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HEREDITARY EQUALITY OF DOMINATION AND EXPONENTIAL DOMINATION

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Abstract

We characterize a large subclass of the class of those graphs G for which the exponential domination number of H equals the domination number of H for every induced subgraph H of G.

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1. Introduction

Domination in graphs is an important area within graph theory, and an astounding variety of different domination parameters are known [6]. Essentially all of these parameters involve merely local conditions, which makes them amenable to similar approaches and arguments. In [5] Dankelmann *et al.* introduce a truly non-local variant of domination, the so-called exponential domination, where the influence of vertices extends to any arbitrary distance within the graph but decays exponentially with that distance. There is relatively few research concerning exponential domination [1–4], and even apparently basic results require new and careful arguments.

As follows easily from the precise definitions given below, the exponential domination number of any graph is at most its domination number. Bessy *et al.* [4] show that computing the exponential domination number is APX-hard for subcubic graphs and describe an efficient algorithm for subcubic trees, but the complexity for general trees is unknown. It is not even known how to decide efficiently for a given tree T whether its exponential domination number $\gamma_e(T)$ equals its domination number $\gamma(T)$. In [8] we study relations between the different parameters of exponential domination and domination. Next to several bounds, we obtain a constructive characterization of the subcubic trees T with $\gamma_e(T) = \gamma(T)$. In view of the efficient algorithms to determine both parameters for such trees, the existence of a constructive characterization is not surprising, but, as said a few lines above, already for general trees all techniques from [3, 4, 8] completely fail.

Note that, since adding a universal vertex to any graph results in a graph G with $\gamma_e(G) = \gamma(G)$, the class of all graphs G that satisfy $\gamma_e(G) = \gamma(G)$ is not hereditary, and does not have a simple structure. The difficulty to decide whether $\gamma_e(G) = \gamma(G)$ for a given graph G motivates the study of the hereditary class $\mathcal{G}$ of graphs that satisfy this equality, that is, $\mathcal{G}$ is the set of those graphs G such that $\gamma_e(H) = \gamma(H)$ for every induced subgraph H of G. As for the well-known class of perfect graphs, the class $\mathcal{G}$ can be characterized by minimal forbidden induced subgraphs.

In the present paper we obtain such a characterization for a large subclass of $\mathcal{G}$, and pose several related conjectures.

Before we proceed to our results, we collect some notation. We consider finite, simple, and undirected graphs, and use standard terminology. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order $n(G)$ of G is the number of vertices of G. For a vertex u of G, the neighborhood of u in G and the degree of u in G are denoted by $N_G(u)$ and $d_G(u)$, respectively. The distance $\text{dist}_G(X, Y)$ between two sets X and Y of vertices in G is the minimum length of a path in G between a vertex in X and a vertex in Y. If no such path exists, then let $\text{dist}_G(X, Y) = \infty$.

Let D be a set of vertices of a graph G. The set D is a dominating set of G [6] if every vertex of G not in D has a neighbor in D. The domination number $\gamma(G)$ of G is the minimum size of a dominating set of G. For two vertices u and v of G, let $\text{dist}_{(G,D)}(u, v)$ be the minimum length of a path P in G between u and v such that D contains exactly one endvertex of P but no internal vertex of P. If no such path exists, then let $\text{dist}_{(G,D)}(u, v) = \infty$. Note that, if u and v are distinct vertices in D, then $\text{dist}_{(G,D)}(u, u) = 0$ and $\text{dist}_{(G,D)}(u, v) = \infty$. For a vertex u of G, let

$$w_{(G,D)}(u) = \sum_{v \in D} \left(\frac{1}{2}\right)^{\mathrm{dist}_{(G,D)}(u,v)-1},$$

where $\left(\frac{1}{2}\right)^{\infty} = 0$. Dankelmann *et al.* [5] define the set D to be an *exponential dominating set* of G if $w_{(G,D)}(u) \geq 1$ for every vertex u of G, and the *exponential domination number* $\gamma_e(G)$ of G as the minimum size of an exponential dominating set of G. Note that $w_{(G,D)}(u) = 2$ for $u \in D$, and that $w_{(G,D)}(u) \geq 1$ for every vertex u that has a neighbor in D, which implies $\gamma_e(G) \leq \gamma(G)$.

