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MATCHINGS EXTEND TO HAMILTONIAN CYCLES IN 5-CUBE [1]

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Abstract

Ruskey and Savage asked the following question: Does every matching in a hypercube Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Fink confirmed that every perfect matching can be extended to a Hamiltonian cycle of Q_n, thus solved Kreweras' conjecture. Also, Fink pointed out that every matching can be extended to a Hamiltonian cycle of Q_n for $n \in \{2, 3, 4\}$. In this paper, we prove that every matching in Q_5 can be extended to a Hamiltonian cycle of Q_5.

Keywords: hypercube, Hamiltonian cycle, matching.

2010 Mathematics Subject Classification: 05C38, 05C45.

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1. Introduction

Let $[n]$ denote the set $\{1, \ldots, n\}$. The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary strings of length n, i.e., $V(Q_n) = \{u : u = u^1 \cdots u^n \text{ and } u^i \in \{0, 1\} \text{ for every } i \in [n]\}$, with two vertices being adjacent whenever the corresponding strings differ in just one position.

The hypercube Q_n is one of the most popular and efficient interconnection networks. It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement dates back to 1872 [9]. Since then, the research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention [2, 3, 4, 6, 12].

A set of edges in a graph G is called a *matching* if no two edges have an endpoint in common. A matching is *perfect* if it covers all vertices of G. A cycle in a graph G is a *Hamiltonian cycle* if every vertex in G appears exactly once in the cycle.

Ruskey and Savage [11] asked the following question: Does every matching in Q_n for $n \geq 2$ extend to a Hamiltonian cycle of Q_n? Kreweras [10] conjectured that every perfect matching of Q_n for $n \geq 2$ can be extended to a Hamiltonian cycle of Q_n. Fink [5, 7] confirmed the conjecture to be true. Let $K(Q_n)$ be the complete graph on the vertices of the hypercube Q_n.

Theorem 1.1 [5, 7]**.** *For every perfect matching M of $K(Q_n)$, there exists a perfect matching F of Q_n, $n \geq 2$, such that $M \cup F$ forms a Hamiltonian cycle of $K(Q_n)$.*

Also, Fink [5] pointed out that the following conclusion holds.

Lemma 1.2 [5]**.** *Every matching in Q_n can be extended to a Hamiltonian cycle of Q_n for $n \in \{2, 3, 4\}$.*

Gregor [8] strengthened Fink's result and obtained that given a partition of the hypercube into subcubes of nonzero dimensions, every perfect matching of the hypercube can be extended on these subcubes to a Hamiltonian cycle if and only if the perfect matching interconnects these subcubes.

The present authors [14] proved that every matching of at most $3n-10$ edges in Q_n can be extended to a Hamiltonian cycle of Q_n for $n \geq 4$.

In this paper, we consider Ruskey and Savage's question and obtain the following result.

Theorem 1.3. *Every matching in Q_5 can be extended to a Hamiltonian cycle of Q_5.*

It is worth mentioning that Ruskey and Savage's question has been recently done for $n = 5$ independently by a computer search [15]. In spite of this, a direct proof is still necessary, as it may serve in a possible solution of the general question.

2. PRELIMINARIES AND LEMMAS

Terminology and notation used in this paper but undefined below can be found in [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a set $F \subseteq E(G)$, let $G - F$ denote the resulting graph after removing all edges in F from G. Let H and H' be two subgraphs of G. We use $H + H'$ to denote the graph with the vertex set $V(H) \cup V(H')$ and edge set $E(H) \cup E(H')$. For $F \subseteq E(G)$, we use $H + F$ to denote the graph with the vertex set $V(H) \cup V(F)$ and edge set $E(H) \cup F$, where $V(F)$ denotes the set of vertices incident with F.

The *distance* between two vertices u and v is the number of edges in a shortest path joining u and v in G, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear.

Let $j \in [n]$. An edge in Q_n is called an *j-edge* if its endpoints differ in the jth position. The set of all j-edges in Q_n is denoted by E_j. Thus, $E(Q_n) = \bigcup_{i=1}^{n} E_i$. Let $Q^0_{n-1,j}$ and $Q^1_{n-1,j}$, with the superscripts j being omitted when the context is clear, be the $(n-1)$-dimensional subcubes of Q_n induced by the vertex sets $\{u \in V(Q_n) : u^j = 0\}$ and $\{u \in V(Q_n) : u^j = 1\}$, respectively. Thus, $Q_n - E_j = Q^0_{n-1} + Q^1_{n-1}$. We say that Q_n *splits* into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} at position j; see Figure 1 for example.

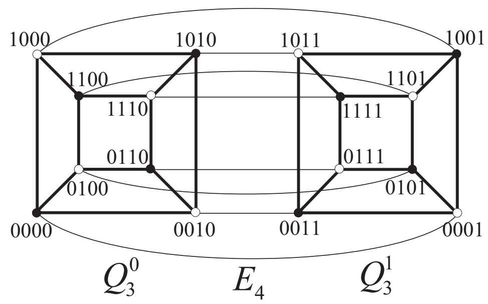

Figure 1. Q_4 splits into two 3-dimensional subcubes Q^0_3 and Q^1_3 at position 4.

The *parity* $p(u)$ of a vertex u in Q_n is defined by $p(u) = \sum_{i=1}^{n} u^i \pmod 2$. Then there are 2^{n-1} vertices with parity 0 and 2^{n-1} vertices with parity 1 in Q_n. Vertices with parity 0 and 1 are called black vertices and white vertices, respectively. Observe that Q_n is bipartite and vertices of each parity form bipartite sets of Q_n. Thus, $p(u) \neq p(v)$ if and only if $d(u, v)$ is odd.

