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THE CROSSING NUMBER OF JOIN OF THE GENERALIZED PETERSEN GRAPH $P(3,1)$ WITH PATH AND CYCLE [1]

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Abstract

There are only few results concerning the crossing numbers of join of some graphs. In this paper, the crossing numbers of join products for the generalized Petersen graph $P(3,1)$ with n isolated vertices as well as with the path P_n on n vertices and with the cycle C_n are determined.

Keywords: crossing number, drawing, join product, generalized Petersen graph.

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1. Introduction

For graph theory terminology not defined here, we direct the reader to [1]. A drawing of a graph $G = (V, E)$ is a mapping ϕ that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, not passing through the image of any other vertex. For simplicity, we impose the following conditions on a drawing: (a) no three edges have an interior point in common, (b) if two edges share an interior point p, then they cross at p, and (c) any two edges of a drawing have only a finite number of crossings (common interior points). The *crossing number*, $cr(G)$, of a graph G is the minimum number of edge crossings in any drawing of G. Let D be a drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. For more about crossing number, we refer the reader to [2] and the references therein.

Let nK_1 denote the graph on n isolated vertices and let P_n and C_n be the path and the cycle on n vertices, respectively. The generalized Petersen graph $P(k, 1)$ for $k \geq 3$ is a graph consisting of an inner cycle C_k and an outer cycle C_k with corresponding vertices in the inner and outer cycles connected with edges. In other words, $P(k, 1)$ is isomorphic to the Cartesian product of C_k with P_2. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

The investigation on the crossing number of a graph is a classical and however very difficult problem (for example, see [2]). In fact, computing the crossing number of a graph is NP-complete [3], and the exact values are known only for very restricted classes of graphs. The join product of two graphs is one of them. Kulli and Muddebihal [4] gave the characterization of all pairs of graphs whose join is a planar graph. It thus seems natural to inquire about crossing numbers of join product of graphs. Very recently, some results concerning crossing numbers for join products of graphs were obtained. Using Kleitman's result [5], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [6, 7]. Moreover, the exact values for crossing numbers of $G + nK_1$ and $G + P_n$ for all graphs G of order at most four were given in [8]. The crossing numbers of the graphs $G + nK_1$ and $G + P_n$ were also known for very few graphs G of order five and six, see [9, 10].

The crossing numbers of the Cartesian product of the graph $P(3, 1)$ with P_n were determined in [11]. In this contribution, we determine the crossing numbers for the join of the graph $P(3, 1)$ with nK_1 in Section 3. This result enables us, in

Section 4 and 5, to give the crossing numbers of $P(3,1) + P_n$ and $P(3,1) + C_n$. In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs [5]. More precisely, he proved that if $m \leq 6$, then

$$cr(K_{m,n}) = Z(m,n),$$

where $Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

The following formulas, which can be shown easily, are usually used in the proofs of our results.

$$cr_D(A \cup B) = cr_D(A) + cr_D(B) + cr_D(A,B), \tag{1.1}$$

$$cr_D(A, B \cup C) = cr_D(A,B) + cr_D(A,C), \tag{1.2}$$

where A, B and C are mutually disjoint subsets of E.

2. Some Definitions and Lemmas

The graph $P(3,1)$ consists of two 3-cycles, denoted by C_3', C_3'', respectively, and of three independent edges joining the cycles C_3' and C_3''. The graph $P(3,1) + nK_1$ in Figure 1 consists of one copy of the graph $P(3,1)$ and n vertices $z_1, z_2, \ldots, z_n$, where every vertex z_i is adjacent to every vertex of $P(3,1)$. Let for $i = 1, 2, \ldots, n$, $E(z_i)$ denote the subgraph induced by six edges incident with the vertex z_i. For convenience, we shall call the edges of C_3' and C_3'' blue, the edges joining the cycles C_3' and C_3'' red, and the edges of $E(z_i)$, $i = 1, 2, \ldots, n$, black.

For the simpler labelling, let H_n denote the graph $P(3,1)+nK_1$ in this paper. In Figure 1 one can easily see that

$$P(3,1) + nK_1 = H_n = P(3,1) \cup K_{6,n} = P(3,1) \cup \left(\bigcup_{i=1}^{n} E(z_i) \right). \tag{2.1}$$

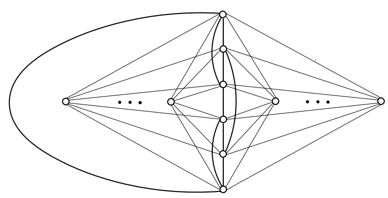

Figure 1. The drawing of the graph $P(3,1) + nK_1$.

Lemma 1. *Let D be an optimal drawing of the graph $P(3,1) + nK_1$, then the following properties are satisfied.*

(1) *Red edges do not cross each other in D;*
(2) *Blue edges do not cross each other in D.*

Proof. (1) If there are two red edges which cross each other, as shown in Figure 2(a), then such a crossing can be removed without introducing additional crossings into the drawing D, see Figure 2(b). It is not difficult to show that the modified drawing is still a good drawing of $P(3,1)+nK_1$, which contradicts our assumption of the drawing D.

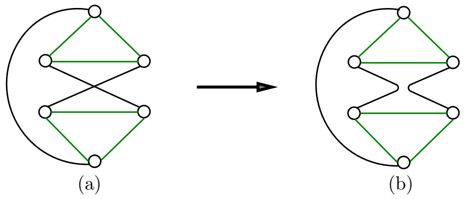

Figure 2. Removing the crossings between red edges.

(2) As in any good drawing the edges of a 3-cycle are pairwise non-crossing, it remains to show that the edges of C_3' and C_3'' do not cross each other. We will prove it by using reduction to absurdity. One can easily verify that all the possible subdrawings of $C_3' \cup C_3''$ are illustrated in Figure 3, if the edges of C_3' and C_3'' cross each other in D. And the three red edges can only be the following six cases: (14)(26)(35), (14)(25)(36), (15)(24)(36), (15)(26)(34), (16)(24)(35), (16)(25)(34).

