# THE CROSSING NUMBER OF JOIN OF THE GENERALIZED PETERSEN GRAPH $P(3,1)$ WITH PATH AND CYCLE ${ }^{1}$ 

ZhangDong Ouyang<br>Department of Mathematics Hunan First Normal University Changsha 410205, P.R. China<br>e-mail: oymath@163.com<br>Jing Wang<br>Department of Mathematics and Information Sciences<br>Changsha University<br>Changsha 410003, P.R. China<br>e-mail: wangjing1001@hotmail.com<br>AND<br>YuanQiu Huang<br>Department of Mathematics<br>Hunan Normal University<br>Changsha 410081, P.R. China<br>e-mail: hyqq@hunnu.edu.cn


#### Abstract

There are only few results concerning the crossing numbers of join of some graphs. In this paper, the crossing numbers of join products for the generalized Petersen graph $P(3,1)$ with $n$ isolated vertices as well as with the path $P_{n}$ on $n$ vertices and with the cycle $C_{n}$ are determined. Keywords: crossing number, drawing, join product, generalized Petersen graph.


2010 Mathematics Subject Classification: 05C10, 05C38.

[^0]
## 1. Introduction

For graph theory terminology not defined here, we direct the reader to [1]. A drawing of a graph $G=(V, E)$ is a mapping $\phi$ that assigns to each vertex in $V$ a distinct point in the plane and to each edge $u v$ in $E$ a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, not passing through the image of any other vertex. For simplicity, we impose the following conditions on a drawing: (a) no three edges have an interior point in common, (b) if two edges share an interior point $p$, then they cross at $p$, and (c) any two edges of a drawing have only a finite number of crossings (common interior points). The crossing number, $\operatorname{cr}(G)$, of a graph $G$ is the minimum number of edge crossings in any drawing of $G$. Let $D$ be a drawing of the graph $G$, we denote the number of crossings in $D$ by $\operatorname{cr}_{D}(G)$. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a good drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. For more about crossing number, we refer the reader to [2] and the references therein.

Let $n K_{1}$ denote the graph on $n$ isolated vertices and let $P_{n}$ and $C_{n}$ be the path and the cycle on $n$ vertices, respectively. The generalized Petersen graph $P(k, 1)$ for $k \geq 3$ is a graph consisting of an inner cycle $C_{k}$ and an outer cycle $C_{k}$ with corresponding vertices in the inner and outer cycles connected with edges. In other words, $P(k, 1)$ is isomorphic to the Cartesian product of $C_{k}$ with $P_{2}$. The join product of two graphs $G_{1}$ and $G_{2}$, denoted by $G_{1}+G_{2}$, is obtained from vertex-disjoint copies of $G_{1}$ and $G_{2}$ by adding all edges between $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$.

The investigation on the crossing number of a graph is a classical and however very difficult problem (for example, see [2]). In fact, computing the crossing number of a graph is NP-complete [3], and the exact values are known only for very restricted classes of graphs. The join product of two graphs is one of them. Kulli and Muddebihal [4] gave the characterization of all pairs of graphs whose join is a planar graph. It thus seems natural to inquire about crossing numbers of join product of graphs. Very recently, some results concerning crossing numbers for join products of graphs were obtained. Using Kleitman's result [5], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in $[6,7]$. Moreover, the exact values for crossing numbers of $G+n K_{1}$ and $G+P_{n}$ for all graphs $G$ of order at most four were given in [8]. The crossing numbers of the graphs $G+n K_{1}$ and $G+P_{n}$ were also known for very few graphs $G$ of order five and six, see $[9,10]$.

The crossing numbers of the Cartesian product of the graph $P(3,1)$ with $P_{n}$ were determined in [11]. In this contribution, we determine the crossing numbers for the join of the graph $P(3,1)$ with $n K_{1}$ in Section 3. This result enables us, in

Section 4 and 5 , to give the crossing numbers of $P(3,1)+P_{n}$ and $P(3,1)+C_{n}$. In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs [5]. More precisely, he proved that if $m \leq 6$, then

$$
c r\left(K_{m, n}\right)=Z(m, n)
$$

where $Z(m, n)=\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$.
The following formulas, which can be shown easily, are usually used in the proofs of our results.

$$
\begin{gather*}
c r_{D}(A \cup B)=c r_{D}(A)+c r_{D}(B)+c r_{D}(A, B),  \tag{1.1}\\
c r_{D}(A, B \cup C)=c r_{D}(A, B)+c r_{D}(A, C), \tag{1.2}
\end{gather*}
$$

where $A, B$ and $C$ are mutually disjoint subsets of $E$.

## 2. Some Definitions and Lemmas

The graph $P(3,1)$ consists of two 3 -cycles, denoted by $C_{3}^{\prime}, C_{3}^{\prime \prime}$, respectively, and of three independent edges joining the cycles $C_{3}^{\prime}$ and $C_{3}^{\prime \prime}$. The graph $P(3,1)+n K_{1}$ in Figure 1 consists of one copy of the graph $P(3,1)$ and $n$ vertices $z_{1}, z_{2}, \ldots, z_{n}$, where every vertex $z_{i}$ is adjacent to every vertex of $P(3,1)$. Let for $i=1,2, \ldots, n$, $E\left(z_{i}\right)$ denote the subgraph induced by six edges incident with the vertex $z_{i}$. For convenience, we shall call the edges of $C_{3}^{\prime}$ and $C_{3}^{\prime \prime}$ blue, the edges joining the cycles $C_{3}^{\prime}$ and $C_{3}^{\prime \prime}$ red, and the edges of $E\left(z_{i}\right), i=1,2, \ldots, n$, black.