The following Figure 1 contains forbidden induced subgraphs that relate to the considered subclasses of $\mathcal{G}$. Recall that P_k and C_k denote the path and cycle of order k, respectively.

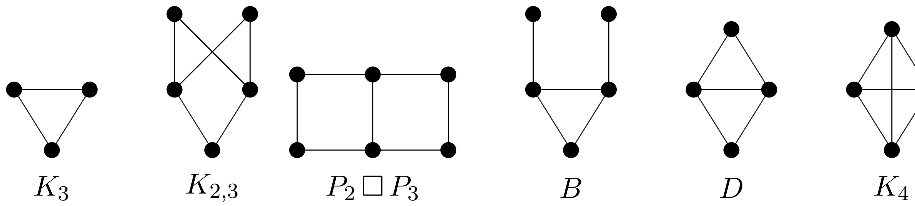

Figure 1. The graphs K_3, $K_{2,3}$, $P_2 \,\square\, P_3$, B (bull), D (diamond), and K_4.

Our main result is the following.

Theorem 1. *If G is a $\{B, D, K_4, K_{2,3}, P_2 \,\square\, P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1, \ldots, F_5\}$-free* (*cf. Figure* 2).

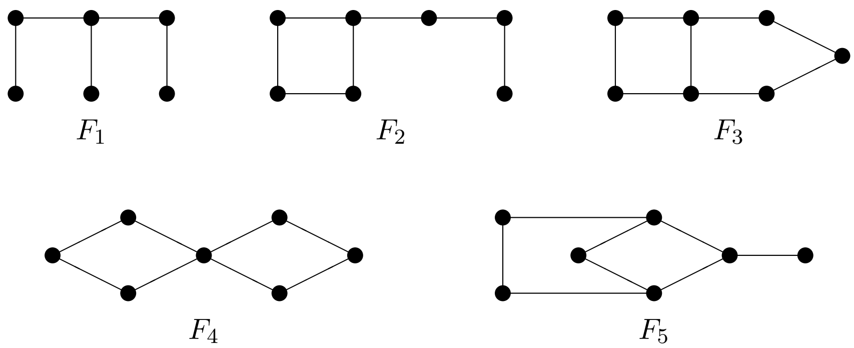

Figure 2. The graphs $F_1, \ldots, F_5$.

Since all graphs in $\{B, D, K_4, K_{2,3}, P_2 \,\square\, P_3\} \cup \{F_2, \ldots, F_5\}$ have girth at most 4, where the girth of a graph is the minimum length of a cycle in it, Theorem 1 has the following immediate corollary.

Corollary 2. *If G is a graph of girth at least 5, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1\}$-free.*

For the trees in $\mathcal{G}$, we achieve a complete characterization.

Corollary 3. *If T is a tree, then $\gamma(F) = \gamma_e(F)$ for every induced subgraph F of T if and only if T is $\{P_7, F_1\}$-free.*

All proofs and our conjectures are postponed to the next section.

2. Proofs and Conjectures

We split the proof of Theorem 1 into the triangle-free case and the non-triangle-free case. The triangle-free case is considered in the following lemma.

Lemma 4. *If G is a $\{K_3, K_{2,3}, P_2 \,\square\, P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1, \ldots, F_5\}$-free.*

Proof. Since $\gamma(H) > \gamma_e(H)$ for every graph H in $\{P_7, C_7, F_1, \ldots, F_5\}$, necessity follows. In order to prove sufficiency, suppose that G is a $\{K_3, K_{2,3}, P_2 \,\square\, P_3\} \cup \{P_7, C_7, F_1, \ldots, F_5\}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of G, we have $\gamma(H) = \gamma_e(H)$ for every proper induced subgraph H of G. Clearly, G is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$. Since G is $\{P_7, C_7\}$-free, either G is a tree or the girth g of G is at most 6.