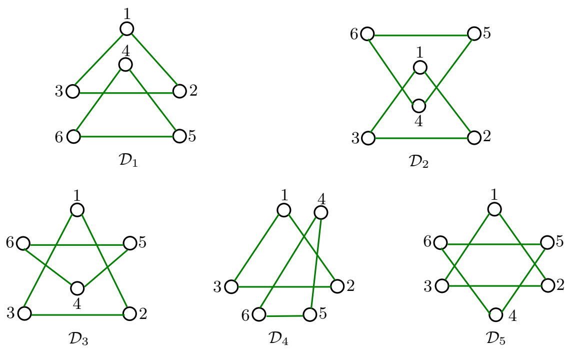

Figure 3. The possible subdrawings of $C_3' \cup C_3''$, if $cr_D(C_3', C_3'') \neq 0$.

If the three red edges are (14)(26)(35), one can modify the subdrawing $\mathcal{D}_i$, $i=1,2,\ldots,5$, to obtain a new drawing, as is shown in Figure 4(a), Figure 5(a), Figure 6(a), Figure 7(c) and Figure 8(b), respectively.

If the three red edges are (14)(25)(36), one can modify the subdrawing $\mathcal{D}_i$, $i=1,2,\ldots,5$, to obtain a new drawing, as is shown in Figure 4(b), Figure 5(d), Figure 6(b), Figure 7(b) and Figure 8(a), respectively.

If the three red edges are (15)(24)(36), one can modify the subdrawing $\mathcal{D}_i$, $i = 1, 2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(a), Figure 5(c), Figure 6(a), Figure 7(a) and Figure 8(b), respectively.

If the three red edges are (15)(26)(34), one can modify the subdrawing $\mathcal{D}_i$, $i = 1, 2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(b), Figure 5(b), Figure 6(c), Figure 7(b) and Figure 8(c), respectively.

If the three red edges are (16)(24)(35), one can modify the subdrawing $\mathcal{D}_i$, $i = 1, 2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(c), Figure 5(b), Figure 6(b), Figure 7(d) and Figure 8(c), respectively.

If the three red edges are (16)(25)(34), one can modify the subdrawing $\mathcal{D}_i$, $i = 1, 2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(d), Figure 5(a), Figure 6(d), Figure 7(c) and Figure 8(d), respectively.

It is not difficult to show that these subdrawings which are modified as above ways are still good drawings of $P(3, 1)$, and the crossings are reduced at least one, which contradicts the optimality of D. ■

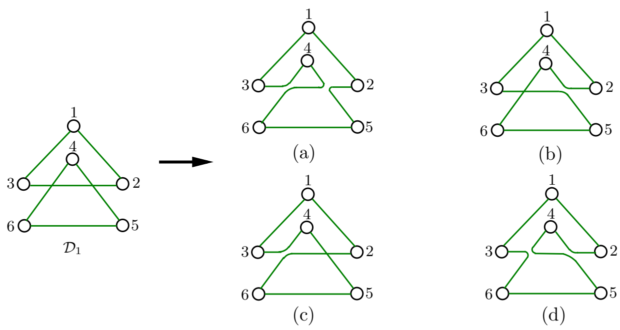

Figure 4. Removing the crossings between blue edges.

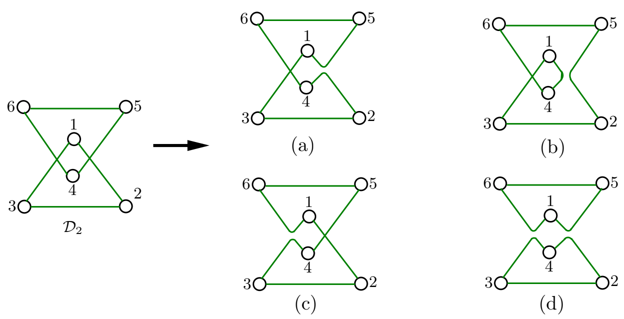

Figure 5. Removing the crossings between blue edges.

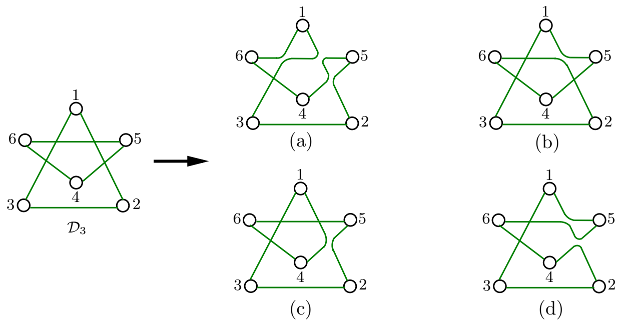

Figure 6. Removing the crossings between blue edges.

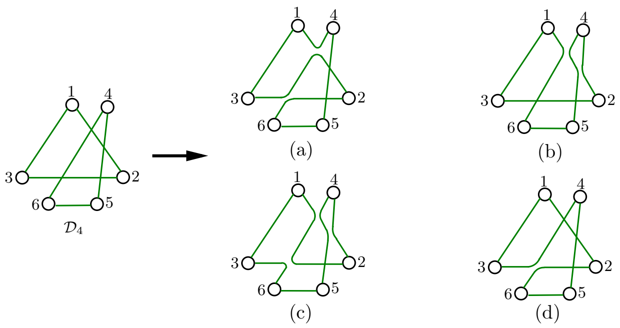

Figure 7. Removing the crossings between blue edges.