For the simpler labelling, let $H_{n}$ denote the graph $P(3,1)+n K_{1}$ in this paper. In Figure 1 one can easily see that

$$
\begin{equation*}
P(3,1)+n K_{1}=H_{n}=P(3,1) \cup K_{6, n}=P(3,1) \cup\left(\bigcup_{i=1}^{n} E\left(z_{i}\right)\right) \tag{2.1}
\end{equation*}
$$



Figure 1. The drawing of the graph $P(3,1)+n K_{1}$.

Lemma 1. Let $D$ be an optimal drawing of the graph $P(3,1)+n K_{1}$, then the following properties are satisfied.
(1) Red edges do not cross each other in D;
(2) Blue edges do not cross each other in D.

Proof. (1) If there are two red edges which cross each other, as shown in Figure 2(a), then such a crossing can be removed without introducing additional crossings into the drawing $D$, see Figure 2(b). It is not difficult to show that the modified drawing is still a good drawing of $P(3,1)+n K_{1}$, which contradicts our assumption of the drawing $D$.


Figure 2. Removing the crossings between red edges.
(2) As in any good drawing the edges of a 3 -cycle are pairwise non-crossing, it remains to show that the edges of $C_{3}^{\prime}$ and $C_{3}^{\prime \prime}$ do not cross each other. We will prove it by using reduction to absurdity. One can easily verify that all the possible subdrawings of $C_{3}^{\prime} \cup C_{3}^{\prime \prime}$ are illustrated in Figure 3, if the edges of $C_{3}^{\prime}$ and $C_{3}^{\prime \prime}$ cross each other in $D$. And the three red edges can only be the following six cases: $(14)(26)(35),(14)(25)(36),(15)(24)(36),(15)(26)(34),(16)(24)(35),(16)(25)(34)$.


Figure 3. The possible subdrawings of $C_{3}^{\prime} \cup C_{3}^{\prime \prime}$, if $\operatorname{cr}_{D}\left(C_{3}^{\prime}, C_{3}^{\prime \prime}\right) \neq 0$.
If the three red edges are $(14)(26)(35)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(a), Figure 5(a), Figure 6(a), Figure 7(c) and Figure 8(b), respectively.

If the three red edges are $(14)(25)(36)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(b), Figure 5(d), Figure 6(b), Figure 7(b) and Figure 8(a), respectively.

If the three red edges are $(15)(24)(36)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(a), Figure 5(c), Figure 6(a), Figure 7(a) and Figure 8(b), respectively.

If the three red edges are $(15)(26)(34)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(b), Figure 5(b), Figure 6(c), Figure 7(b) and Figure 8(c), respectively.

If the three red edges are $(16)(24)(35)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(c), Figure 5(b), Figure 6(b), Figure 7(d) and Figure 8(c), respectively.

If the three red edges are $(16)(25)(34)$, one can modify the subdrawing $\mathcal{D}_{i}$, $i=1,2, \ldots, 5$, to obtain a new drawing, as is shown in Figure 4(d), Figure 5(a), Figure 6(d), Figure 7(c) and Figure 8(d), respectively.

It is not difficult to show that these subdrawings which are modified as above ways are still good drawings of $P(3,1)$, and the crossings are reduced at least one, which contradicts the optimality of $D$.


Figure 4. Removing the crossings between blue edges.


Figure 5. Removing the crossings between blue edges.



(c)

(b)

(d)

Figure 6. Removing the crossings between blue edges.


Figure 7. Removing the crossings between blue edges.

(c)

(b)

(d)

Figure 8. Removing the crossings between blue edges.

Remark 2. It is easily seen that the conclusion of Lemma 1 also applies to the graphs $P(3,1)+P_{n}$ and $P(3,1)+C_{n}$.

Lemma 3. Let $D$ be an optimal drawing of the graph $P(3,1)+n K_{1}$. Then all the possible subdrawings of $P(3,1)$ induced by $D$ are that shown in Figure 9.

Proof. By Lemma 1, it is not difficult to show that the claim follows, and the details are left to the reader.


Figure 9. The possible subdrawings of $P(3,1)$ in the optimal drawing $P(3,1)+n K_{1}$.

Remark 4. It is easily seen that the conclusion of Lemma 3 also applies to the graph $P(3,1)+P_{n}$ and $P(3,1)+C_{n}$.