Suppose that G is a tree. If G has at most one vertex of degree at least 3, then, since G is $\{P_7, F_1\}$-free, it arises from a path $P : u_1 \cdots u_\ell$ with $\ell \leq 6$ by attaching further endvertices to u_2. Since $\ell \leq 6$, the set $\{u_2, u_{\ell-1}\}$ is a dominating set of G, which contradicts $\gamma(G) \geq 3$. Hence, G has at least two vertices of degree at least 3. Let $P : u_1 \cdots u_\ell$ be a shortest path in G between two such vertices. Since G is F_1-free, it arises from P by attaching at least two further endvertices to u_1 and at least two further endvertices to u_ℓ. Since G is P_7-free, we obtain $\ell \leq 4$. This implies that the set $\{u_1, u_\ell\}$ is a dominating set of G, which contradicts $\gamma(G) \geq 3$. Hence, we may assume that G is not a tree. Let $C : x_1 x_2 x_3 \cdots x_g x_1$ be a shortest cycle of G, where we consider the indices modulo g. Let $R = V(G) \setminus V(C)$.

Suppose $g = 6$. Since $\gamma(C_6) = \gamma_e(C_6) = 2$, some vertex y in R has a neighbor x_i on C. Since $g = 6$, the vertex y has no further neighbor on C, implying that $G[\{y, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}] = F_1$, contradicting the fact that G is F_1-free. Hence, $g < 6$.

Suppose $g = 5$. This implies that no vertex in R has more than one neighbor on C. If some vertex z has distance 2 from $V(C)$ in G and $x_i yz$ is a path in G, then $G[\{z, y, x_{i-1}, x_i, x_{i+1}, x_{i+2}\}] = F_1$, which is a contradiction. Hence, every vertex in R has a unique neighbor on C. Suppose that there is some $i \in [5]$ such that x_i has a neighbor y_i in R and x_{i+1} has a neighbor y_{i+1} in R. Since $g = 5$, we note that $y_i \neq y_{i+1}$ and that the vertex y_i is not adjacent to y_{i+1}, implying that $G[\{x_{i-2}, x_{i-1}, x_i, x_{i+1}, y_i, y_{i+1}\}] = F_1$, which is a contradiction. This implies the existence of some index $i \in [5]$ such that $\{x_i, x_{i+2}\}$ is a dominating set of G, which contradicts $\gamma(G) \geq 3$. Hence, $g \leq 4$. Since G is K_3-free, this implies that $g = 4$.

Since G is $\{K_3, K_{2,3}\}$-free, no vertex in R has more than one neighbor on C, and since G is F_2-free, no vertex in R has distance more than 2 from $V(C)$.

Suppose that some vertex z has distance 2 from $V(C)$. Let $x_1 yz$ be a path in G. Suppose that x_2 has a neighbor u in R. Recall that u is not adjacent to any other vertex on C. Since G is $P_2 \Box P_3$-free, the vertex u is not adjacent to y. If u is not adjacent to z, then $G[\{u, x_1, x_2, x_4, y, z\}] = F_1$, which is a contradiction. If u is adjacent to z, then $G[V(C) \cup \{u, y, z\}] = F_3$, which is a contradiction. Hence, by symmetry, we obtain $d_G(x_2) = d_G(x_4) = 2$.

Suppose that x_1 has a neighbor u in $R \setminus \{y\}$. Since G is $\{K_3, K_{2,3}\}$-free, the vertex u is not adjacent to any vertex in $\{x_2, x_3, x_4, y\}$. If u is not adjacent to z, then $G[\{x_1, x_2, x_3, u, y, z\}] = F_1$, which is a contradiction. If u is adjacent to z, then $G[V(C) \cup \{u, y, z\}] = F_4$, which is a contradiction. Hence, we obtain $d_G(x_1) = 3$.

Since $\{x_3, y\}$ is not a dominating set of G, and no vertex in R has distance more than 2 from $V(C)$, the degrees of x_1, x_2, and x_4 imply the existence of a path $x_3 uv$, where v has distance 2 to $V(C)$, and v is not adjacent to y. Since $G[\{v, u, x_3, x_2, x_1, y, z\}]$ is neither P_7 nor C_7, the vertex u is adjacent to y or z. If u is adjacent to z, then, because G is K_3-free, $G[\{u, v, y, z, x_3, x_2\}] = F_1$, which is a contradiction. Hence, the vertex u is adjacent to y. If v is adjacent to z, then $G[\{u, v, y, z, x_1, x_2, x_3\}] = F_3$, which is a contradiction. Hence, the vertex v is not adjacent to z, and $G[\{u, v, y, z, x_3, x_2\}] = F_1$, which is a contradiction. Hence, every vertex in R has a unique neighbor on C.