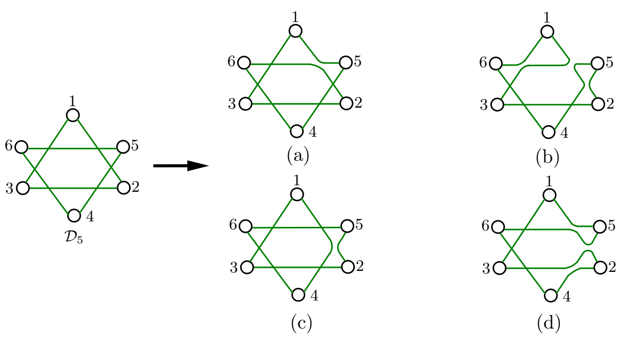

Figure 8. Removing the crossings between blue edges.

Remark 2. It is easily seen that the conclusion of Lemma 1 also applies to the graphs $P(3,1)+P_n$ and $P(3,1)+C_n$.

Lemma 3. *Let D be an optimal drawing of the graph $P(3,1)+nK_1$. Then all the possible subdrawings of $P(3,1)$ induced by D are that shown in Figure 9.*

Proof. By Lemma 1, it is not difficult to show that the claim follows, and the details are left to the reader. ■

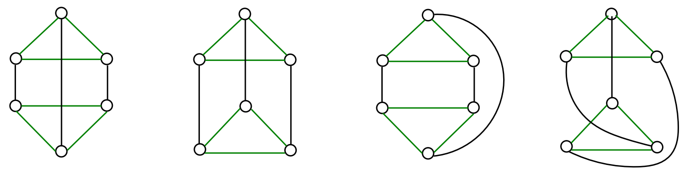

Figure 9. The possible subdrawings of $P(3,1)$ in the optimal drawing $P(3,1)+nK_1$.

Remark 4. It is easily seen that the conclusion of Lemma 3 also applies to the graph $P(3,1)+P_n$ and $P(3,1)+C_n$.

Lemma 5. $cr(P(3,1)+K_1)=2$.

Proof. A suitable subdrawing of $P(3,1)+K_1$ induced from the drawing of $P(3,1)+nK_1$ in Figure 1 shows that its crossing number is at most 2. To prove the reverse inequality we assume that there is a drawing of the graph $P(3,1)+K_1$ with fewer than two crossings and let D be such a drawing. As the graph $P(3,1)+K_1$ contains a subdivision of the complete bipartite graph $K_{3,3}$ with $cr(K_{3,3})=1$, and therefore the drawing D contains exactly one crossing. By Lemma 1, the red edges do not cross each other in D, that is to say, one of blue or black edge must be crossed. There is a contradiction since removing any blue or black edge of the graph $P(3,1)+K_1$ results in a graph containing a subdivision of $K_{3,3}$. ■

Lemma 6. $cr(P(3,1)+2K_1)=4$.

Proof. A suitable subdrawing of $P(3,1)+2K_1$ induced from the drawing of $P(3,1)+nK_1$ in Figure 1 shows that its crossing number is at most 4. Assume now that there is a drawing D of the graph $P(3,1)+2K_1$ with fewer than four crossings. By Lemma 1, the red edges do not cross each other in any optimal drawing of $P(3,1)+2K_1$, and hence, $cr_D(P(3,1)+2K_1)=3$ since removing any blue or black edge of the graph $P(3,1)+2K_1$ results in a graph containing a subdivision of $K_{3,4}$ with $cr(K_{3,4})=2$.

We claim that at least one of the three crossings in D does not appear on black edges, since deleting any three black edges from the graph $P(3,1)+2K_1$ results in a graph containing $K_{3,3}$ as a subgraph. Thus, $cr_D(P(3,1)) \geq 1$ and by Lemma 3, the subdrawing of $P(3,1)$ must be drawn as one of D_1, D_2 and D_4 in Figure 9. It is not difficult to find that the blue edges cross the red edges at least once, and the black edges must be crossed at least once. However, one can easily verify that the deleting of any two edges which one is blue and other one is black from the graph $P(3,1)+2K_1$ results in a graph containing $K_{3,4}$ as a subgraph, a contradiction. ■

Lemma 7. *Let D be a good drawing of the graph $P(3,1)+nK_1$ in which for some $i \in \{1,2,\ldots,n\}$, and for all $j=1,2,\ldots,n$, $j \neq i$, $cr_D(P(3,1) \cup E(z_i), E(z_j)) \geq 5$. If $cr_D(P(3,1) \cup E(z_i), E(z_j)) > 5$ for k different subgraphs $E(z_j)$, then $cr_D(P(3,1) + nK_1) \geq Z(6,n)+2n+k$.*

Proof. Without loss of generality, assume that the edges of $P(3,1) \cup E(z_1)$ are crossed in D at least five times by the edges of every subgraph $E(z_j)$, $j=2,3,\ldots,n$, and that k of the subgraphs $E(z_j)$ cross the edges of $P(3,1) \cup E(z_1)$ more than five times. As $H_n = K_{6,n-1} \cup P(3,1) \cup E(z_1)$ and $P(3,1) \cup E(z_1) = P(3,1)+K_1$, by (1.1) (1.2) and Lemma 5, we have

$$\begin{aligned} cr_D(H_n) &= cr_D\left(\bigcup_{j=2}^{n} E(z_j)\right) + cr_D(P(3,1) \cup E(z_1)) \\ &+ \sum_{j=2}^{n} cr_D\left(E(z_j), P(3,1) \cup E(z_1)\right) \\ &\geq Z(6,n-1) + 2 + 5(n-1) + k \geq Z(6,n) + 2n + k, \end{aligned}$$

as desired. ■

The proofs of the main results in Section 5 are based on the next lemma which was proved in [6].

Lemma 8. *Let D be a good drawing of $mK_1 + C_n$, $m \geq 2$, $n \geq 3$, in which no edge of C_n is crossed, and C_n does not separate the other vertices of the graph. Then, for all $z_i, z_j \in V(mK_1)$, $z_i \neq z_j$, two subgraphs $E(z_i)$ and $E(z_j)$ cross each other in D at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times.*

3. The Crossing Numbers of $P(3,1)+nK_1$

Theorem 9. $cr(P(3,1)+nK_1) = Z(6,n)+2n$.