Lemma 5. $\operatorname{cr}\left(P(3,1)+K_{1}\right)=2$.
Proof. A suitable subdrawing of $P(3,1)+K_{1}$ induced from the drawing of $P(3,1)+n K_{1}$ in Figure 1 shows that its crossing number is at most 2. To prove the reverse inequality we assume that there is a drawing of the graph $P(3,1)+K_{1}$ with fewer than two crossings and let $D$ be such a drawing. As the graph $P(3,1)+K_{1}$ contains a subdivision of the complete bipartite graph $K_{3,3}$ with $\operatorname{cr}\left(K_{3,3}\right)=1$, and therefore the drawing $D$ contains exactly one crossing. By Lemma 1, the red edges do not cross each other in $D$, that is to say, one of blue or black edge must be crossed. There is a contradiction since removing any blue or black edge of the graph $P(3,1)+K_{1}$ results in a graph containing a subdivision of $K_{3,3}$.

Lemma 6. $\operatorname{cr}\left(P(3,1)+2 K_{1}\right)=4$.
Proof. A suitable subdrawing of $P(3,1)+2 K_{1}$ induced from the drawing of $P(3,1)+n K_{1}$ in Figure 1 shows that its crossing number is at most 4. Assume now that there is a drawing $D$ of the graph $P(3,1)+2 K_{1}$ with fewer than four crossings. By Lemma 1, the red edges do not cross each other in any optimal drawing of $P(3,1)+2 K_{1}$, and hence, $\operatorname{cr}_{D}\left(P(3,1)+2 K_{1}\right)=3$ since removing any blue or black edge of the graph $P(3,1)+2 K_{1}$ results in a graph containing a subdivision of $K_{3,4}$ with $\operatorname{cr}\left(K_{3,4}\right)=2$.

We claim that at least one of the three crossings in $D$ does not appear on black edges, since deleting any three black edges from the graph $P(3,1)+2 K_{1}$ results in a graph containing $K_{3,3}$ as a subgraph. Thus, $\operatorname{cr}_{D}(P(3,1)) \geq 1$ and by Lemma 3, the subdrawing of $P(3,1)$ must be drawn as one of $D_{1}, D_{2}$ and $D_{4}$ in Figure 9. It is not difficult to find that the blue edges cross the red edges at least once, and the black edges must be crossed at least once. However, one can easily verify that the deleting of any two edges which one is blue and other one is black from the graph $P(3,1)+2 K_{1}$ results in a graph containing $K_{3,4}$ as a subgraph, a contradiction.

Lemma 7. Let $D$ be a good drawing of the graph $P(3,1)+n K_{1}$ in which for some $i \in\{1,2, \ldots, n\}$, and for all $j=1,2, \ldots, n, j \neq i, r_{D}\left(P(3,1) \cup E\left(z_{i}\right), E\left(z_{j}\right)\right) \geq 5$. If cr ${ }_{D}\left(P(3,1) \cup E\left(z_{i}\right), E\left(z_{j}\right)\right)>5$ for $k$ different subgraphs $E\left(z_{j}\right)$, then $\operatorname{cr}_{D}(P(3,1)$ $\left.+n K_{1}\right) \geq Z(6, n)+2 n+k$.

Proof. Without loss of generality, assume that the edges of $P(3,1) \cup E\left(z_{1}\right)$ are crossed in $D$ at least five times by the edges of every subgraph $E\left(z_{j}\right)$, $j=2,3, \ldots, n$, and that $k$ of the subgraphs $E\left(z_{j}\right)$ cross the edges of $P(3,1) \cup E\left(z_{1}\right)$ more than five times. As $H_{n}=K_{6, n-1} \cup P(3,1) \cup E\left(z_{1}\right)$ and $P(3,1) \cup E\left(z_{1}\right)=$ $P(3,1)+K_{1}$, by (1.1) (1.2) and Lemma 5, we have

$$
\begin{aligned}
c r_{D}\left(H_{n}\right) & =c r_{D}\left(\bigcup_{j=2}^{n} E\left(z_{j}\right)\right)+c r_{D}\left(P(3,1) \cup E\left(z_{1}\right)\right) \\
& +\sum_{j=2}^{n} c r_{D}\left(E\left(z_{j}\right), P(3,1) \cup E\left(z_{1}\right)\right) \\
& \geq Z(6, n-1)+2+5(n-1)+k \geq Z(6, n)+2 n+k,
\end{aligned}
$$

as desired.
The proofs of the main results in Section 5 are based on the next lemma which was proved in [6].

Lemma 8. Let $D$ be a good drawing of $m K_{1}+C_{n}, m \geq 2, n \geq 3$, in which no edge of $C_{n}$ is crossed, and $C_{n}$ does not separate the other vertices of the graph. Then, for all $z_{i}, z_{j} \in V\left(m K_{1}\right), z_{i} \neq z_{j}$, two subgraphs $E\left(z_{i}\right)$ and $E\left(z_{j}\right)$ cross each other in $D$ at least $\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times.

## 3. The Crossing Numbers of $P(3,1)+n K_{1}$

Theorem 9. $\operatorname{cr}\left(P(3,1)+n K_{1}\right)=Z(6, n)+2 n$.

