_{2}=2$ and $n_{1}=n_{2}=2$, one of the graphs in Figure 4 or 5 will be obtained by adding an edge to $T(1,2)$ or $T(2,2)$, respectively. The ( 1,2 )-rainbow-path colorings given in Figures 4 and 5 show that all these graphs have $(1,2)$-rainbow connection number no more than $n-2$.


Figure 4. Graphs obtained by adding an edge to $T(1,2)$.
For all the other situations, i.e., $n_{1}=1, n_{2} \geq 3$ or $n_{1}=2, n_{2} \geq 3$ or $n_{1} \geq 3$, $n_{2} \geq 3$, Figure 6, Figure 7 and Figure 8 give all the graphs obtained by adding
an edge to $T\left(1, n_{2} \geq 3\right), T\left(2, n_{2} \geq 3\right)$ and $T\left(n_{1} \geq 3, n_{2} \geq 3\right)$, respectively. We give (1,2)-rainbow-path colorings for these graphs showed in Figure 6, Figure 7 and Figure 8. One can easily check that all these graphs have (1,2)-rainbow connection number no more than $n-2$.



Figure 5. Graphs obtained by adding an edge to $T(2,2)$.

From the discussions all above, we come to a conclusion that if $r c_{1,2}(G)=$ $n-1$, then $G \in \mathcal{G}_{1}=\left\{S_{n}(n \geq 2), T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)\right\}$.




Figure 6 . Graphs obtained by adding an edge to $T\left(1, n_{2} \geq 3\right)$.

$r c_{1,2} \leq n-3$

$r c_{1,2} \leq n-3$

$r c_{1,2} \leq n-3$

$r c_{1,2} \leq 4=n-3$

$r c_{1,2} \leq n-3$

Figure 7. Graphs obtained by adding an edge to $T\left(2, n_{2} \geq 3\right)$.


Figure 8. Graphs obtained by adding an edge to $T\left(n_{1} \geq 3, n_{2} \geq 3\right)$.
Proof of (ii). One can easily check that $r c_{1,2}(G)=n-2$ for any graph $G \in \mathcal{G}_{2}$. Hence, it remains to show the converse. Since $\operatorname{rc}_{1,2}(G)=n-2$, $n-2 \leq r c_{1,2}(T) \leq n-1$. Thus, Theorem 2.2 implies that any spanning tree $T$ of $G$ must be an element of the set $\left\{S_{n}(n \geq 2), T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right), T^{1}\left(n_{1}, n_{2}\right)\right.$ $\left.\left(n_{1}, n_{2} \geq 1\right)\right\}$.

If $G$ is a tree, then $G \cong T^{1}\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right) \subseteq \mathcal{G}_{2}$. Next we suppose that $G$ is not a tree. Then $G$ can be constructed from $S_{n}(n \geq 2), T\left(n_{1}, n_{2}\right)$ $\left(n_{1}, n_{2} \geq 1\right)$ or $T^{1}\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ by adding edges. In the proof of (i), we listed eight graphs with (1,2)-rainbow connection number $n-2$, which are $C_{3}, C_{4}, G_{1}, G_{3}, G_{4}, G_{6}, G_{7}$ and $G_{8}$, respectively. Furthermore, all graphs obtained by adding an edge to $S_{n}(n \geq 2)$ or $T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ except these eight ones have (1,2)-rainbow connection number no more than $n-3$. Therefore, the graph $G$ can be constructed from $C_{3}, C_{4}, G_{1}, G_{3}, G_{4}, G_{6}, G_{7}, G_{8}$ or $T^{1}\left(n_{1}, n_{2}\right)$ $\left(n_{1}, n_{2} \geq 1\right)$ by adding edges.


Figure 9 . Graphs obtained by adding an edge to $T^{1}\left(n_{1} \geq 2, n_{2} \geq 2\right)$.