A *u, v-path* is a path with endpoints u and v, denoted by P_{uv} when we specify a particular such path. We say that a spanning subgraph of G whose components are k disjoint paths is a *spanning k-path* of G. A spanning 1-path thus is simply a spanning or Hamiltonian path. We say that a path P (respectively, a cycle C) passes through a set M of edges if $M \subseteq E(P)$ (respectively, $M \subseteq E(C)$).

Lemma 2.1 [13]**.** *Let u, v, x, y be pairwise distinct vertices in Q_3 with $p(u) = p(v) \neq p(x) = p(y)$ and $d(u, x) = d(v, y) = 1$. If M is a matching in $Q_3 - \{u, v\}$, then there exists a spanning 2-path $P_{ux} + P_{vy}$ in Q_3 passing through M.*

Lemma 2.2 [13]**.** *For $n \in \{3, 4\}$, let $u, v \in V(Q_n)$ be such that $p(u) \neq p(v)$. If M be a matching in $Q_n - u$, then there exists a Hamiltonian path in Q_n joining u and v passing through M.*

3. Proof of Theorem 1.3

Let M be a matching in Q_5. If M is a perfect matching, then the theorem holds by Theorem 1.1. So in the following, we only need to consider the case that M is not perfect. Since Q_5 has 2^5 vertices, we have $|M| \leq 15$.

Choose a position $j \in [5]$ such that $|M \cap E_j|$ is as small as possible. Then $|M \cap E_j| \leq 3$. Without loss of generality, we may assume $j = 5$. Split Q_5 into Q_4^0 and Q_4^1 at position 5. Then $Q_5 - E_5 = Q_4^0 + Q_4^1$. Let $\alpha \in \{0, 1\}$. Observe that every vertex $u_\alpha \in V(Q_4^\alpha)$ has in $Q_4^{1-\alpha}$ a unique neighbor, denoted by $u_{1-\alpha}$. Let $M_\alpha = M \cap E(Q_4^\alpha)$. We distinguish four cases to consider.

Case 1. $M \cap E_5 = \emptyset$. We say that a vertex u is covered by M if $u \in V(M)$. Otherwise, we say that u is uncovered by M. Since M is not perfect in Q_5, there exists a vertex uncovered by M. By symmetry we may assume that the uncovered vertex lies in Q_4^0, and denote it by u_0. In other words, $u_0 \in V(Q_4^0) \setminus V(M)$. First apply Lemma 1.2 to find a hamiltonian cycle C_1 in Q_4^1 passing through M_1. Let v_1 be a neighbor of u_1 on C_1 such that $u_1v_1 \notin M$. Since M is a matching, this is always possible. Since $p(u_0) \neq p(v_0)$ and M_0 is a matching in $Q_4^0 - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0v_0}$ in Q_4^0 passing through M_0. Hence $P_{u_0v_0} + C_1 + \{u_0u_1, v_0v_1\} - u_1v_1$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 2.

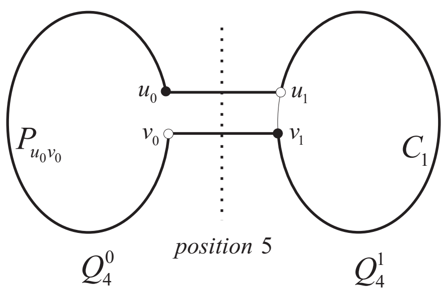

Figure 2. Illustration for Case 1.

Case 2. $|M \cap E_5| = 1$. Let $M \cap E_5 = \{u_0u_1\}$, where $u_\alpha \in V(Q_4^\alpha)$. Let v_α be a neighbor of u_α in Q_4^α for $\alpha \in \{0,1\}$. Then $p(u_0) \neq p(v_0)$ and $p(u_1) \neq p(v_1)$. Since $u_0u_1 \in M \cap E_5$, we have $u_\alpha \notin V(M_\alpha)$ for every $\alpha \in \{0,1\}$. In other words, M_α is a matching in $Q_4^\alpha - u_\alpha$. By Lemma 2.2 there exist Hamiltonian paths $P_{u_\alpha v_\alpha}$ in Q_4^α passing through M_α for every $\alpha \in \{0,1\}$. Hence $P_{u_0v_0} + P_{u_1v_1} + \{u_0u_1, v_0v_1\}$ is a Hamiltonian cycle in Q_5 passing through M.

Case 3. $|M \cap E_5| = 2$. Let $M \cap E_5 = \{u_0u_1, v_0v_1\}$, where $u_\alpha, v_\alpha \in V(Q_4^\alpha)$. If $p(u_0) \neq p(v_0)$, then $p(u_1) \neq p(v_1)$, the proof is similar to Case 2. So in the following we may assume $p(u_0) = p(v_0)$. Now $p(u_1) = p(v_1)$. In Q_4^α, since there are already matched two vertices with the same color, we have $|M_\alpha| \leq 6$ for every $\alpha \in \{0,1\}$. Thus, $\sum_{i\in[4]} |M \cap E_i| = |M_0| + |M_1| \leq 12$ and $|M| \leq 14$.

Choose a position $k \in [4]$ such that $|M \cap E_k|$ is as small as possible. Then $|M \cap E_k| \leq 3$. Without loss of generality, we may assume $k = 4$. Let $\alpha \in \{0,1\}$. Split Q_4^α into $Q_3^{\alpha 0}$ and $Q_3^{\alpha 1}$ at position 4. For clarity, we write $Q_3^{\alpha 0}$ and $Q_3^{\alpha 1}$ as $Q_3^{\alpha L}$ and $Q_3^{\alpha R}$, respectively, see Figure 3. Then $Q_4^\alpha - E_4 = Q_3^{\alpha L} + Q_3^{\alpha R}$. Let $M_{\alpha\delta} = M_\alpha \cap E(Q_3^{\alpha\delta})$ for every $\delta \in \{L, R\}$. Note that every vertex $s_{\alpha L} \in V(Q_3^{\alpha L})$ has in $Q_3^{\alpha R}$ a unique neighbor, denoted by $s_{\alpha R}$, and every vertex $t_{\alpha R} \in V(Q_3^{\alpha R})$ has in $Q_3^{\alpha L}$ a unique neighbor, denoted by $t_{\alpha L}$.