Proof. The drawing in Figure 1 shows that $c r\left(P(3,1)+n K_{1}\right) \leq Z(6, n)+2 n$ and that the theorem is true if the equality holds. We prove the reverse inequality by induction on $n$. By Lemmas 5 and 6 , the theorem is true for $n=1,2$. Suppose now that for $n \geq 3$

$$
\begin{equation*}
\operatorname{cr}\left(H_{n-2}\right) \geq Z(6, n-2)+2(n-2), \tag{3.1}
\end{equation*}
$$

and consider such an optimal drawing $D$ of $H_{n}$ that

$$
\begin{equation*}
\operatorname{cr}_{D}\left(H_{n}\right) \leq Z(6, n)+2 n-1 . \tag{3.2}
\end{equation*}
$$

The following claim is critical.
Claim 10. $\operatorname{cr}_{D}\left(E\left(z_{i}\right), E\left(z_{j}\right)\right) \geq 1$ for all $i, j=1,2, \ldots, n, i \neq j$.
Proof. Assume that there are at least two different subgraphs $E\left(z_{i}\right)$ and $E\left(z_{j}\right)$ that do not cross each other in $D$. Without loss of generality, let $\operatorname{cr}_{D}\left(E\left(z_{1}\right)\right.$, $\left.E\left(z_{2}\right)\right)=0$. Let $x_{i}, x_{i}^{\prime}, y_{i}, i=1,2,3$ denote the number of crossings between the nine edges of $P(3,1)$ and $E\left(z_{1}\right) \cup E\left(z_{2}\right)$, respectively (see Figure 10). It is clear that $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right) \cup E\left(z_{2}\right)\right)=\sum_{i=1}^{3}\left(x_{i}+x_{i}^{\prime}+y_{i}\right)$.

It is not a difficult task to show that there is at least one crossing between the edges of each 3 -cycle and $E\left(z_{1}\right) \cup E\left(z_{2}\right)$. Thus, it follows that

$$
\begin{aligned}
x_{1}+x_{2}+x_{3} & \geq 1, \\
x_{1}^{\prime}+x_{2}^{\prime}+x_{3}^{\prime} & \geq 1 .
\end{aligned}
$$

By implication,

$$
\begin{equation*}
\sum_{i=1}^{3}\left(x_{i}+x_{i}^{\prime}\right)=2+\alpha \tag{3.3}
\end{equation*}
$$

where $\alpha \geq 0$.
On the other hand, it is not a difficult task to show that there are at least two crossings between the edges of each 4 -cycle and $E\left(z_{1}\right) \cup E\left(z_{2}\right)$. Thus, it follows that

$$
\begin{aligned}
& x_{1}+x_{1}^{\prime}+y_{1}+y_{2} \geq 2, \\
& x_{2}+x_{2}^{\prime}+y_{2}+y_{3} \geq 2, \\
& x_{3}+x_{3}^{\prime}+y_{1}+y_{3} \geq 2 .
\end{aligned}
$$

By implication,

$$
\begin{equation*}
\sum_{i=1}^{3}\left(x_{i}+x_{i}^{\prime}\right)+2 \sum_{i=1}^{3} y_{i}=6+\beta \tag{3.4}
\end{equation*}
$$

where $\beta \geq 0$.

Therefore, by combining (3.3) and (3.4), we have

$$
\sum_{i=1}^{3}\left(x_{i}+x_{i}^{\prime}+y_{i}\right)=4+\frac{1}{2}(\alpha+\beta) \geq 4
$$

which implies that $c r_{D}\left(P(3,1), E\left(z_{1}\right) \cup E\left(z_{2}\right)\right) \geq 4$.
For $3 \leq i \leq n, E\left(z_{1}\right) \cup E\left(z_{2}\right) \cup E\left(z_{i}\right)$ is isomorphic to $K_{3,6}$. Hence, by (1.1), (1.2) and the assumptions we have $c r_{D}\left(E\left(z_{1}\right) \cup E\left(z_{2}\right), E\left(z_{i}\right)\right) \geq c r\left(K_{3,6}\right)=6$.

Then, by (1.1), (1.2) and (3.1) we have

$$
\begin{aligned}
c r_{D}\left(H_{n}\right) & =c r_{D}\left(H_{n-2}\right)+c r_{D}\left(E\left(z_{1}\right) \cup E\left(z_{2}\right)\right)+c r_{D}\left(P(3,1), E\left(z_{1}\right) \cup E\left(z_{2}\right)\right) \\
& +\sum_{i=3}^{n} c r_{D}\left(E\left(z_{i}\right), E\left(z_{1}\right) \cup E\left(z_{2}\right)\right) \\
& \geq Z(6, n-2)+2(n-2)+4+6(n-2)=Z(6, n)+2 n
\end{aligned}
$$

which contradicts with (3.2). This proves the claim.
Now we continue with the proof of the theorem. From (1.1) and (1.2) it follows that

$$
\begin{equation*}
c r_{D}\left(H_{n}\right)=c r_{D}(P(3,1))+c r_{D}\left(\bigcup_{i=1}^{n} E\left(z_{i}\right)\right)+\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \tag{3.5}
\end{equation*}
$$

Since $\bigcup_{i=1}^{n} E\left(z_{i}\right)$ is isomorphic to $K_{6, n}$, we have $c r_{D}\left(\bigcup_{i=1}^{n} E\left(z_{i}\right)\right) \geq Z(6, n)$. Hence, by (3.2) and (3.5) we get

$$
\begin{equation*}
c r_{D}(P(3,1))+\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \leq 2 n-1 \tag{3.6}
\end{equation*}
$$

Therefore $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \leq 1$ for some $1 \leq i \leq n$. Without loss of generality, we assume that $c r_{D}\left(P(3,1), E\left(z_{1}\right)\right) \leq 1$ and let $F=P(3,1) \cup E\left(z_{1}\right)$. There are two cases to be considered: Case 1. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=0$ and Case 2. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=1$.