Proof. The drawing in Figure 1 shows that $cr(P(3,1)+nK_1) \leq Z(6,n)+2n$ and that the theorem is true if the equality holds. We prove the reverse inequality by induction on n. By Lemmas 5 and 6, the theorem is true for $n = 1, 2$. Suppose now that for $n \geq 3$

$$cr(H_{n-2}) \geq Z(6, n-2) + 2(n-2), \tag{3.1}$$

and consider such an optimal drawing D of H_n that

$$cr_D(H_n) \leq Z(6,n) + 2n - 1. \tag{3.2}$$

The following claim is critical.

Claim 10. *$cr_D(E(z_i), E(z_j)) \geq 1$ for all $i, j = 1, 2, \ldots, n, i \neq j$.*

Proof. Assume that there are at least two different subgraphs $E(z_i)$ and $E(z_j)$ that do not cross each other in D. Without loss of generality, let $cr_D(E(z_1), E(z_2)) = 0$. Let $x_i, x'_i, y_i, i = 1, 2, 3$ denote the number of crossings between the nine edges of $P(3,1)$ and $E(z_1) \cup E(z_2)$, respectively (see Figure 10). It is clear that $cr_D(P(3,1), E(z_1) \cup E(z_2)) = \sum_{i=1}^{3}(x_i + x'_i + y_i)$.

It is not a difficult task to show that there is at least one crossing between the edges of each 3-cycle and $E(z_1) \cup E(z_2)$. Thus, it follows that

$$x_1 + x_2 + x_3 \geq 1,$$
$$x'_1 + x'_2 + x'_3 \geq 1.$$

By implication,

$$\sum_{i=1}^{3} \left(x_i + x'_i\right) = 2 + \alpha, \tag{3.3}$$

where $\alpha \geq 0$.

On the other hand, it is not a difficult task to show that there are at least two crossings between the edges of each 4-cycle and $E(z_1) \cup E(z_2)$. Thus, it follows that

$$x_1 + x'_1 + y_1 + y_2 \geq 2,$$
$$x_2 + x'_2 + y_2 + y_3 \geq 2,$$
$$x_3 + x'_3 + y_1 + y_3 \geq 2.$$

By implication,

$$\sum_{i=1}^{3} \left(x_i + x'_i\right) + 2\sum_{i=1}^{3} y_i = 6 + \beta, \tag{3.4}$$

where $\beta \geq 0$.

Therefore, by combining (3.3) and (3.4), we have

$$\sum_{i=1}^{3}\big(x_i + x'_i + y_i\big) = 4 + \frac{1}{2}(\alpha + \beta) \geq 4,$$

which implies that $cr_D(P(3,1), E(z_1) \cup E(z_2)) \geq 4$.

For $3 \leq i \leq n$, $E(z_1) \cup E(z_2) \cup E(z_i)$ is isomorphic to $K_{3,6}$. Hence, by (1.1), (1.2) and the assumptions we have $cr_D(E(z_1) \cup E(z_2), E(z_i)) \geq cr(K_{3,6}) = 6$.

Then, by (1.1), (1.2) and (3.1) we have

$$\begin{aligned} cr_D(H_n) &= cr_D(H_{n-2}) + cr_D(E(z_1) \cup E(z_2)) + cr_D(P(3,1), E(z_1) \cup E(z_2)) \\ &\quad + \sum_{i=3}^{n} cr_D(E(z_i), E(z_1) \cup E(z_2)) \\ &\geq Z(6, n-2) + 2(n-2) + 4 + 6(n-2) = Z(6,n) + 2n, \end{aligned}$$

which contradicts with (3.2). This proves the claim. □

Now we continue with the proof of the theorem. From (1.1) and (1.2) it follows that

$$cr_D(H_n) = cr_D(P(3,1)) + cr_D\left(\bigcup_{i=1}^{n} E(z_i)\right) + \sum_{i=1}^{n} cr_D(P(3,1), E(z_i)). \tag{3.5}$$

Since $\bigcup_{i=1}^{n} E(z_i)$ is isomorphic to $K_{6,n}$, we have $cr_D\left(\bigcup_{i=1}^{n} E(z_i)\right) \geq Z(6,n)$. Hence, by (3.2) and (3.5) we get

$$cr_D(P(3,1)) + \sum_{i=1}^{n} cr_D(P(3,1), E(z_i)) \leq 2n - 1. \tag{3.6}$$

Therefore $cr_D(P(3,1), E(z_i)) \leq 1$ for some $1 \leq i \leq n$. Without loss of generality, we assume that $cr_D(P(3,1), E(z_1)) \leq 1$ and let $F = P(3,1) \cup E(z_1)$. There are two cases to be considered: *Case* 1. $cr_D(P(3,1), E(z_1)) = 0$ and *Case* 2. $cr_D(P(3,1), E(z_1)) = 1$.

Case 1. $cr_D(P(3,1), E(z_1)) = 0$. From $cr_D(P(3,1), E(z_1)) = 0$ we can conclude that the subdrawing of $P(3,1)$ induced by D has a region with all vertices of $P(3,1)$ on its boundary. From Lemma 3, the subdrawing of $P(3,1)$ must be D_1 in Figure 9, and F must be drawn as in Figure 11.

If z_i for $2 \leq i \leq n$ lies in any region being not marked with ★, we can check that

$$cr_D(F, E(z_i)) \geq 5.$$

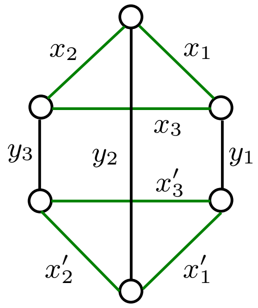

Figure 10. Marking the numbers of crossings for the graph $P(3,1)$.