Since $\gamma(G) > 2$, we may assume that x_i has a neighbor y_i in R for $i \in [3]$. Since G is $P_2 \Box P_3$-free, the vertex y_2 is not adjacent to y_1 or y_3. Since G is F_5-free, the vertex y_1 is not adjacent to y_3. Now, $G[\{x_1, x_2, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction, and completes the proof. ■

With Lemma 4 at hand, we now proceed to the proof of Theorem 1.

***Proof of Theorem* 1.** Necessity follows as above. In order to prove sufficiency, suppose that G is a $\{B, D, K_4, K_{2,3}, P_2 \Box P_3\} \cup \{P_7, C_7, F_1, \ldots, F_5\}$-free graph with $\gamma(G) > \gamma_e(G)$ of minimum order. By the choice of G, we have $\gamma(H) = \gamma_e(H)$

for every proper induced subgraph H of G. Clearly, G is connected. Since $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$, we obtain $\gamma_e(G) \geq 2$ and $\gamma(G) \geq 3$.

By Lemma 4, G is not K_3-free, that is, the girth of G is 3.

We proceed with a series of claims. Let F be the graph that is obtained from the triangle $x_1x_2x_3$ and the path $y_1y_2y_3$ by adding the edge x_1y_1.

Claim 1. *F is not an induced subgraph of G.*

Proof. Suppose that F is an induced subgraph of G. Since $\{x_1, y_2\}$ is not a dominating set of G, there is a vertex u at distance 2 from the set $\{x_1, y_2\}$ in G.

We proceed with three subclaims.

Claim 1.1. *The vertex u is not adjacent to x_2 or x_3.*

Proof. Suppose that u is adjacent to x_2. Since G is D-free, the vertex u is not adjacent to x_3, and, since G is B-free, u is adjacent to y_1. If u is not adjacent to y_3, then $G[\{u, x_1, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If u is adjacent to y_3, then $G[\{u, x_1, x_2, y_1, y_2, y_3\}] = P_2 \,\Box\, P_3$, which is a contradiction. Hence, by symmetry, we obtain that u is not adjacent to x_2 or x_3. $\square$

Claim 1.2. *The vertex u is not adjacent to y_1.*

Proof. Suppose that u is adjacent to y_1. Since G is F_1-free, the vertex u is adjacent to y_3. Since $\{x_1, y_3\}$ is not a dominating set of G, there is a vertex v at distance 2 from the set $\{x_1, y_3\}$. Suppose that v is adjacent to x_2. Since G is D-free, the vertex v is not adjacent to x_3, and, since G is B-free, v is adjacent to y_1. If v is adjacent to u, then $G[\{u, v, x_2, y_1, y_3\}] = B$, which is a contradiction. Hence, v is not adjacent to u, and, by symmetry, v is also not adjacent to y_2. Therefore, $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = F_4$, which is a contradiction. Thus, by symmetry, v is not adjacent to x_2 or x_3. Next, suppose that v is adjacent to y_1. Since G is F_1-free, the vertex v is adjacent to both u and y_2, which yields the contradiction $G[\{u, v, y_1, y_2\}] = D$. Thus, v is not adjacent to y_1. Suppose that v is adjacent to u. If v is adjacent to y_2, then $G[\{u, v, y_1, y_2, y_3\}] = K_{2,3}$, which is a contradiction. If v is not adjacent to y_2, then $G[\{u, v, x_1, x_2, y_1, y_2\}] = F_1$, which is a contradiction. Thus, by symmetry, v is not adjacent to u or y_2, implying that v is at distance 2 from the set $\{u, x_1, x_2, x_3, y_1, y_2, y_3\}$.

Since the vertex v is at distance 2 from the set $\{x_1, y_3\}$ in G, there is a neighbor v' of v, that is adjacent to x_1 or to y_3 or to both x_1 and y_3. First, suppose that v' is not adjacent to x_1, implying that v' is adjacent to y_3. Suppose that v' is adjacent to y_1. If v' is adjacent to u, then $G[\{u, v', y_1, y_3\}] = D$, which is a contradiction. Thus, by symmetry, v' is adjacent to neither u nor y_2, implying that $G[\{u, v', y_1, y_2, y_3\}] = K_{2,3}$, which is a contradiction. Thus, v' is not adjacent to y_1. Since G is B-free, v' is not adjacent to u or y_2, which yields the contradiction $G[\{v, v', x_1, x_2, y_1, y_2, y_3\}] = P_7$. Therefore, v' is adjacent to x_1.