Case 1. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=0$. From $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=0$ we can conclude that the subdrawing of $P(3,1)$ induced by $D$ has a region with all vertices of $P(3,1)$ on its boundary. From Lemma 3 , the subdrawing of $P(3,1)$ must be $D_{1}$ in Figure 9, and $F$ must be drawn as in Figure 11.

If $z_{i}$ for $2 \leq i \leq n$ lies in any region being not marked with $\star$, we can check that

$$
c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 5
$$



Figure 10. Marking the numbers of crossings for the graph $P(3,1)$.


Figure 11. The subdrawings of $F$ in the drawing of $P(3,1)+n K_{1}$.

If $z_{i}$ for $2 \leq i \leq n$ lies in any region marked with $\star$, we have

$$
c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 4
$$

since there are four vertices of $P(3,1)$ which are not on the boundary of the region marked with $\star$ and the boundary of this region is formed by the edges of $P(3,1)$. By Claim 10, it follows that

$$
\begin{equation*}
c r_{D}\left(F, E\left(z_{i}\right)\right)=c r_{D}\left(P(3,1), E\left(z_{i}\right)\right)+c r_{D}\left(E\left(z_{1}\right), E\left(z_{i}\right)\right) \geq 4+1=5 \tag{3.7}
\end{equation*}
$$

Therefore, we know that $c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 5$ for all $i=2, \ldots, n$. From Figure 11 , it is known that $c r_{D}(F)=2$. Hence, by Lemma 7 we have $c r_{D}\left(H_{n}\right) \geq$ $Z(6, n)+2 n$, this contradicts our assumption about the drawing $D$.

Case 2. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=1$. For this case, there exists a region in the subdrawing of $P(3,1)$ induced by $D$ such that its boundary contains at least 5 vertices of $P(3,1)$. By Lemma 3, the subdrawing of $P(3,1)$ must be drawn as one of $D_{1}$ and $D_{2}$ in Figure 9. For $D_{1}, z_{1}$ must be placed in the unique region with all vertices of $P(3,1)$ on its boundary. However, one can easily verify that, in this case, the edges of $E\left(z_{1}\right)$ cross the edges of $P(3,1)$ at least two times or 0 times. Hence, the subdrawing of $P(3,1)$ induced by $D$ must be $D_{2}$, and the graph $F$ must be drawn as $D_{2}^{\prime}$ or $D_{2}^{\prime \prime}$ in Figure 12. From now on, we make the following assumption on the subindex $i, 2 \leq i \leq n$, in the rest of this Section.

We first consider $D_{2}^{\prime}$. If $z_{i}$ lies in any region which is not marked with $\star$, one can prove that

$$
\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right) \geq 6
$$

If $z_{i}$ lies in any region marked with $\star$, we have

$$
c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 3
$$



Figure 12. The possible subdrawings of $F$ in the drawing of $P(3,1)+n K_{1}$.
since there are at least three vertices of $P(3,1)$ which are not on the boundary of the region marked with $\star$ and the boundary of this region is formed by the edges of $P(3,1)$. By Claim 10, it follows that

$$
\begin{equation*}
\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right)=\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right)+\operatorname{cr}_{D}\left(E\left(z_{1}\right), E\left(z_{i}\right)\right) \geq 3+1=4 . \tag{3.8}
\end{equation*}
$$

Let $l_{1}$ be the number of vertices $z_{i}$ which lies in the region marked with $\star$. Hence,

$$
\begin{equation*}
\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 3 l_{1}+\left(n-l_{1}\right) \tag{3.9}
\end{equation*}
$$

since $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 1$ for $1 \leq i \leq n$. Thus, from (3.6) and $\operatorname{cr}_{D}(P(3,1))=$ 1 , it follows that $l_{1} \leq\left\lfloor\frac{n-2}{2}\right\rfloor$. The similar calculating as in the proof of Lemma 7 gives the following formula.

$$
\begin{aligned}
\operatorname{cr}_{D}\left(H_{n}\right) & =c r_{D}\left(\bigcup_{i=2}^{n} E\left(z_{i}\right)\right)+c r_{D}(F)+\sum_{i=2}^{n} c r_{D}\left(E\left(z_{i}\right), F\right) \\
& \geq Z(6, n-1)+2+4 l_{1}+6\left(n-1-l_{1}\right)=Z(6, n-1)+6 n-2 l_{1}-4 \\
& \geq Z(6, n-1)+6 n-2\left\lfloor\frac{n-2}{2}\right\rfloor-4 \geq Z(6, n)+2 n
\end{aligned}
$$

This contradiction completes the proof for $D_{2}^{\prime}$.
Finally, we consider $D_{2}^{\prime \prime}$. If $z_{i}$ lies in any region which is not marked with and $\mathbf{\Delta}$, one can check that

$$
c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 6 .
$$

If $z_{i}$ lies in any region marked with $\star$, then similarly to the proof of the claim of (3.8), one can prove that

$$
\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 3, \text { and } c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 4 .
$$

If $z_{i}$ lies in any region marked with $\mathbf{\Delta}$, one can show that

$$
c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 5 .
$$

Moreover, it is easy to see that if $\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right)=5$, then the edges of $E\left(z_{i}\right)$ cross the edges of $P(3,1)$ exactly once or three times.