Considering graphs constructed from $C_{3}, C_{4}, G_{1}, G_{3}, G_{4}, G_{6}, G_{7}$ or $G_{8}$ by adding edges, we find only another two graphs $G_{2}, G_{5}$ with $r c_{1,2}\left(G_{2}\right)=2=$ $\left|V\left(G_{2}\right)\right|-2$ and $r c_{1,2}\left(G_{5}\right)=3=\left|V\left(G_{5}\right)\right|-2$. All others have (1,2)-rainbow connection number no more than $n-3$. Now we focus on the graphs obtained by adding an edge to $T^{1}\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$. For the cases $n_{1}=n_{2}=1, n_{1}=1$, $n_{2} \geq 2$ and $n_{1} \geq 2, n_{2}=1$, we find another graph $C_{5}$ such that $r c_{1,2}\left(C_{5}\right)=n-2$ with similar analysis as in the proof of (i). Denote by $e$ the new edge added to $T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ or $T^{1}\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ and $T\left(n_{1}, n_{2}\right)+e, T^{1}\left(n_{1}, n_{2}\right)+e$ the newly obtained graphs. For the case $n_{1} \geq 2, n_{2} \geq 2$, we consider cases depending on whether the pendent vertex $u_{0}$ in $T^{1}\left(n_{1}, n_{2}\right)$ is an end vertex of $e$ or
not. It is obvious that if $u_{0} \notin e$, then $T^{1}\left(n_{1}, n_{2}\right)+e \backslash u_{0} \cong T\left(n_{1}, n_{2}\right)+e$. The proof of (i) suggests that we only need to consider the case when $T^{1}\left(n_{1}, n_{2}\right)+e \backslash u_{0} \cong G_{8}$. It is easy to check that $r c_{1,2}\left(T^{1}\left(n_{1}, n_{2}\right)+e\right)=n-3<n-2$ for this case. If $u_{0} \in e$, then one of the graphs in Figure 9 will be obtained by adding an edge to $T^{1}\left(n_{1}, n_{2}\right)$. However, all these graphs have ( 1,2 )-rainbow connection number no more than $n-3$ (as colored in the figure). Thus, we complete the proof of (ii).

Theorem 4.2. Let $G$ and $\bar{G}$ be connected graphs on $n$ vertices. Then $r_{1,2}(G)+$ $r_{c_{1,2}}(\bar{G}) \leq n+2$ and the equality holds if and only if $G$ or $\bar{G}$ is isomorphic to a double star, i.e., $G \cong T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ or $\bar{G} \cong T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$.

Proof. Since both $G$ and $\bar{G}$ are connected, we have $n \geq 4$ and $\Delta(G), \Delta(\bar{G}) \leq$ $n-2$. Let $G$ be the double star with center vertices $u, v$ and $N_{G}(u) \backslash v=A$, $N_{G}(v) \backslash u=B$. So, $\bar{G}[A \cup B]$ is a clique and $N_{\bar{G}}(u)=B, N_{\bar{G}}(v)=A$. Certainly, all edges of $G$ must have distinct colors so we consider colorings of $\bar{G}$. Color all edges incident to $v$ with 1 , all edges incident to $u$ with 2 and edges in $\bar{G}[A \cup B]$ with 3. This coloring shows that $r c_{1,2}(\bar{G}) \leq 3$. Since $u$ and $v$ are at distance 3 in $\bar{G}$, we get that $r c_{1,2}(\bar{G})=3$ and so $r c_{1,2}(G)+r c_{1,2}(\bar{G})=n+2$. Now, we must show that $r c_{1,2}(G)+r c_{1,2}(\bar{G})<n+2$ for all other connected graphs $G$ and $\bar{G}$. One can easily check that this is true for $n=4,5$. So we consider $n \geq 6$ in the following.

If $G$ or $\bar{G}$ has $(1,2)$-rainbow connection number $n-1$ or $n-2$, i.e., $G \in$ $\mathcal{G}_{1} \cup \mathcal{G}_{2} \backslash T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$ or $\bar{G} \in \mathcal{G}_{1} \cup \mathcal{G}_{2} \backslash T\left(n_{1}, n_{2}\right)\left(n_{1}, n_{2} \geq 1\right)$, then $r c_{1,2}(G)+r c_{1,2}(\bar{G})<n+2$ by simple examination. Hence, we can assume that $2 \leq r c_{1,2}(G) \leq n-3$ and $2 \leq r c_{1,2}(\bar{G}) \leq n-3$.