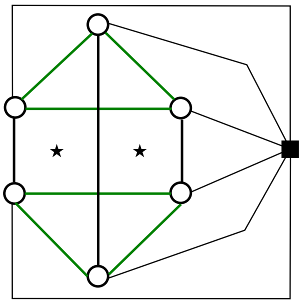

Figure 11. The subdrawings of F in the drawing of $P(3,1)+nK_1$.

If z_i for $2 \le i \le n$ lies in any region marked with ★, we have

$$cr_D(P(3,1), E(z_i)) \ge 4,$$

since there are four vertices of $P(3,1)$ which are not on the boundary of the region marked with ★ and the boundary of this region is formed by the edges of $P(3,1)$. By Claim 10, it follows that

$$cr_D(F, E(z_i)) = cr_D(P(3,1),\, E(z_i)) + cr_D(E(z_1),\, E(z_i)) \ge 4+1 = 5. \tag{3.7}$$

Therefore, we know that $cr_D(F, E(z_i)) \ge 5$ for all $i = 2, \dots, n$. From Figure 11, it is known that $cr_D(F) = 2$. Hence, by Lemma 7 we have $cr_D(H_n) \ge Z(6,n) + 2n$, this contradicts our assumption about the drawing D.

Case 2. $cr_D(P(3,1), E(z_1)) = 1$. For this case, there exists a region in the subdrawing of $P(3,1)$ induced by D such that its boundary contains at least 5 vertices of $P(3,1)$. By Lemma 3, the subdrawing of $P(3,1)$ must be drawn as one of D_1 and D_2 in Figure 9. For D_1, z_1 must be placed in the unique region with all vertices of $P(3,1)$ on its boundary. However, one can easily verify that, in this case, the edges of $E(z_1)$ cross the edges of $P(3,1)$ at least two times or 0 times. Hence, the subdrawing of $P(3,1)$ induced by D must be D_2, and the graph F must be drawn as D_2' or D_2'' in Figure 12. From now on, we make the following assumption on the subindex i, $2 \le i \le n$, in the rest of this Section.

We first consider D_2'. If z_i lies in any region which is not marked with ★, one can prove that

$$cr_D(F, E(z_i)) \ge 6.$$

If z_i lies in any region marked with ★, we have

$$cr_D(P(3,1), E(z_i)) \ge 3,$$

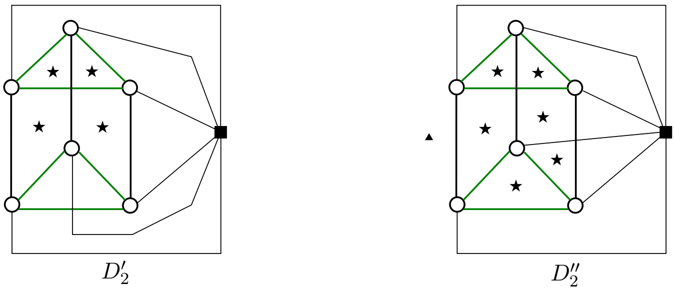

Figure 12. The possible subdrawings of F in the drawing of $P(3,1)+nK_1$.

since there are at least three vertices of $P(3,1)$ which are not on the boundary of the region marked with ★ and the boundary of this region is formed by the edges of $P(3,1)$. By Claim 10, it follows that

$$cr_D(F, E(z_i)) = cr_D(P(3,1), E(z_i)) + cr_D(E(z_1), E(z_i)) \geq 3+1 = 4. \tag{3.8}$$

Let l_1 be the number of vertices z_i which lies in the region marked with ★. Hence,

$$\sum_{i=1}^{n} cr_D(P(3,1), E(z_i)) \geq 3l_1 + (n-l_1), \tag{3.9}$$

since $cr_D(P(3,1), E(z_i)) \geq 1$ for $1 \leq i \leq n$. Thus, from (3.6) and $cr_D(P(3,1)) = 1$, it follows that $l_1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor$. The similar calculating as in the proof of Lemma 7 gives the following formula.

$$\begin{aligned} cr_D(H_n) &= cr_D\left(\bigcup_{i=2}^{n} E(z_i)\right) + cr_D(F) + \sum_{i=2}^{n} cr_D(E(z_i), F) \\ &\geq Z(6, n-1) + 2 + 4l_1 + 6(n-1-l_1) = Z(6, n-1) + 6n - 2l_1 - 4 \\ &\geq Z(6, n-1) + 6n - 2\left\lfloor \frac{n-2}{2} \right\rfloor - 4 \geq Z(6,n) + 2n. \end{aligned}$$

This contradiction completes the proof for D_2'.

Finally, we consider D_2''. If z_i lies in any region which is not marked with ★ and ▲, one can check that

$$cr_D(F, E(z_i)) \geq 6.$$

If z_i lies in any region marked with ★, then similarly to the proof of the claim of (3.8), one can prove that

$$cr_D(P(3,1), E(z_i)) \geq 3, \text{ and } cr_D(F, E(z_i)) \geq 4.$$

If z_i lies in any region marked with ▲, one can show that

$$cr_D(F, E(z_i)) \geq 5.$$

Moreover, it is easy to see that if $cr_D(F, E(z_i)) = 5$, then the edges of $E(z_i)$ cross the edges of $P(3,1)$ exactly once or three times.

Let k_1 be the number of vertices z_i which lies in the region marked with ★ and in the region marked with ▲ for which the edges of $E(z_i)$ cross the edges of $P(3,1)$ exactly three times. Let k_2 be the number of vertices z_i which lies in the region marked with ▲ for which $cr_D(F, E(z_i)) = 5$ and the edges of $E(z_i)$ cross the edges of $P(3,1)$ exactly once. Our next analysis depends on whether $k_2 = 0$ or not.

If $k_2 = 0$, then the similar calculating as in the proof of the case for D_2' results in a contradiction again, and the details are omited.