Let $k_{1}$ be the number of vertices $z_{i}$ which lies in the region marked with $\star$ and in the region marked with $\mathbf{\Delta}$ for which the edges of $E\left(z_{i}\right)$ cross the edges of $P(3,1)$ exactly three times. Let $k_{2}$ be the number of vertices $z_{i}$ which lies in the region marked with $\mathbf{\Delta}$ for which $\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right)=5$ and the edges of $E\left(z_{i}\right)$ cross the edges of $P(3,1)$ exactly once. Our next analysis depends on whether $k_{2}=0$ or not.

If $k_{2}=0$, then the similar calculating as in the proof of the case for $D_{2}^{\prime}$ results in a contradiction again, and the details are omited.

If $k_{2} \geq 1$, then without loss of generality, assume that $z_{2}$ lies in the region marked with $\mathbf{\Delta}$ for which $\operatorname{cr}_{D}\left(F, E\left(z_{2}\right)\right)=5$ and the edges of $E\left(z_{2}\right)$ cross the edges of $P(3,1)$ exactly once. For the case, the unique subdrawing of $P(3,1) \cup E\left(z_{1}\right) \cup$ $E\left(z_{2}\right)$ is shown in Figure 13. For simpler labelling, let $W=P(3,1) \cup E\left(z_{1}\right) \cup E\left(z_{2}\right)$. If $z_{i}$ lies in any region which is not marked with $\star$ and $\boldsymbol{\Delta}$, one can verify that

$$
c r_{D}\left(W, E\left(z_{i}\right)\right) \geq 11
$$

If $z_{i}$ lies in any region marked with $\star$, then the similar discussion as in the proof of (3.8) gives that

$$
\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 3, \text { and } \operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right) \geq 5 .
$$

If $z_{i}$ lies in any region marked with $\mathbf{\Delta}$, one can show that

$$
\operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right) \geq 9 .
$$

Moreover, it is easy to check that if $\operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right)=9$, then the edges of $E\left(z_{i}\right)$ cross the edges of $P(3,1)$ at least three times, otherwise $\operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right) \geq 11$.

Let $h_{1}$ be the number of vertices $z_{i}$ which lies in the region marked with $\star$. Let $h_{2}$ be the number of vertices $z_{i}$ which lies in the region marked with $\boldsymbol{\Delta}$ and $\operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right)=9$. Note also that $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right)=1, i=1,2$ and $\operatorname{cr}_{D}(P(3,1))=1$; similarly to the proof of (3.9), we obtain that $h_{1}+h_{2} \leq\left\lfloor\frac{n-2}{2}\right\rfloor$. Thus, using similar arguments as in the proof of Lemma 7 and noting that $c r_{D}(W)$ $=7$, we can also get the following formula

$$
\begin{aligned}
c r_{D}\left(H_{n}\right) & =c r_{D}\left(\bigcup_{i=3}^{n} E\left(z_{i}\right)\right)+c r_{D}(W)+\sum_{i=3}^{n} c r_{D}\left(E\left(z_{i}\right), W\right) \\
& \geq Z(6, n-2)+7+5 h_{1}+9 h_{2}+11\left(n-2-h_{1}-h_{2}\right) \\
& =Z(6, n-2)+11 n-6 h_{1}-2 h_{2}-15 \\
& \geq Z(6, n-2)+11 n-6\left\lfloor\frac{n-2}{2}\right\rfloor-15>Z(6, n)+2 n .
\end{aligned}
$$

This contradiction completes the proof.


Figure 13. The subdrawings of $W$ in the drawing of $P(3,1)+n K_{1}$.

## 4. The Crossing Numbers of $P(3,1)+P_{n}$

The graph $P(3,1)+P_{n}$ contains $P(3,1)+n K_{1}$ as a subgraph. For the subgraphs of the graph $P(3,1)+P_{n}$ which are also subgraphs of the graph $P(3,1)+n K_{1}$, we will use the same notation as before. Let $P_{n}^{*}$ denote the path on $n$ vertices of $P(3,1)+P_{n}$ not belonging to the subgraph $P(3,1)$. One can easily see that

$$
\begin{equation*}
P(3,1)+P_{n}=P(3,1) \cup K_{6, n} \cup P_{n}^{*}=P(3,1) \cup\left(\bigcup_{i=1}^{n} E\left(z_{i}\right)\right) \cup P_{n}^{*} . \tag{4.1}
\end{equation*}
$$

The graph $P(3,1)+P_{1}$ is isomorphic to $P(3,1)+K_{1}$ and $\operatorname{cr}\left(P(3,1)+K_{1}\right)=2$. For $n \geq 2$ we have the next result.