$H_{1}^{7}$

$H_{2}^{7}$

Figure 10. Graphs for the proof of Theorem 4.2.
Suppose first that both $G$ and $\bar{G}$ are 2 -connected. For $n=6$, it is easy to check that $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq 3+3<8=n+2$. And for $n \geq 9$, Theorem 2.4 implies that $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq 5+5=10<11 \leq n+2$. Then what remains are the cases $n=7$ and $n=8$. For convenience, we denote the circumference of $G$ by $c(G)$. We first suppose $n=7$. Obviously $4 \leq c(G) \leq 7$. If $c(G)=7$, then $C_{7}$ is a spanning subgraph of $G$ and $r c_{1,2}(G) \leq r c_{1,2}\left(C_{7}\right)=3$. If $c(G)=6$, then $G$ has a traceable spanning subgraph which is composed of $C_{6}$ by adding an open ear of length two. Thus, $r_{1,2}(G) \leq 3$. If $c(G)=5$, then $G$ contains $H_{1}^{7}$ or $H_{2}^{7}$
(see Figure 10) as a spanning subgraph. Since $H_{1}^{7}$ is traceable and $r c_{1,2}\left(H_{2}^{7}\right) \leq 3$, we have $r c_{1,2}(G) \leq 3$. For the case $c(G)=4, G$ contains $K_{2,5}$ as its spanning subgraph, which contradicts the assumption that $\bar{G}$ is connected. Therefore, all 2 -connected graphs of order $n=7$ with connected complementary graphs has $(1,2)$-rainbow connection number no more than 3 . Hence, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq$ $3+3<9=n+2$. With similar analysis as for the situation $n=7$, we can also draw the conclusion that $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq 3+3<10=n+2$ for $n=8$.

Now we consider the case where at least one of $G$ and $\bar{G}$ has at least one cut vertex. Without loss of generality, suppose that $G$ has at least one cut vertex. We distinguish the following two cases.

Case 1. $G$ has a cut vertex $u$ such that $G-u$ has at least three components. Let $G_{1}, G_{2}, \ldots, G_{k}(k \geq 3)$ be the components of $G-u$, and let $n_{i}$ be the number of vertices of $G_{i}$ for $i=1,2, \ldots, k$ with $n_{1} \leq n_{2} \leq \cdots \leq n_{k}$. Since $\Delta(G) \leq n-2$, $n_{k} \geq 2$. The complementary graph $\bar{G} \backslash u$ contains $K_{n_{k}, n-n_{k}-1}$ as a spanning subgraph and both $n_{k} \geq 2$ and $n-n_{k}-1 \geq 2$. By Theorem 2.3, there exists a (1,2)-rainbow-path 3-coloring of $K_{n_{k}, n-n_{k}-1}$ using elements in [3]. Then, if we color the edges incident to $u$ in $\bar{G}$ with color 4 , then we obtain a (1,2)-rainbowpath 4-coloring of $\bar{G}$. Therefore, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq(n-3)+4=n+1<n+2$.

Case 2. Each cut vertex $u$ of $G$ satisfies that $G-u$ has only two components.
Let $G_{1}, G_{2}$ be the two components of $G-u$, and let $n_{i}$ be the number of vertices of $G_{i}$ for $i=1,2$ with $n_{1} \leq n_{2}$. Since $n \geq 6$, we have $n_{2} \geq 2$.

Subcase 2.1. $n_{1} \geq 2$. The complementary graph $\bar{G} \backslash u$ contains $K_{n_{1}, n_{2}}$ as a spanning subgraph. By Theorem 2.3, there is a coloring of $K_{n_{1}, n_{2}}$ with colors in [3], and we color the edges incident to $u$ in $\bar{G}$ with color 4 . This gives a (1,2)-rainbow-path 4-coloring of $\bar{G}$. As a result, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq n-3+4=$ $n+1<n+2$ as desired.

Subcase 2.2. $n_{1}=1$, i.e., each cut vertex of $G$ is incident with a pendent edge.