If $k_2 \geq 1$, then without loss of generality, assume that z_2 lies in the region marked with ▲ for which $cr_D(F, E(z_2)) = 5$ and the edges of $E(z_2)$ cross the edges of $P(3,1)$ exactly once. For the case, the unique subdrawing of $P(3,1) \cup E(z_1) \cup E(z_2)$ is shown in Figure 13. For simpler labelling, let $W = P(3,1) \cup E(z_1) \cup E(z_2)$. If z_i lies in any region which is not marked with ★ and ▲, one can verify that

$$cr_D(W, E(z_i)) \geq 11.$$

If z_i lies in any region marked with ★, then the similar discussion as in the proof of (3.8) gives that

$$cr_D(P(3,1), E(z_i)) \geq 3, \text{ and } cr_D(W, E(z_i)) \geq 5.$$

If z_i lies in any region marked with ▲, one can show that

$$cr_D(W, E(z_i)) \geq 9.$$

Moreover, it is easy to check that if $cr_D(W, E(z_i)) = 9$, then the edges of $E(z_i)$ cross the edges of $P(3,1)$ at least three times, otherwise $cr_D(W, E(z_i)) \geq 11$.

Let h_1 be the number of vertices z_i which lies in the region marked with ★. Let h_2 be the number of vertices z_i which lies in the region marked with ▲ and $cr_D(W, E(z_i)) = 9$. Note also that $cr_D(P(3,1),\ E(z_i)) = 1$, $i = 1,2$ and $cr_D(P(3,1)) = 1$; similarly to the proof of (3.9), we obtain that $h_1 + h_2 \leq \left\lfloor \frac{n-2}{2} \right\rfloor$. Thus, using similar arguments as in the proof of Lemma 7 and noting that $cr_D(W) = 7$, we can also get the following formula

$$\begin{aligned} cr_D(H_n) &= cr_D\left(\bigcup_{i=3}^{n} E(z_i)\right) + cr_D(W) + \sum_{i=3}^{n} cr_D(E(z_i), W) \\ &\geq Z(6, n-2) + 7 + 5h_1 + 9h_2 + 11(n - 2 - h_1 - h_2) \\ &= Z(6, n-2) + 11n - 6h_1 - 2h_2 - 15 \\ &\geq Z(6, n-2) + 11n - 6\left\lfloor \frac{n-2}{2} \right\rfloor - 15 > Z(6,n) + 2n. \end{aligned}$$

This contradiction completes the proof. ■

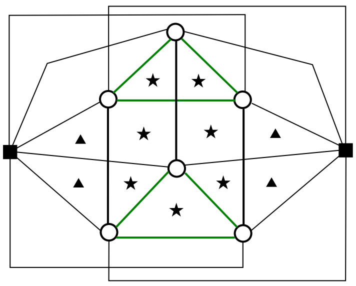

Figure 13. The subdrawings of W in the drawing of $P(3,1)+nK_1$.

4. The Crossing Numbers of $P(3,1)+P_n$

The graph $P(3,1)+P_n$ contains $P(3,1)+nK_1$ as a subgraph. For the subgraphs of the graph $P(3,1)+P_n$ which are also subgraphs of the graph $P(3,1)+nK_1$, we will use the same notation as before. Let P_n^* denote the path on n vertices of $P(3,1)+P_n$ not belonging to the subgraph $P(3,1)$. One can easily see that

$$P(3,1)+P_n = P(3,1)\cup K_{6,n}\cup P_n^* = P(3,1)\cup\left(\bigcup_{i=1}^{n} E(z_i)\right)\cup P_n^*. \tag{4.1}$$

The graph $P(3,1)+P_1$ is isomorphic to $P(3,1)+K_1$ and $cr(P(3,1)+K_1)=2$. For $n\geq 2$ we have the next result.

Theorem 11. *$cr(P(3,1)+P_n)=Z(6,n)+2n+1$, for $n\geq 2$.*

Proof. Figure 1 shows the drawing of the graph $P(3,1)+nK_1$ with $Z(6,n)+2n$ crossings. One can easily see that in this drawing it is possible to add $n-1$ edges which form the path P_n^* on the vertices of nK_1 in such a way that only one edge of P_n^* is crossed by an edge of $P(3,1)$. Hence, $cr(P(3,1)+P_n)\leq Z(6,n)+2n+1$. To prove the reverse inequality we assume that there is an optimal drawing of the graph $P(3,1)+P_n$ with fewer than $Z(6,n)+2n+1$ crossings and let D be such a drawing. Since the graph $P(3,1)+P_n$ contains $P(3,1)+nK_1$ as a subgraph, by Theorem 9, $cr(P(3,1)+P_n)=Z(6,n)+2n$ and therefore, no edge of the path P_n^* is crossed in D, which implies that all vertices z_i, $i=1,2,\ldots,n$, are placed in the same region of the subdrawing of $P(3,1)$ induced by D.

The similar argument as in the proof of (3.6) gives that

$$cr_D(P(3,1))+\sum_{i=1}^{n} cr_D(P(3,1),E(z_i))\leq 2n. \tag{4.2}$$

Therefore $cr_D(P(3,1),E(z_i))\leq 2$ for some $1\leq i\leq n$. Without loss of generality, we assume that $cr_D(P(3,1),E(z_1))\leq 2$ and let $F=P(3,1)\cup E(z_1)$. By the similar discussion as in the proof of Theorem 9, the following three cases are considered.

Case 1. $cr_D(P(3,1), E(z_1)) = 0$. One can easily verify that, in this case, in the subdrawing of $P(3,1)$ induced by D there are all six vertices of $P(3,1)$ on the boundary of one, say unbounded, region and, in D, all vertices z_i, $i = 1, 2, \ldots, n$, are placed in this region. From Remark 4, such unique subdrawing of $P(3,1)$ must be D_1 in Figure 9, and F is drawn as in Figure 11.