Since G is $\{B, K_4\}$-free, the vertex v' is not adjacent to x_2 or x_3. Since G is B-free, the vertex v' is not adjacent to y_1. Suppose that v' is not adjacent to y_3. If v' is not adjacent to u, then $G[\{u, v, v', x_1, x_2, y_1\}] = F_1$, which is a contradiction. Thus, by symmetry, v' is adjacent to both u and y_2, which yields the contradiction $G[\{u, v', y_1, y_2, y_3\}] = K_{2,3}$. Thus, v' is adjacent to y_3. Since G is F_1-free, the vertex v' is adjacent to both u and y_2, implying that $G[\{u, v, v', y_1, y_3\}] = B$, which is a contradiction. Therefore, u is not adjacent to y_1. □

Claim 1.3. *The vertex u is not adjacent to y_3.*

Proof. Suppose that u is adjacent to y_3. Since $\{x_1, y_3\}$ is not a dominating set of G, there is a vertex v at distance 2 from the set $\{x_1, y_3\}$ in G.

Suppose that v is adjacent to x_2. Since G is D-free, the vertex v is not adjacent to x_3. Since G is B-free, v is adjacent to y_1. If v is not adjacent to y_2, then $G[\{v, x_1, x_3, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If v is adjacent to y_2, then we get the contradiction $G[\{x_2, v, y_1, y_2, y_3\}] = B$. Therefore, by symmetry, v is not adjacent to x_2 or x_3.

Next, suppose that v is adjacent to y_1. If v is not adjacent to y_2, then $G[\{v, x_1, x_2, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If v is adjacent to y_2, then $G[\{v, x_1, y_1, y_2, y_3\}] = B$, which is a contradiction. Thus, v is not adjacent to y_1.

Next, suppose that v is adjacent to y_2. If v is not adjacent to u, then $G[\{u, v, x_1, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If v is adjacent to u, then $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = F_2$, which is a contradiction. Therefore, v is not adjacent to y_2, implying that v is at distance 2 from the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$.

If v is adjacent to u, then $G[\{u, v, x_1, x_2, y_1, y_2, y_3\}] = P_7$, which is a contradiction. Hence, v is not adjacent to u. Since the vertex v is at distance 2 from the set $\{x_1, y_3\}$ in G, there is a neighbor v' of v that is adjacent to x_1 or to y_3 or to both x_1 and y_3. Note that $v' \neq u$.

First, suppose that v' is not adjacent to x_1, implying that v' is adjacent to y_3. If v' is not adjacent to y_2, then analogous arguments as in Claim 1.1 and Claim 1.2 (with the vertex u replaced by the vertex v') show that y_3 is the only vertex in the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ that is adjacent to v'. This in turn implies that $G[\{v, v', x_1, x_2, y_1, y_2, y_3\}] = P_7$, which is a contradiction. Hence, v' is adjacent to y_2. If v' is adjacent to y_1, then $G[\{v', y_1, y_2, y_3\}] = D$, which is a contradiction. Thus, v' is not adjacent to y_1, implying that $G[\{v, v', y_1, y_2, y_3\}] = B$, which is a contradiction. Therefore, v' is adjacent to x_1.

Since G is $\{D, K_4\}$-free, the vertex v' is not adjacent to x_2 or x_3. If v' is adjacent to y_1, then $G[\{v, v', x_1, x_2, y_1\}] = B$, which is a contradiction. Thus, v' is not adjacent to y_1. If v' is not adjacent to y_2, then $G[\{v, v', x_1, x_2, y_1, y_2\}] = F_1$, which is a contradiction. Thus, v' is adjacent to y_2. If v' is not adjacent to y_3, then $G[\{v, v', x_1, x_2, y_2, y_3\}] = F_1$, which is a contradiction. Thus, v' is adjacent to y_3, implying that $G[\{v, v', y_1, y_2, y_3\}] = B$, which is a contradiction. Therefore, u is not adjacent to y_3. □