By symmetry, we may assume $|M_0 \cap E_4| \leq |M_1 \cap E_4|$. Since $|M \cap E_4| = |M_0 \cap E_4| + |M_1 \cap E_4| \leq 3$, we have $|M_0 \cap E_4| \leq 1$. Since $u_0 \in V(Q_4^0)$, without loss of generality we may assume $u_0 \in V(Q_3^{0L})$. Now $u_1 \in V(Q_3^{1L})$. We distinguish two cases to consider.

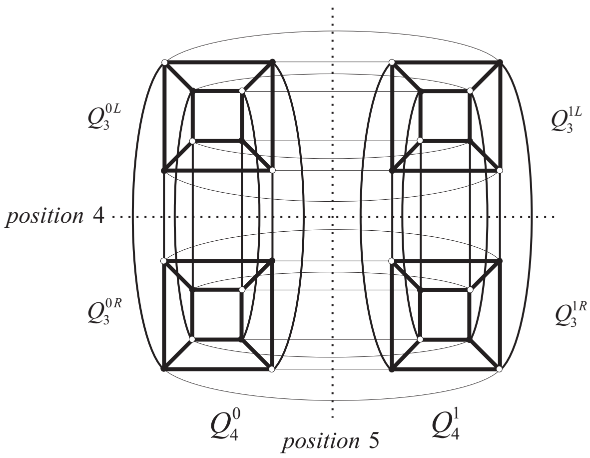

Figure 3. Q_5 splits into four 3-dimensional subcubes Q_3^{0L}, Q_3^{0R}, Q_3^{1L} and Q_3^{1R}.

Subcase 3.1. $v_0 \in V(Q_3^{0R})$. Now $v_1 \in V(Q_3^{1R})$.

Subcase 3.1.1. $M_0 \cap E_4 = \emptyset$. Apply Lemma 1.2 to find a Hamiltonian cycle C_1 in Q_4^1 passing through M_1. Since u_1 has only one neighbor in Q_3^{1R}, we may choose a neighbor x_1 of u_1 on C_1 such that $x_1 \in V(Q_3^{1L})$. Similarly, we may choose a neighbor y_1 of v_1 on C_1 such that $y_1 \in V(Q_3^{1R})$. Since $\{u_0u_1, v_0v_1\} \subseteq M$, we have $\{u_1x_1, v_1y_1\} \cap M = \emptyset$. Since $p(u_0) \neq p(x_0)$ and M_{0L} is a matching in $Q_3^{0L} - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0x_0}$ in Q_3^{0L} passing through M_{0L}. Similarly, there exists a Hamiltonian path $P_{v_0y_0}$ in Q_3^{0R} passing through M_{0R}. Hence $P_{u_0x_0} + P_{v_0y_0} + C_1 + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1\} - \{u_1x_1, v_1y_1\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 4.

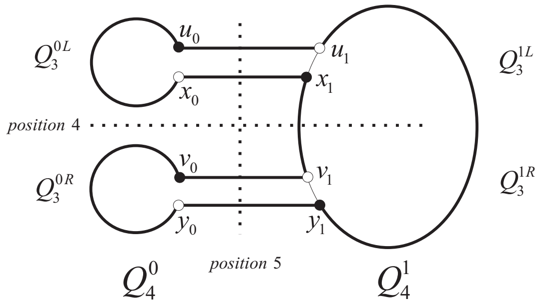

Figure 4. Illustration for Subcase 3.1.1.

Subcase 3.1.2. $|M_0 \cap E_4| = 1$. Now $1 \leq |M_1 \cap E_4| \leq 2$. Let $M_0 \cap E_4 = \{s_{0L}s_{0R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$. Since $p(u_0) = p(v_0)$ and $p(s_{0L}) \neq p(s_{0R})$, without loss of generality, we may assume $p(u_0) = p(v_0) = p(s_{0L}) \neq p(s_{0R})$.

First, we claim that there exists a Hamiltonian cycle C_1 in Q_4^1 passing through M_1 such that the two neighbors of v_1 on C_1 both belong to $V(Q_3^{1R})$.

If $|M_1 \cap E_4| = 2$, then $|M \cap E_4| = 3$. So $|M \cap E_i| = 3$ for every $i \in [4]$. Since $|M_\alpha| \leq 6$ for every $\alpha \in \{0, 1\}$ and $|M_0| + |M_1| = \sum_{i \in [4]} |M \cap E_i|$, we have $|M_\alpha| = 6$ for every $\alpha \in \{0, 1\}$. Let $M_1 \cap E_4 = \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$, where $a_{1\delta}, b_{1\delta} \in V(Q_3^{1\delta})$. Then $p(a_{1\delta}) \neq p(b_{1\delta})$ for every $\delta \in \{L, R\}$. (Otherwise, if $p(a_{1\delta}) = p(b_{1\delta})$, then $|M_{1\delta}| \leq 2$. Moreover, either $p(u_1) = p(a_{1L}) = p(b_{1L})$ or $p(v_1) = p(a_{1R}) = p(b_{1R})$, so $|M_{1L}| \leq 1$ or $|M_{1R}| \leq 1$. Thus, $|M_1| \leq 2 + 1 + 2 = 5$, a contradiction).

If $|M_1 \cap E_4| = 1$, let $M_1 \cap E_4 = \{a_{1L}a_{1R}\}$, where $a_{1\delta} \in V(Q_3^{1\delta})$. Since a_{1R} has three neighbors in Q_3^{1R}, we may choose a neighbor b_{1R} of a_{1R} in Q_3^{1R} such that $b_{1R} \neq v_1$. Now $p(a_{1\delta}) \neq p(b_{1\delta})$ for every $\delta \in \{L, R\}$.