Theorem 11. $\operatorname{cr}\left(P(3,1)+P_{n}\right)=Z(6, n)+2 n+1$, for $n \geq 2$.
Proof. Figure 1 shows the drawing of the graph $P(3,1)+n K_{1}$ with $Z(6, n)+2 n$ crossings. One can easily see that in this drawing it is possible to add $n-1$ edges which form the path $P_{n}^{*}$ on the vertices of $n K_{1}$ in such a way that only one edge of $P_{n}^{*}$ is crossed by an edge of $P(3,1)$. Hence, $\operatorname{cr}\left(P(3,1)+P_{n}\right) \leq Z(6, n)+2 n+1$. To prove the reverse inequality we assume that there is an optimal drawing of the graph $P(3,1)+P_{n}$ with fewer than $Z(6, n)+2 n+1$ crossings and let $D$ be such a drawing. Since the graph $P(3,1)+P_{n}$ contains $P(3,1)+n K_{1}$ as a subgraph, by Theorem $9, \operatorname{cr}\left(P(3,1)+P_{n}\right)=Z(6, n)+2 n$ and therefore, no edge of the path $P_{n}^{*}$ is crossed in $D$, which implies that all vertices $z_{i}, i=1,2, \ldots, n$, are placed in the same region of the subdrawing of $P(3,1)$ induced by $D$.

The similar argument as in the proof of (3.6) gives that

$$
\begin{equation*}
\operatorname{cr}_{D}(P(3,1))+\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \leq 2 n . \tag{4.2}
\end{equation*}
$$

Therefore $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \leq 2$ for some $1 \leq i \leq n$. Without loss of generality, we assume that $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right) \leq 2$ and let $F=P(3,1) \cup E\left(z_{1}\right)$. By the
similar discussion as in the proof of Theorem 9, the following three cases are considered.

Case 1. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=0$. One can easily verify that, in this case, in the subdrawing of $P(3,1)$ induced by $D$ there are all six vertices of $P(3,1)$ on the boundary of one, say unbounded, region and, in $D$, all vertices $z_{i}, i=1,2, \ldots, n$, are placed in this region. From Remark 4, such unique subdrawing of $P(3,1)$ must be $D_{1}$ in Figure 9, and $F$ is drawn as in Figure 11.

If $z_{i}$ for $2 \leq i \leq n$ lies in the unbounded region of $P(3,1)$, we can check that

$$
\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right) \geq 6 .
$$

From Figure 11, it is known that $c r_{D}(F)=2$, which, together with Lemma 7 , contradicts the assumption.

Case 2. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=1$. By the similar analysis as in Case 2 of Section 3, we can know that the subdrawing of $P(3,1)$ induced by $D$ must be $D_{2}$ in Figure 9, and the graph $F$ must be drawn as one of $D_{2}^{\prime}$ and $D_{2}^{\prime \prime}$ in Figure 12. Moreover, all vertices $z_{i}, i=2,3, \ldots, n$, are placed in the same region of $D_{2}$ as the vertex $z_{1}$.

For $D_{2}^{\prime}$, it is not difficult to verify that

$$
\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right) \geq 6,
$$

for $2 \leq i \leq n$, if $z_{i}$ lies in the same region of $D_{2}$ as the vertex $z_{1}$. Note also that $\operatorname{cr}_{D}(F)=2$, and the same contradiction can be obtained as in the above case.

For $D_{2}^{\prime \prime}$, One can easily verify that

$$
c r_{D}\left(F, E\left(z_{i}\right)\right) \geq 5
$$

for $2 \leq i \leq n$, if $z_{i}$ lies in the same region of $D_{2}$ as the vertex $z_{1}$. Note that $c r_{D}(F)=2$, and it follows from Lemma 7 that $\operatorname{cr}_{D}\left(P(3,1)+P_{n}\right) \geq Z(6, n-1)+$ $2+5(n-1)>Z(6, n)+2 n$ for even $n$, a contradiction.

For odd $n$, it follows that $\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right)=5$ for $2 \leq i \leq n$, from Lemma 7 . In addition, it is easy to see that if $\operatorname{cr}_{D}\left(F, E\left(z_{i}\right)\right)=5$, then the edges of $E\left(z_{i}\right)$ cross the edges of $P(3,1)$ exactly once or three times. Moreover, if $c r_{D}\left(E\left(z_{i}\right)\right.$, $P(3,1))=1$, then $c r_{D}\left(E\left(z_{i}\right), E\left(z_{1}\right)\right)=4$, and if $\operatorname{cr}_{D}\left(E\left(z_{i}\right), P(3,1)\right)=3$, then $c r_{D}\left(E\left(z_{i}\right), E\left(z_{1}\right)\right)=2$, for $2 \leq i \leq n$. Let $\alpha_{1}$ be the number of vertices $z_{i}, 2 \leq i \leq$ $n$, with $c r_{D}\left(E\left(z_{i}\right), P(3,1)\right)=1$ and $c r_{D}\left(E\left(z_{i}\right), E\left(z_{1}\right)\right)=4$. Let $\alpha_{2}$ be the number of vertices $z_{i}, 2 \leq i \leq n$, with $c r_{D}\left(E\left(z_{i}\right), P(3,1)\right)=3$ and $\operatorname{cr}_{D}\left(E\left(z_{i}\right), E\left(z_{1}\right)\right)=2$.