We return to the proof of Claim 1. By Claims 1.1, 1.2 and 1.3, the vertex u is at distance 2 from the set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$. Since the vertex u is at distance 2 from the set $\{x_1, y_2\}$ in G, there is a neighbor u' of u that is adjacent to x_1 or to y_2 or to both x_1 and y_2. First, suppose that u' is adjacent to x_1. Analogously as above, since G is $\{B, D, K_4\}$-free, the vertex u' is not adjacent to x_2, x_3 and y_1. If u' is not adjacent to y_2, then $G[\{u, u', x_1, x_2, y_1, y_2\}] = F_1$, which is a contradiction. Thus, u' is adjacent to y_2. If u' is adjacent to y_3, then $G[\{u, u', y_1, y_2, y_3\}] = B$, while, if u' is not adjacent to y_3, then $G[\{u, u', x_1, x_2, y_2, y_3\}] = F_1$. Since both cases produce a contradiction, we deduce that u' is not adjacent to x_1, implying that u' is adjacent to y_2. Since G is B-free, u' is not adjacent to y_1. If u' is not adjacent to y_3, then $G[\{u, u', x_1, y_1, y_2, y_3\}] = F_1$, which is a contradiction. If u' is adjacent to y_3, then $G[\{u, u', y_1, y_2, y_3\}] = B$, which is a contradiction. This completes the proof of Claim 1. □

Claim 2. *If C is an arbitrary triangle in G, then every vertex is within distance 2 from $V(C)$.*

Proof. Let $C : x_1x_2x_3$ be a triangle in G. Suppose that there is a vertex y_3 at distance 3 from $V(C)$ in G. Let $x_1y_1y_2y_3$ be a shortest path in G from y_3 to $V(C)$. Since G is $\{D, K_4\}$-free, the vertex y_1 is adjacent to neither x_2 nor x_3, implying that F is an induced subgraph of G, which contradicts Claim 1. □

Claim 3. *Every triangle contains at least one vertex of degree exactly 2 in G.*

Proof. Let $C : x_1x_2x_3$ be a triangle in G. Suppose that every vertex on C has degree at least 3 in G. Let $y_1, y_2, y_3 \in V(G) \setminus V(C)$ be neighbors of x_1, x_2, x_3, respectively. Since G is $\{D, K_4\}$-free, x_i is the only neighbor of y_i in $V(C)$ for $i \in [3]$. Since G is B-free, the vertices y_1, y_2 and y_3 induce a triangle C' in G. Suppose that there is a vertex $y \in V(G) \setminus (V(C) \cup V(C'))$ that is adjacent to a vertex on C, say x_1. Since G is $\{D, K_4\}$-free, x_1 is the only neighbor of y on C, and y is non-adjacent to some vertex y_j on C' with $j \in \{2, 3\}$, which implies the contradiction that $G[\{x_1, x_2, x_3, y, y_j\}] = B$. Hence, each vertex on C has degree exactly 3 in G. By symmetry, each vertex on C' has degree exactly 3 in G. Thus, $G = P_2 \,\square\, C_3$, implying that $\gamma(G) = \gamma_e(G) = 2$, which is a contradiction. This completes the proof of Claim 3. □

Claim 4. *Every triangle contains two vertices of degree exactly 2 in G.*

Proof. Let $C : x_1x_2x_3$ be a triangle in G and let $R = V(G) \setminus V(C)$. By Claim 3, the triangle C contains at least one vertex of degree exactly 2 in G. Renaming vertices if necessary, we may assume that x_1 has degree 2 in G. Suppose that both x_2 and x_3 have degree at least 3 in G. Since G is D-free, the vertices x_2 and x_3 have no common neighbor in R. Further, since G is B-free, every neighbor of