For the above two cases, since $M_{1\delta}$ is a matching in $Q_3^{1\delta} - a_{1\delta}$, by Lemma 2.2 there exist Hamiltonian paths $P_{a_{1\delta}b_{1\delta}}$ in $Q_3^{1\delta}$ passing through $M_{1\delta}$ for every $\delta \in \{L, R\}$. Let $C_1 = P_{a_{1L}b_{1L}} + P_{a_{1R}b_{1R}} + \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$. In the former case, since $\{v_0v_1, a_{1L}a_{1R}, b_{1L}b_{1R}\} \subseteq M$, we have $v_1 \notin \{a_{1R}, b_{1R}\}$. In the latter case, since $\{v_0v_1, a_{1L}a_{1R}\} \subseteq M$, we have $v_1 \neq a_{1R}$, and therefore, $v_1 \notin \{a_{1R}, b_{1R}\}$. Hence C_1 is a Hamiltonian cycle in Q_4^1 passing through M_1 such that the two neighbors of v_1 on C_1 both belong to $V(Q_3^{1R})$, see Figure 5(1).

Next, choose a neighbor x_1 of u_1 on C_1 such that $x_1 \in V(Q_3^{1L})$ and choose a neighbor y_1 of v_1 on C_1 such that $y_0 \neq s_{0R}$. Since M_{0R} is a matching in $Q_3^{0R} - v_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{v_0y_0}$ in Q_3^{0R} passing through M_{0R}, see Figure 5(2). Since $s_{0R} \notin \{v_0, y_0\}$, we may choose a neighbor t_{0R} of s_{0R} on $P_{v_0y_0}$ such that $t_{0L} \neq x_0$. Now u_0, x_0, s_{0L}, t_{0L} are pairwise distinct vertices in Q_3^{0L}, and $p(u_0) = p(s_{0L}) \neq p(x_0) = p(t_{0L})$, and $d(u_0, x_0) = d(s_{0L}, t_{0L}) = 1$. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, s_{0L}\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0x_0} + P_{s_{0L}t_{0L}}$ in Q_3^{0L} passing through M_{0L}. Hence $P_{u_0x_0} + P_{s_{0L}t_{0L}} + P_{v_0y_0} + C_1 + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1, s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{u_1x_1, v_1y_1, s_{0R}t_{0R}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 5(2).

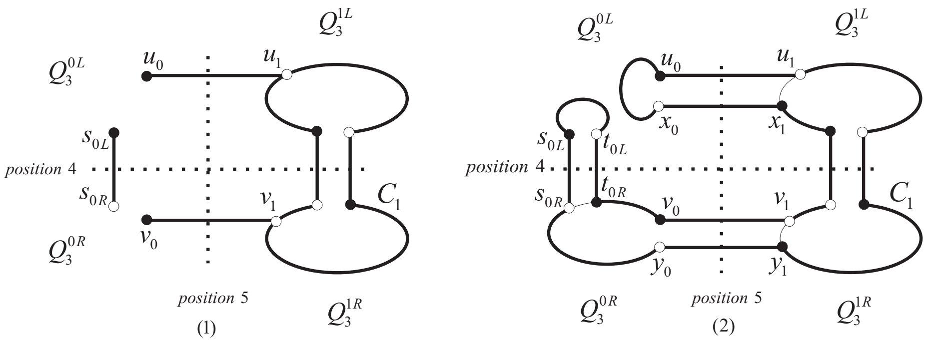

Figure 5. Illustration for Subcase 3.1.2.

Subcase 3.2. $v_0 \in V(Q_3^{0L})$. Now $v_1 \in V(Q_3^{1L})$. Let x_1 be the unique vertex in Q_3^{1L} satisfying $d(x_1, v_1) = 3$. Then $d(x_1, u_1) = 1$. Since M_1 is a matching in $Q_4^1 - u_1$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_1x_1}$ in Q_4^1 passing through M_1. Since v_1 has only one neighbor in Q_3^{1R}, we may choose a neighbor y_1 of v_1 on $P_{u_1x_1}$ such that $y_1 \in V(Q_3^{1L})$. Since $d(x_1, v_1) = 3$, we have $y_1 \neq x_1$. Then u_1, x_1, v_1, y_1 are pairwise distinct vertices, and $p(u_1) = p(v_1) \neq p(x_1) = p(y_1)$, and $d(u_1, x_1) = d(v_1, y_1) = 1$, and the same properties also hold for the corresponding vertices u_0, x_0, v_0, y_0. If we can find a spanning 2-path $P'_{u_0x_0} + P'_{v_0y_0}$ in Q_4^0 passing through M_0, then $P'_{u_0x_0} + P'_{v_0y_0} + P_{u_1x_1} + \{u_0u_1, x_0x_1, v_0v_1, y_0y_1\} - v_1y_1$ is a Hamiltonian cycle in Q_5 passing through M. So in the following, we only need to show that the desired spanning 2-path $P'_{u_0x_0} + P'_{v_0y_0}$ exists. We distinguish several cases to consider.

Subcase 3.2.1. $|M_0 \cap E_4| = 1$. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, v_0\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L}. Let $M_0 \cap E_4 = \{s_{0L}s_{0R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$. Without loss of generality assume $s_{0L} \in V(P_{v_0y_0})$. Choose a neighbor t_{0L} of s_{0L} on $P_{v_0y_0}$. Since $s_{0L}s_{0R} \in M$, we have $s_{0L}t_{0L} \notin M$. Since M_{0R} is a matching in $Q_3^{0R} - s_{0R}$, by Lemma 2.2 there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}. Let $P'_{u_0x_0} = P_{u_0x_0}$ and $P'_{v_0y_0} = P_{v_0y_0} + P_{s_{0R}t_{0R}} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - s_{0L}t_{0L}$. Then $P'_{u_0x_0} + P'_{v_0y_0}$ is the desired spanning 2-path in Q_4^0, see Figure 6.