Since $n \geq 6$, we have $n_{2} \geq 4$. Let $\left\{u_{1}, u_{2}, \ldots, u_{\ell}\right\}$ be the set of all cut vertices of $G$, and let $u_{1} v_{1}, u_{2} v_{2}, \ldots, u_{\ell} v_{\ell}$ be the pendent edges incident to these cut vertices in $G$. Set $H=G \backslash\left\{v_{1}, v_{2}, \ldots, v_{\ell}\right\}$, so $H$ is 2-connected. By Theorem 2.4, we know that $r c_{1,2}(H) \leq 5$.

If $\ell \geq 2$, then $\bar{G} \backslash\left\{u_{1}, u_{2}\right\}$ contains $K_{2, n-4}$ as a spanning subgraph. By Theorem 2.3, there is a coloring of $K_{2, n-4}$ using colors from [3], and we color the edges incident to $u_{1}$ or $u_{2}$ in $\bar{G}$ with color 4 . One can easily check this is a (1,2)-rainbow-path 4-coloring of $\bar{G}$. Thus, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq(n-3)+4=$ $n+1<n+2$.

Thus, we may assume $\ell=1$, so $r c_{1,2}(G) \leq r c_{1,2}(H)+1 \leq 6$. Since $\bar{G}$ is connected, $\left|N_{\bar{G}}\left(u_{1}\right)\right| \geq 1$ and $\bar{G}$ contains $G^{1}, G^{2}$ or $G^{3}$ (see Figure 11) as a spanning subgraph. We first suppose that $G^{1}$ is a spanning subgraph of $\bar{G}$.

Let $H_{1}, \ldots, H_{5}$ be as in Figure 12. If $\bar{G} \cong H_{1}$, then it is easy to verify that $r c_{1,2}(G)+r c_{1,2}(\bar{G})=3+3=6<8=n+2$ for $n=6$ and $r c_{1,2}(G)+r c_{1,2}(\bar{G})=$ $4+3=7<9=n+2$ for $n=7$. If $\bar{G} \cong H_{1}$ and $n \geq 8$, the coloring depicted in Figure 12 shows that $r c_{1,2}(\bar{G}) \leq n-4$. In addition, if we color $u_{1} v_{1}$ with color 1 , other edges incident to $u_{1}$ with color 2 and all other edges color 3 in $G$, then we get a (1,2)-rainbow-path 3-coloring of $G$. Consequently, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq$ $(n-4)+3=n-1<n+2$. Next we consider the situation $H_{1} \varsubsetneqq \bar{G}$. Adding an edge to $G^{1}$, we arrive at some graph in $\left\{H_{2}, H_{3}, H_{4}, H_{5}\right\}$ depicted in Figure 12.

$G^{1}$

$G^{2}$

$G^{3}$

Figure 11. Graphs for the proof of Theorem 4.2.


Figure 12. Graphs for the proof of Theorem 4.2.

If $\bar{G} \cong H_{5}$, then $r c_{1,2}(\bar{G}) \leq n-4$ by the coloring in Figure 12. In order to color $G$, we color $u_{1} v_{1}$ with color 1 and other edges incident to $u_{1}$ with color 2 . Additionally, we color edges incident to $x$ ( $y$ is the same) with colors 1,3 such that both 1 and 3 appear and all other edges with color 2 in $G$. Thus, we get a $(1,2)$-rainbow-path 3 -coloring of $G$ and so $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq 3+(n-4)=$ $n-1<n+2$. If $\bar{G}$ is not isomorphic to $H_{5}$, then $\bar{G}$ has $H_{2}, H_{3}$ or $H_{4}$ as its spanning subgraph. As is depicted in Figure 12, $r c_{1,2}\left(H_{i}\right) \leq n-5(2 \leq i \leq 4)$ for $n \geq 9$. Therefore, $r c_{1,2}(G)+r c_{1,2}(\bar{G}) \leq 6+(n-5)=n+1<n+2$ for $n \geq 9$. For the situation $6 \leq n \leq 8$, we can verify the result depending on the circumference of $H=G \backslash u_{1}$ similarly as above. Hence, if $G^{1}$ is a spanning subgraph of $G$, then $r c_{1,2}(G)+r c_{1,2}(\bar{G})<n+2$. By the same method, we can draw the same conclusion for $G^{2}$ or $G^{3}$ as a spanning subgraph of $G$. Therefore, we complete the proof.

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