x_2 in R is adjacent to every neighbor of x_3 in R. Hence, since G is $\{D, K_{2,3}\}$-free, the degrees of x_2 and x_3 are exactly 3 in G. Let y_2 and y_3 in R be neighbors of x_2 and x_3, respectively. Recall that $\gamma(G) \geq 3$. Let w_2 be a vertex not dominated by $\{x_2, y_3\}$, and let w_3 be a vertex not dominated by $\{x_3, y_2\}$. By Claim 2, the vertex w_2 is within distance 2 from $V(C)$, implying that w_2 is adjacent to y_2. Analogously, the vertex w_3 is adjacent to y_3. Note that $w_2 \neq w_3$. If w_2 is adjacent to w_3, then $G[\{w_2, w_3, x_2, x_3, y_2, y_3\}] = P_2 \,\Box\, P_3$. If w_2 is not adjacent to w_3, then $G[\{w_2, w_3, x_1, x_2, y_2, y_3\}] = F_1$. Both cases produce a contradiction, which completes the proof of Claim 4. $\square$

Let $C : x_1x_2x_3$ be a triangle in G. By Claim 4, we may assume, renaming vertices if necessary, that x_2 and x_3 have degree 2 in G. Since $\gamma(G) \geq 3$, the vertex x_1 does not dominate $V(G)$. Let $D_2 = V(G) \setminus N_G[x_1]$. Claim 2 implies that every vertex in D_2 is at distance exactly 2 from x_1 in G. Let D_1 be the set of neighbors in $V(G) \setminus D_2$ of the vertices in D_2. Note that $D_1 \subset N_G(x_1)$. By Claim 4, the set D_1 is independent.

Claim 5. *Every vertex in D_2 has exactly one neighbor in D_1.*

Proof. Since D_1 is an independent set, and, since G is $K_{2,3}$-free, every vertex in D_2 has at most two neighbors in D_1. Suppose that a vertex w_1 in D_2 has two neighbors y_1, y_2 in D_1. Since $\{x_1, y_1\}$ is not a dominating set of G, there is a vertex $w_2 \in D_2$ that is not adjacent to y_1.

Claim 5.1. *The vertex w_2 is not adjacent to y_2.*

Proof. Suppose that w_2 is adjacent to y_2. Since $\{x_1, y_2\}$ is not a dominating set, there is a vertex w_3 in D_2 that is not adjacent to y_2. Suppose that w_3 is adjacent to y_1. If w_3 is not adjacent to w_2, then $G[\{x_1, x_2, y_1, y_2, w_2, w_3\}] = F_1$, which is a contradiction. Hence, w_3 is adjacent to w_2. If w_3 is adjacent to w_1, then, since G is D-free, w_1 is not adjacent to w_2, implying that $G[\{x_1, y_1, w_1, w_2, w_3\}] = B$, which is a contradiction. Thus, w_3 is not adjacent to w_1. If w_1 is not adjacent to w_2, then $G[\{x_1, x_2, y_1, w_1, w_2, w_3\}] = F_1$, while, if w_1 is adjacent to w_2, then $G[\{x_1, y_2, w_1, w_2, w_3\}] = B$. Since both cases produce a contradiction, we deduce that w_3 is not adjacent to y_1. Since G is $\{D, P_2 \,\Box\, P_3\}$-free, the vertex w_3 is therefore adjacent to at most one of w_1 and w_2.

Let y_3 be a neighbor of w_3 in D_1. As observed earlier, every vertex in D_2 has at most two neighbors in D_1. In particular, w_1 is not adjacent to y_3. If w_3 is not adjacent to w_1, then $G[\{x_1, x_2, y_1, y_3, w_1, w_3\}] = F_1$, which is a contradiction. Thus, w_3 is adjacent to w_1, implying that w_3 is not adjacent to w_2. If w_2 is not adjacent to y_3, then $G[\{x_1, x_2, y_2, y_3, w_2, w_3\}] = F_1$, which is a contradiction. Hence, w_2 is adjacent to y_3. If w_1 and w_2 are not adjacent, then $G[\{x_1, y_1, y_2, y_3, w_1, w_2\}] = P_2 \,\Box\, P_3$, which is a contradiction. Hence, w_1 and w_2

are adjacent, implying that $G[\{x_1, y_2, w_1, w_2, w_3\}] = B$, which is a contradiction. Therefore, w_2 is not adjacent to y_2. $\square$