If z_i for $2 \le i \le n$ lies in the unbounded region of $P(3,1)$, we can check that

$$cr_D(F, E(z_i)) \ge 6.$$

From Figure 11, it is known that $cr_D(F) = 2$, which, together with Lemma 7, contradicts the assumption.

Case 2. $cr_D(P(3,1), E(z_1)) = 1$. By the similar analysis as in Case 2 of Section 3, we can know that the subdrawing of $P(3,1)$ induced by D must be D_2 in Figure 9, and the graph F must be drawn as one of D_2' and D_2'' in Figure 12. Moreover, all vertices z_i, $i = 2, 3, \ldots, n$, are placed in the same region of D_2 as the vertex z_1.

For D_2', it is not difficult to verify that

$$cr_D(F, E(z_i)) \ge 6,$$

for $2 \le i \le n$, if z_i lies in the same region of D_2 as the vertex z_1. Note also that $cr_D(F) = 2$, and the same contradiction can be obtained as in the above case.

For D_2'', One can easily verify that

$$cr_D(F, E(z_i)) \ge 5,$$

for $2 \le i \le n$, if z_i lies in the same region of D_2 as the vertex z_1. Note that $cr_D(F) = 2$, and it follows from Lemma 7 that $cr_D(P(3,1) + P_n) \ge Z(6, n-1) + 2 + 5(n-1) > Z(6,n) + 2n$ for even n, a contradiction.

For odd n, it follows that $cr_D(F, E(z_i)) = 5$ for $2 \le i \le n$, from Lemma 7. In addition, it is easy to see that if $cr_D(F, E(z_i)) = 5$, then the edges of $E(z_i)$ cross the edges of $P(3,1)$ exactly once or three times. Moreover, if $cr_D(E(z_i), P(3,1)) = 1$, then $cr_D(E(z_i), E(z_1)) = 4$, and if $cr_D(E(z_i), P(3,1)) = 3$, then $cr_D(E(z_i), E(z_1)) = 2$, for $2 \le i \le n$. Let α_1 be the number of vertices z_i, $2 \le i \le n$, with $cr_D(E(z_i), P(3,1)) = 1$ and $cr_D(E(z_i), E(z_1)) = 4$. Let α_2 be the number of vertices z_i, $2 \le i \le n$, with $cr_D(E(z_i), P(3,1)) = 3$ and $cr_D(E(z_i), E(z_1)) = 2$.

Using the fact that $cr_D(E(z_1), P(3,1)) = 1$ and $cr_D(P(3,1)) = 1$, together with (4.2) we have

$$\alpha_1 + 3\alpha_2 + 2 \le 2n,$$

which implies that $\alpha_1 \neq 0$. Assume, without loss of generality, that $cr_D(E(z_2), P(3,1)) = 1$ and $cr_D(E(z_2), E(z_1)) = 4$. For this case, the unique subdrawing of $P(3,1) \cup E(z_1) \cup E(z_2)$ is shown in Figure 13. To simplify the notation, let $W = P(3,1) \cup E(z_1) \cup E(z_2)$.

For $3 \leq i \leq n$, if z_i lies in the unique region of D_2 with five vertices of $P(3,1)$ on its boundary as the vertices z_1 and z_2, one can easily verify that

$$cr_D(W, E(z_i)) \geq 9.$$

Note that $cr_D(W) = 7$, it follows that

$$\begin{aligned} cr_D(P(3,1) + P_n) &= cr_D\left(\bigcup_{i=3}^{n} E(z_i)\right) + cr_D(W) + \sum_{i=3}^{n} cr_D\left(E(z_i), W\right) \\ &\geq Z(6, n-2) + 7 + 9(n-2) > Z(6,n) + 2n, \end{aligned}$$

a contradiction again.

Case 3. $cr_D(P(3,1), E(z_1)) = 2$. For this case, by (4.2), $cr_D(P(3,1), E(z_i)) = 2$ for all $i = 2, 3, \ldots, n$ and $cr_D(P(3,1)) = 0$. Up to the isomorphism, there is unique possible subdrawings of $P(3,1)$ induced by D as D_3 shown in Figure 9. Consider the symmetry of the drawing of D_3, suppose that all vertices z_i, $1 \leq i \leq n$, are placed in some one of three regions with four vertices of $P(3,1)$ on its boundary. The edges of $E(z_i)$ divide this region as shown in Figure 14(a) or (b). One can easily verify that $cr_D(E(z_j), E(z_i)) \geq 3$, for all $i, j = 1, 2, \ldots, n, j \neq i$, if the vertex z_j is placed in the same region of D_3 as the vertex z_i. Thus, in D there are at least $3\binom{n}{2} + 2n > Z(6,n) + 2n$ crossings, which contradicts the assumption. This completes the proof. ■

Figure 14. The possible placements of $E(z_i)$ inside the region with four vertices of $P(3,1)$ on its boundary.

5. THE CROSSING NUMBERS OF $P(3,1) + C_n$

The graph $P(3,1)+C_n$ contains both $P(3,1)+nK_1$ and $P(3,1)+P_n$ as a subgraph. Let C_n^* denote the subgraph of $P(3,1)+C_n$ induced on the vertices not belonging to the subgraph $P(3,1)$. For $i = 1, 2, \dots, 6$, let a_i denote the six vertices of $P(3,1)$, and $E(a_i)$ denote the subgraph induced by n edges of $K_{6,n}$ incident with the vertex a_i, respectively. One can easily see that

$$P(3,1) + C_n = P(3,1) \cup K_{6,n} \cup C_n^* = P(3,1) \cup \left(\bigcup_{i=1}^{n} E(z_i) \right) \cup C_n^*. \tag{5.1}$$

On the other hand, the graph $P(3,1) + C_n$ contains the graph $6K_1 + C_n^*$ as a subgraph and

$$P(3,1) + C_n = P(3,1) \cup \left(\bigcup_{i=1}^{6} E(a_i) \right) \cup C_n^*. \tag{5.2}$$