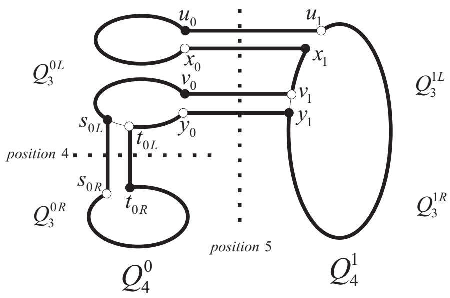

Figure 6. Illustration for Subcase 3.2.1.

Subcase 3.2.2. $M_0 \cap E_4 = \emptyset$. It suffices to consider the case that M_{0L} is maximal in $Q_3^{0L} - \{u_0, v_0\}$ and M_{0R} is maximal in Q_3^{0R}. In Q_3^{0L}, since $p(u_0) = p(v_0)$, we have u_0, v_0 are different in two positions, so there is one possibility of $\{u_0, v_0\}$ up to isomorphism. Since $d(x_0, v_0) = 3$, the vertex x_0 is fixed by v_0. Since $d(y_0, v_0) = 1$, there are two choices of y_0 up to isomorphism. Thus, there are two possibilities of $\{u_0, v_0, x_0, y_0\}$ up to isomorphism, see Figure 7$(a)(b)$. When $\{u_0, v_0, x_0, y_0\}$ is the case (a), since M_{0L} is a maximal matching in $Q_3^{0L} - \{u_0, v_0\}$, there are three possibilities of M_{0L} up to isomorphism, see Figure 7(1)–(3). When $\{u_0, v_0, x_0, y_0\}$ is the case (b), there are seven possibilities of M_{0L}, see Figure 7(4)–(10). In Q_3^{0R}, there are three non-isomorphic maximal matchings, denoted by P_1, P_2 and P_3, see Figure 8.

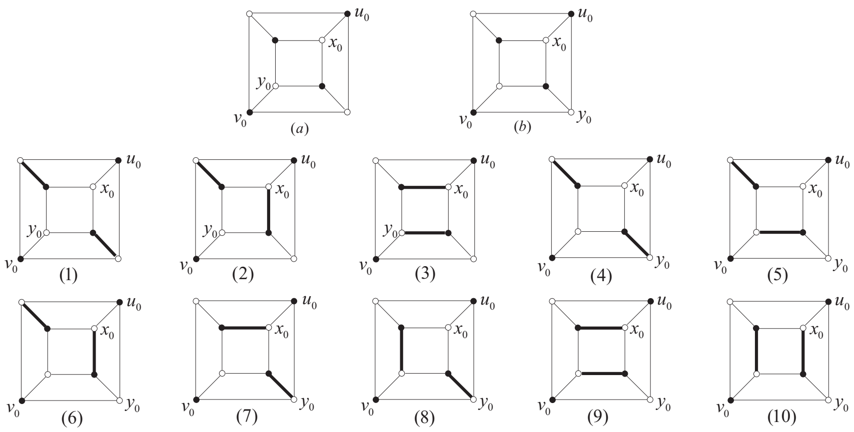

Figure 7. All possibilities of $\{u_0, v_0, x_0, y_0, M_{0L}\}$ up to isomorphism.

Before the proof, we point out that if M_{0R} is isomorphic to the matching P_1 or P_2, then there exists a Hamiltonian cycle in Q_3^{0R} passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$, see Figure 9.

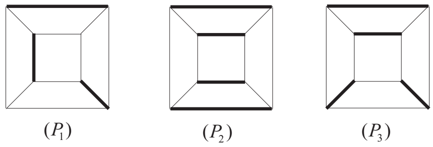

Figure 8. Three non-isomorphic maximal matchings in Q_3^{0R}.

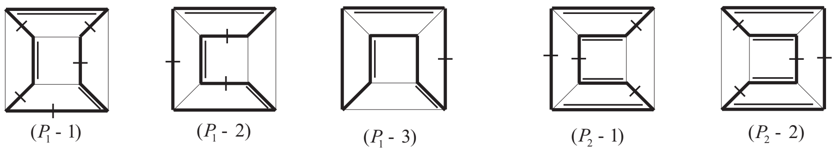

Figure 9. Hamiltonian cycles passing through $M_{0R} \cup \{e\}$ for any $e \notin M_{0R}$ in Q_3^{0R} when M_{0R} is isomorphic to P_1 or P_2.

First, suppose that M_{0R} is isomorphic to P_1. Since M_{0L} is a matching in $Q_3^{0L} - \{u_0, v_0\}$, by Lemma 2.1 there exists a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L}. Since $|E(P_{u_0x_0} + P_{v_0y_0})| = 6 > |M_{0L}| + |M_{0R}|$, there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_{0R}t_{0R} \notin M_{0R}$. Choose a Hamiltonian cycle C_{0R} in Q_3^{0R} passing through $M_{0R} \cup \{s_{0R}t_{0R}\}$. Hence $P_{u_0x_0} + P_{v_0y_0} + C_{0R} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{s_{0L}t_{0L}, s_{0R}t_{0R}\}$ is the desired spanning 2-path in Q_4^0. (Note that the construction is similar to Subcase 3.2.1, so the readers may refer to the construction in Figure 6.)

Next, suppose that M_{0R} is isomorphic to P_2. We say that a set S of edges crosses a position i if $S \cap E_i \neq \emptyset$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (2)–(10) in Figure 7, then we may choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} such that the set $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ crosses at least two positions, see Figure 10(2)–(10). Since all the edges in M_{0R} lie in the same position, there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ such that $s_{0R}t_{0R} \notin M_{0R}$. If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to the case (1) in Figure 7, then we may choose two different spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} such that the two sets $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ cross two different positions, see Figure 10(1–1), (1–2), and therefore, at least one of them is different from the position in which M_{0R} lies. Thus, we may choose a suitable spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ such that there exists an edge $s_{0L}t_{0L} \in E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ and $s_{0R}t_{0R} \notin M_{0R}$. The remaining construction is similar to the above case.