Recall that w_2 is not adjacent to y_1. By Claim 5.1, the vertex w_2 is not adjacent to y_2. Let y_4 be a neighbor of w_2 in D_1. Since every vertex in D_2 has at most two neighbors in D_1, the vertex w_1 is not adjacent to y_4. If w_1 is not adjacent to w_2, then $G[\{x_1, x_2, y_1, y_4, w_1, w_2\}] = F_1$, which is a contradiction. Hence, w_1 is adjacent to w_2. As $\{x_1, w_1\}$ is not a dominating set of G, there is a vertex w_4 in D_2 that is not adjacent to w_1. Since G is F_1-free, the vertex w_4 is adjacent to y_1 or to y_2 or to both y_1 and y_2. If w_4 is adjacent to y_1 and y_2, then $G[\{x_1, y_1, y_2, w_1, w_4\}] = K_{2,3}$, which is a contradiction. Hence, by symmetry, we may assume that w_4 is adjacent to y_1, but not to y_2. If w_4 is adjacent to w_2, then we get the contradiction $G[\{x_1, y_1, y_2, w_1, w_2, w_4\}] = P_2 \square P_3$. If w_4 is not adjacent to w_2, then $G[\{x_1, x_2, y_1, y_4, w_2, w_4\}] = F_1$, which is a contradiction, and completes the proof of Claim 5. $\square$

Let $D_1 = \{y_1, \ldots, y_k\}$, and, for $i \in [k]$, let w_i be a neighbor of y_i in D_2. If $k = 1$, then $\{x_1, y_1\}$ is a dominating set of G, which is a contradiction. Hence, $k \geq 2$. By Claim 5, the vertex y_i is the only neighbor of w_i in D_1 for $i \in [k]$. Since G is F_1-free, the vertices $w_1, \ldots, w_k$ induce a clique in G. Thus, by Claim 4, we obtain $k \leq 2$. This implies $k = 2$. Since G is F_1-free, each neighbor of y_i in D_2 is adjacent to every neighbor of y_{3-i} in D_2 for $i \in [2]$. If y_1 and y_2 both have only one neighbor in D_2, then $\{x_1, w_1\}$ is a dominating set of G, which is a contradiction. Hence, by symmetry, we may assume that the vertex y_1 has two neighbors w_1 and w_1' in D_2. Both w_1 and w_1' are adjacent to w_2. Since G is D-free, w_1 and w_1' are not adjacent. Since $\{x_1, w_2\}$ is not a dominating set of G, the vertex y_2 has a neighbor w_2' in D_2 that is different from w_2 and not adjacent to w_2. Thus, $G[\{w_1, w_1', w_2, w_2', y_1\}] = K_{2,3}$, which is a contradiction, and completes the proof of Theorem 1. $\blacksquare$

We close with a number of conjectures.

Conjecture 5. *There is a finite set $\mathcal{F}$ of graphs such that some graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\mathcal{F}$-free.*

Conjecture 6. *The set $\mathcal{F}$ in Conjecture* 5 *can be chosen such that $\gamma(F) = 3$ and $\gamma_e(F) = 2$ for every graph F in $\mathcal{F}$.*

Our proof of Theorem 1 actually implies that every component of a graph that is $\{B, D, K_4, K_{2,3}, P_2 \square P_3\} \cup \{P_7, C_7, F_1, \ldots, F_5\}$-free has domination number at most 2. Wolk [9] showed that the largest hereditary class of graphs for which every component has domination number 1 is the class of $\{P_4, C_4\}$-free graphs. A complete characterization of the largest hereditary class of graphs for which every component has domination number at most 2 in terms of minimal forbidden induced subgraphs seems to be a challenging and interesting problem, to which our results indirectly contribute.

Similar to the definition of an exponential dominating set, Dankelmann *et al.* [5] define a set D of vertices of a graph G to be a *porous exponential dominating set* of G if $w^*_{(G,D)}(u) \geq 1$ for every vertex u of G, where $w^*_{(G,D)}(u) = \sum_{v\in D}\left(\frac{1}{2}\right)^{\text{dist}_G(u,v)-1}$. They define the *porous exponential domination number* $\gamma_e^*(G)$ of G as the minimum size of a porous exponential dominating set of G. Clearly, $\gamma_e^*(G) \leq \gamma_e(G) \leq \gamma(G)$ for every graph G.

Conjecture 7. *A graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if $\gamma(H) = \gamma_e^*(H)$ for every induced subgraph H of G.*

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