Theorem 12. *$cr(P(3,1) + C_n) = Z(6,n) + 2n + 3$, for $n \geq 3$.*

Proof. In the drawing in Figure 1 it is possible to add n edges in such a way that they, together with the vertices of nK_1, form the cycle C_n^* and that the edges of C_n^* are crossed only three times. Hence, $cr(P(3,1) + C_n) \leq Z(6,n) + 2n + 3$. To prove the reverse inequality, assume that there is an optimal drawing of the graph $P(3,1) + C_n$ with at most $Z(6,n) + 2n + 2$ crossings and let D be such a drawing. Since the graph $P(3,1) + C_n$ contains $P(3,1) + P_n$ as a subgraph, by Theorem 11, $cr_D(P(3,1) + C_n) = Z(6,n) + 2n + 1$ or $Z(6,n) + 2n + 2$, and by Theorem 9, in D there are at most two crossings on the edges of C_n^*, otherwise deleting the edges from C_n^* results in an drawing of the graph $P(3,1) + nK_1$ fewer than $Z(6,n) + 2n$ crossings.

We claim that the edges of C_n^* do not cross each other, otherwise one can modify the drawing in a sufficiently small neighborhood of the crossing point resulting in a new good drawing of $P(3,1) + C_n$ as shown in Figure 15, and the crossings are reduced at least one. As $P(3,1)$ is a 3-connected graph, all vertices of $P(3,1)$ are placed in the same region in the view of the subdrawing of C_n^* induced by D, otherwise in D there are at least three crossings on the edges of C_n^*. The edges of C_n^* are not crossed by the edges of $P(3,1)$, otherwise $cr_D(C_n^*, P(3,1)) \geq 2$ and $cr_D(E(a_i), C_n^*) = 0$ for all $i = 1, 2, \dots, 6$, and then, it follows from Lemma 8 that in D there are at least $\binom{6}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor > Z(6,n) + 2n + 2$ crossings. This implies that all vertices z_i, $i = 1, 2, \dots, n$, are placed in the same region in the view of the subdrawing of $P(3,1)$ induced by D.

We conclude that the edges of C_n^* are crossed at least once in D, otherwise it follows from Lemma 8 that in D there are at least $\binom{6}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor > Z(6,n) + 2n + 2$ crossings, a contradiction.

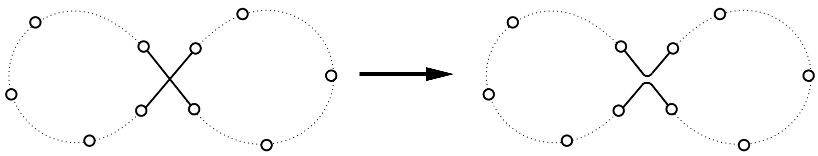

Figure 15. Removing the self-crossings on the edges C_n^*.

Claim 13.

$$\begin{aligned} cr_D(P(3,1)) + \sum_{i=1}^{6} cr_D(P(3,1), E(a_i)) &= cr_D(P(3,1)) + \sum_{i=1}^{n} cr_D(P(3,1), E(z_i)) \\ &\geq n+1. \end{aligned}$$

Proof. From Remark 4, we know that all the possible subdrawings of $P(3,1)$ induced by D are that shown in Figure 9. For D_3 and D_4, it is easily see that every region has at most four vertices of $P(3,1)$ on its boundary, which implies that $cr_D(P(3,1), E(z_i)) \geq 2$, for all $i = 1, 2, \ldots, n$, and the claim follows. For D_2, one can easily see that every region has at most five vertices of $P(3,1)$ on its boundary, which implies that $cr_D(P(3,1), E(z_i)) \geq 1$, for all $i = 1, 2, \ldots, n$. Note that $cr_D(P(3,1)) = 1$, the claim holds. For D_1, there is a region, say unbounded, with all vertices of $P(3,1)$ on its boundary, and other regions, say bounded, have at most two vertices of $P(3,1)$ on its boundary. If all vertices z_i, $i = 1, 2, \ldots, n$, are placed in some bounded region, then the claim holds. Suppose now that all vertices z_i, $i = 1, 2, \ldots, n$, are placed in this unbounded region, and there is at least one subgraph $E(z_i)$ which does not cross $P(3,1)$ in D. Note that the edges of C_n^* are crossed at least once. By the similar analysis as in Case 1 of Section 4, a contradiction appears. This completes the proof of claim. □

Suppose now that in D the edges of C_n^* are crossed exactly once. Without loss of generality, let $cr_D(E(a_1), C_n^*) = 1$. The simple modification of Lemma 8 for this case implies that $cr_D(E(a_1), E(a_i)) \geq \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor$ for $i = 2, 3, \ldots, 6$. And, for $i, j = 2, 3, \ldots, 6$, $i \neq j$, $cr_D(E(a_i), E(a_j)) \geq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ by Lemma 8. Thus, by (5.2) and Claim 13, we have

$$\begin{aligned} cr_D(P(3,1) + C_n) = \; & cr_D(P(3,1)) + \sum_{i=1}^{6} cr_D(P(3,1), E(a_i)) + cr_D\left(\bigcup_{i=1}^{6} E(a_i)\right) \\ & + cr_D\left(P(3,1) \cup \bigcup_{i=1}^{6} E(a_i), C_n^*\right) \geq n + 1 + \binom{5}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \\ & + 5 \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 1 > Z(6,n) + 2n + 2, \end{aligned}$$

a contradiction.

Finally, assume that in D the edges of C_n^* are crossed exactly two times. From (1.1), (1.2) and (5.1), it follows that

$$cr_D(P(3,1)) + \sum_{i=1}^{n} cr_D(P(3,1), E(z_i)) \leq 2n. \tag{5.3}$$

For the case, the similar discussion as in the proof of Theorem 11 gives a contradiction again, and the details are omitted. This completes the proof. ■

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