Last, suppose that M_{0R} is isomorphic to P_3. Without loss of generality, we may assume $M_{0R} \subseteq (E_2 \cup E_3)$.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to the case (5) or (8) in Figure 7, we may choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L} such that $(E(P_{u_0x_0}+P_{v_0y_0})\setminus M_{0L})\cap E_1 \neq \emptyset$, see Figure 11. Let $s_{0L}t_{0L} \in (E(P_{u_0x_0}+P_{v_0y_0})\setminus M_{0L}) \cap E_1$. Then $s_{0R}t_{0R} \in E_1$. One can verify that there exists a Hamiltonian cycle C_{0R} in Q_3^{0R} passing through $M_{0R} \cup \{s_{0R}t_{0R}\}$. Hence $P_{u_0x_0} + P_{v_0y_0} + C_{0R} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{s_{0L}t_{0L}, s_{0R}t_{0R}\}$ is the desired spanning 2-path in Q_4^0.

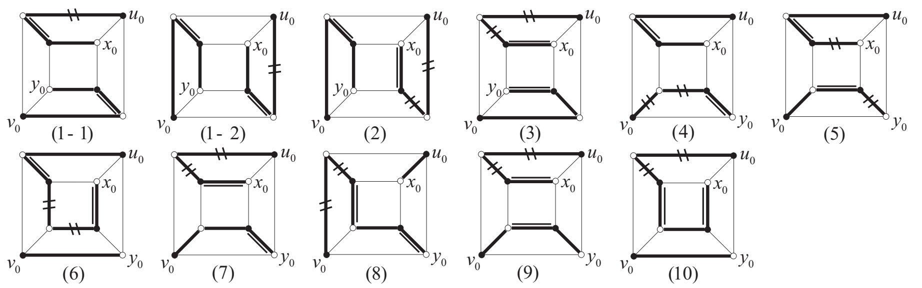

Figure 10. Spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} with the possible edges $s_{0L}t_{0L}$ lined by \\.

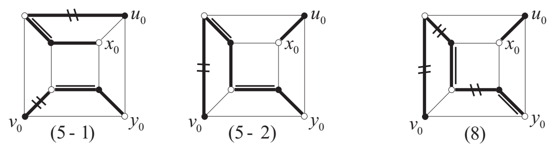

Figure 11. The possible spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ such that $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset$.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (3), (6), (7) or (10) in Figure 7, then choose a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in Q_3^{0L} passing through M_{0L}, see Figure 12(3), (6), (7), (10). If $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 \neq \emptyset$, then the proof is similar to the above case. If $(E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}) \cap E_1 = \emptyset$, then the set $E(P_{u_0x_0} + P_{v_0y_0}) \setminus M_{0L}$ crosses the positions 2 and 3, and therefore, M_{0R} has two choices for every case, see Figure 12. Then we can find a spanning 2-path $P'_{u_0x_0} + P'_{v_0y_0}$ in Q_4^0 passing through M_0, see Figure 12.

If $\{u_0, v_0, x_0, y_0, M_{0L}\}$ is isomorphic to one of the cases (1), (2), (4) or (9) in Figure 7, we observe that there exist two vertices in $V(Q_3^{0L})$ at distance 3, denoted by s_{0L}, t_{0L}, such that there is a spanning 2-path $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L} + s_{0L}t_{0L}$ passing through $M_{0L} \cup \{s_{0L}t_{0L}\}$, see Figure 13. Next, we can verify that there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}. Hence $P_{u_0x_0} + P_{v_0y_0} + P_{s_{0R}t_{0R}} + \{s_{0L}s_{0R}, t_{0L}t_{0R}\} - s_{0L}t_{0L}$ is the desired spanning 2-path in Q_4^0.

Case 4. $|M \cap E_5| = 3$. Let $M \cap E_5 = \{u_0u_1, v_0v_1, w_0w_1\}$, where $u_\alpha, v_\alpha, w_\alpha \in V(Q_4^\alpha)$. Now $|M \cap E_i| = 3$ for every $i \in [5]$ and $|M| = 15$. Hence there are two vertices of $\{u_\alpha, v_\alpha, w_\alpha\}$ in one partite set and one vertex in the other partite set. Otherwise, if $p(u_\alpha) = p(v_\alpha) = p(w_\alpha)$, then $|M_\alpha| \leq 5$, and therefore, $|M| \leq 13$, a contradiction. Without loss of generality, we may assume $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$.

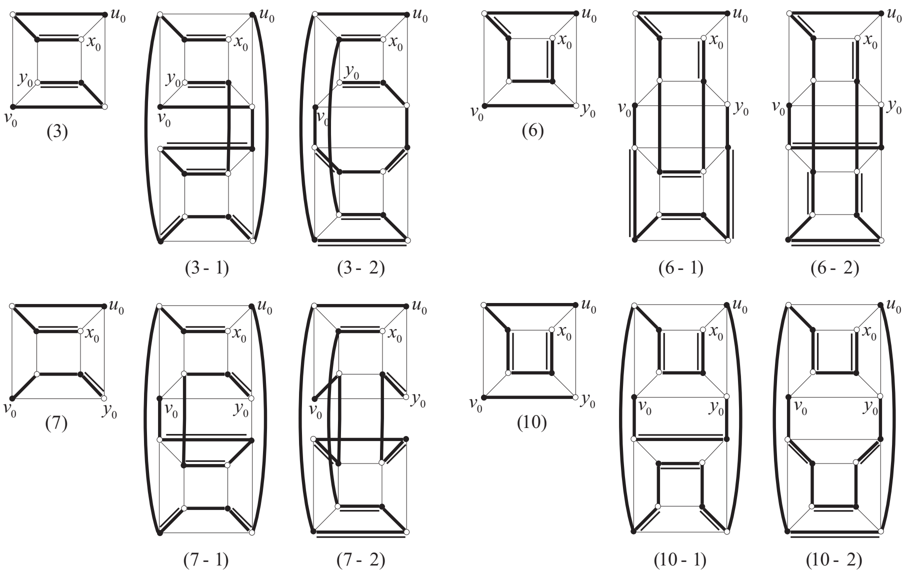

Figure 12. Spanning 2-paths $P'_{u_0x_0} + P'_{v_0y_0}$ in Q_4^0 passing through M_0.

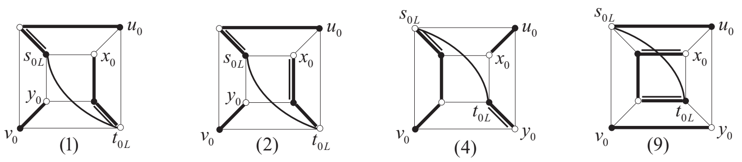

Figure 13. Spanning 2-paths $P_{u_0x_0} + P_{v_0y_0}$ in $Q_3^{0L} + s_{0L}t_{0L}$ passing through $M_{0L} \cup \{s_{0L}t_{0L}\}$.

Split Q_4^α into two 3-cubes $Q_3^{\alpha L}$ and $Q_3^{\alpha R}$ at some position k such that $u_\alpha \in V(Q_3^{\alpha L})$ and $v_\alpha \in V(Q_3^{\alpha R})$. Without loss of generality, we may assume $k = 4$. Since $p(u_\alpha) = p(v_\alpha) \neq p(w_\alpha)$, by symmetry we may assume $w_\alpha \in V(Q_3^{\alpha L})$. Since $|M_0 \cap E_4| + |M_1 \cap E_4| = |M \cap E_4| = 3$, by symmetry we may assume $|M_0 \cap E_4| \leq 1$. Let $M_{\alpha\delta} = M_\alpha \cap E(Q_3^{\alpha\delta})$ for every $\delta \in \{L, R\}$.

Subcase 4.1. $M_0 \cap E_4 = \emptyset$. Since $p(u_1) \neq p(w_1)$ and M_1 is a matching in $Q_4^1 - u_1$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_1w_1}$ in Q_4^1 passing through M_1. Since v_1 has only one neighbor in Q_3^{1L}, we may choose a neighbor y_1 of v_1 on $P_{u_1w_1}$ such that $y_1 \in V(Q_3^{1R})$. Now $y_0 \in V(Q_3^{0R})$ and $p(u_0) = p(v_0) \neq p(w_0) = p(y_0)$. Since M_{0L} is a matching in $Q_3^{0L} - u_0$ and M_{0R} is a matching in $Q_3^{0R} - v_0$, by Lemma 2.2 there exist Hamiltonian paths $P_{u_0w_0}$ in Q_3^{0L} and $P_{v_0y_0}$ in Q_3^{0R} passing through M_{0L} and M_{0R}, respectively. Hence $P_{u_1w_1} + P_{u_0w_0} + P_{v_0y_0} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1\} - v_1y_1$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 14.

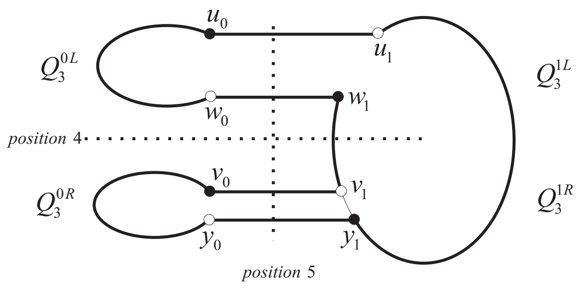

Figure 14. Illustration for Subcase 4.1.

Subcase 4.2. $|M_0 \cap E_4| = 1$. Now $|M_1 \cap E_4| = 2$. Let $M_0 \cap E_4 = \{s_{0L}s_{0R}\}$ and $M_1 \cap E_4 = \{a_{1L}a_{1R}, b_{1L}b_{1R}\}$, where $s_{0\delta} \in V(Q_3^{0\delta})$ and $a_{1\delta}, b_{1\delta} \in V(Q_3^{1\delta})$. Since $|M| = 15$, Q_5 has exactly two vertices uncovered by M, one in Q_3^{0L} and the other in Q_3^{1R}. Thus, $p(a_{1L}) \neq p(b_{1L})$, and $p(v_0) \neq p(s_{0R})$, and M_{1L} is a perfect matching in $Q_3^{1L} - \{u_1, w_1, a_{1L}, b_{1L}\}$. Since $p(u_1) \neq p(w_1)$ and $p(a_{1L}) \neq p(b_{1L})$, without loss of generality, we may assume $p(u_1) = p(b_{1L}) \neq p(w_1) = p(a_{1L})$. Thus, $p(v_1) = p(a_{1R}) \neq p(b_{1R})$.

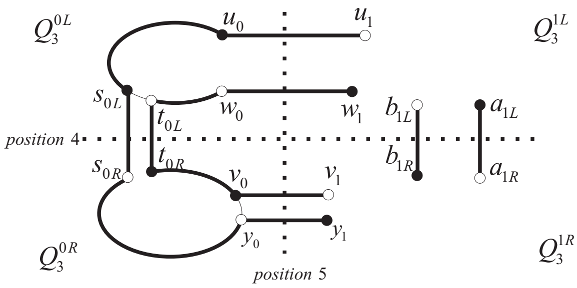

Figure 15. Illustration for Subcase 4.2.

Since $p(u_0) \neq p(w_0)$ and M_{0L} is a matching in $Q_3^{0L} - u_0$, by Lemma 2.2 there exists a Hamiltonian path $P_{u_0w_0}$ in Q_3^{0L} passing through M_{0L}, see Figure 15. Since $s_{0L} \notin \{u_0, w_0\}$, we may choose a neighbor t_{0L} of s_{0L} on $P_{u_0w_0}$ such that $t_{0R} \neq v_0$. Since $p(s_{0R}) \neq p(t_{0R})$ and M_{0R} is a matching in $Q_3^{0R} - s_{0R}$, by Lemma 2.2 there exists a Hamiltonian path $P_{s_{0R}t_{0R}}$ in Q_3^{0R} passing through M_{0R}, see Figure 15. Since $v_0 \notin \{s_{0R}, t_{0R}\}$, we may choose a neighbor y_0 of v_0 on $P_{s_{0R}t_{0R}}$ such that $y_1 \neq b_{1R}$. Now v_1, y_1, a_{1R}, b_{1R} are pairwise distinct vertices, and $p(v_1) = p(a_{1R}) \neq p(y_1) = p(b_{1R})$, and $d(v_1, y_1) = 1$, and M_{1R} is a matching in $Q_3^{1R} - \{v_1, a_{1R}\}$.

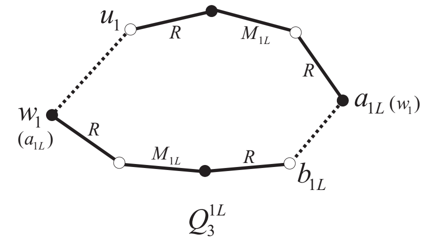

Figure 16. The spanning 2-path $P_{u_1a_{1L}} + P_{w_1b_{1L}}$ (or $P_{u_1w_1} + P_{a_{1L}b_{1L}}$) in Q_3^{1L}.

If $d(a_{1R}, b_{1R}) = 1$, then by Lemma 2.1 there is a spanning 2-path $P_{v_1y_1} + P_{a_{1R}b_{1R}}$ in Q_3^{1R} passing through M_{1R}, see Figure 17(1). Since M_{1L} is a perfect matching in $Q_3^{1L} - \{u_1, w_1, a_{1L}, b_{1L}\}$, we have $M_{1L} \cup \{u_1w_1, a_{1L}b_{1L}\}$ is a perfect matching in $K(Q_3^{1L})$. By Theorem 1.1, there exists a perfect matching R in Q_3^{1L} such that $M_{1L} \cup \{u_1w_1, a_{1L}b_{1L}\} \cup R$ forms a Hamiltonian cycle in $K(Q_3^{1L})$. Hence $M_{1L} \cup R$ forms a spanning 2-path in Q_3^{1L}. Note that each path of the spanning 2-path is an (R, M_{1L})-alternating path beginning with an edge in R and ending with an edge in R. So the number of vertices in each path is even. Since Q_5 is a bipartite graph, the two endpoints of each path have different parities. Hence one path joins the vertices u_1 and a_{1L}, and the other path joins the vertices w_1 and b_{1L}, see Figure 16 for example. Denote the spanning 2-path by $P_{u_1a_{1L}} + P_{w_1b_{1L}}$. Note that $s_{0L}t_{0L} \notin M$ and $v_0y_0 \notin M$. Hence $P_{u_0w_0} + P_{s_{0R}t_{0R}} + P_{u_1a_{1L}} + P_{w_1b_{1L}} + P_{v_1y_1} + P_{a_{1R}b_{1R}} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1, a_{1L}a_{1R}, b_{1L}b_{1R}, s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{v_0y_0, s_{0L}t_{0L}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 17(1).

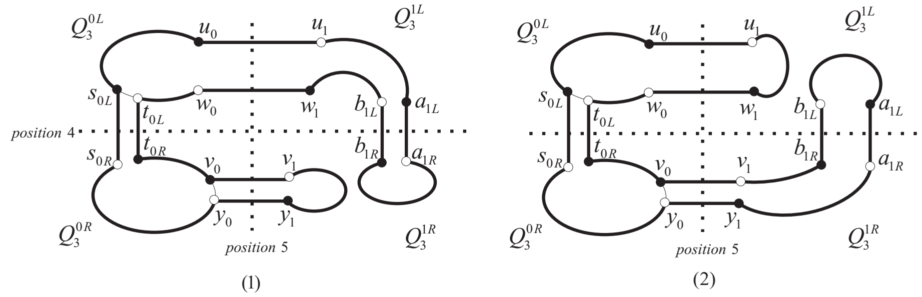

Figure 17. Illustration for Subcase 4.2.

If $d(a_{1R}, b_{1R}) = 3$, then $d(v_1, b_{1R}) = d(a_{1R}, y_1) = 1$. Since M_{1R} is a matching in $Q_3^{1R} - \{v_1, a_{1R}\}$, by Lemma 2.1 there is a spanning 2-path $P_{v_1b_{1R}} + P_{a_{1R}y_1}$ in Q_3^{1R} passing through M_{1R}, see Figure 17(2). Since $M_{1L} \cup \{u_1a_{1L}, w_1b_{1L}\}$ is a perfect matching in $K(Q_3^{1L})$, similar to the above case, there is a spanning 2-path $P_{u_1w_1} + P_{a_{1L}b_{1L}}$ in Q_3^{1L} passing through M_{1L}. Hence $P_{u_0w_0} + P_{s_{0R}t_{0R}} + P_{v_1b_{1R}} + P_{y_1a_{1R}} + P_{u_1w_1} + P_{a_{1L}b_{1L}} + \{u_0u_1, w_0w_1, v_0v_1, y_0y_1, a_{1L}a_{1R}, b_{1L}b_{1R}, s_{0L}s_{0R}, t_{0L}t_{0R}\} - \{v_0y_0, s_{0L}t_{0L}\}$ is a Hamiltonian cycle in Q_5 passing through M, see Figure 17(2). The proof of Theorem 1.3 is complete.

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