Using the fact that $c r_{D}\left(E\left(z_{1}\right), P(3,1)\right)=1$ and $c r_{D}(P(3,1))=1$, together with (4.2) we have

$$
\alpha_{1}+3 \alpha_{2}+2 \leq 2 n,
$$

which implies that $\alpha_{1} \neq 0$. Assume, without loss of generality, that $\operatorname{cr}_{D}\left(E\left(z_{2}\right)\right.$, $P(3,1))=1$ and $\operatorname{cr}_{D}\left(E\left(z_{2}\right), E\left(z_{1}\right)\right)=4$. For this case, the unique subdrawing of $P(3,1) \cup E\left(z_{1}\right) \cup E\left(z_{2}\right)$ is shown in Figure 13. To simplify the notation, let $W=P(3,1) \cup E\left(z_{1}\right) \cup E\left(z_{2}\right)$.

For $3 \leq i \leq n$, if $z_{i}$ lies in the unique region of $D_{2}$ with five vertices of $P(3,1)$ on its boundary as the vertices $z_{1}$ and $z_{2}$, one can easily verify that

$$
\operatorname{cr}_{D}\left(W, E\left(z_{i}\right)\right) \geq 9 .
$$

Note that $\operatorname{cr}_{D}(W)=7$, it follows that

$$
\begin{aligned}
c r_{D}\left(P(3,1)+P_{n}\right) & =c r_{D}\left(\bigcup_{i=3}^{n} E\left(z_{i}\right)\right)+c r_{D}(W)+\sum_{i=3}^{n} c r_{D}\left(E\left(z_{i}\right), W\right) \\
& \geq Z(6, n-2)+7+9(n-2)>Z(6, n)+2 n
\end{aligned}
$$

a contradiction again.
Case 3. $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{1}\right)\right)=2$. For this case, by (4.2), $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right)=$ 2 for all $i=2,3, \ldots, n$ and $\operatorname{cr}_{D}(P(3,1))=0$. Up to the isomorphism, there is unique possible subdrawings of $P(3,1)$ induced by $D$ as $D_{3}$ shown in Figure 9. Consider the symmetry of the drawing of $D_{3}$, suppose that all vertices $z_{i}, 1 \leq$ $i \leq n$, are placed in some one of three regions with four vertices of $P(3,1)$ on its boundary. The edges of $E\left(z_{i}\right)$ divide this region as shown in Figure 14(a) or (b). One can easily verify that $c_{D}\left(E\left(z_{j}\right), E\left(z_{i}\right)\right) \geq 3$, for all $i, j=1,2, \ldots, n, j \neq i$, if the vertex $z_{j}$ is placed in the same region of $D_{3}$ as the vertex $z_{i}$. Thus, in $D$ there are at least $3\binom{n}{2}+2 n>Z(6, n)+2 n$ crossings, which contradicts the assumption. This completes the proof.


Figure 14. The possible placements of $E\left(z_{i}\right)$ inside the region with four vertices of $P(3,1)$ on its boundary.

## 5. The Crossing Numbers of $P(3,1)+C_{n}$

The graph $P(3,1)+C_{n}$ contains both $P(3,1)+n K_{1}$ and $P(3,1)+P_{n}$ as a subgraph. Let $C_{n}^{*}$ denote the subgraph of $P(3,1)+C_{n}$ induced on the vertices not belonging
to the subgraph $P(3,1)$. For $i=1,2, \ldots, 6$, let $a_{i}$ denote the six vertices of $P(3,1)$, and $E\left(a_{i}\right)$ denote the subgraph induced by $n$ edges of $K_{6, n}$ incident with the vertex $a_{i}$, respectively. One can easily see that

$$
\begin{equation*}
P(3,1)+C_{n}=P(3,1) \cup K_{6, n} \cup C_{n}^{*}=P(3,1) \cup\left(\bigcup_{i=1}^{n} E\left(z_{i}\right)\right) \cup C_{n}^{*} . \tag{5.1}
\end{equation*}
$$

On the other hand, the graph $P(3,1)+C_{n}$ contains the graph $6 K_{1}+C_{n}^{*}$ as a subgraph and

$$
\begin{equation*}
P(3,1)+C_{n}=P(3,1) \cup\left(\bigcup_{i=1}^{6} E\left(a_{i}\right)\right) \cup C_{n}^{*} \tag{5.2}
\end{equation*}
$$

Theorem 12. $\operatorname{cr}\left(P(3,1)+C_{n}\right)=Z(6, n)+2 n+3$, for $n \geq 3$.
Proof. In the drawing in Figure 1 it is possible to add $n$ edges in such a way that they, together with the vertices of $n K_{1}$, form the cycle $C_{n}^{*}$ and that the edges of $C_{n}^{*}$ are crossed only three times. Hence, $\operatorname{cr}\left(P(3,1)+C_{n}\right) \leq Z(6, n)+2 n+3$. To prove the reverse inequality, assume that there is an optimal drawing of the graph $P(3,1)+C_{n}$ with at most $Z(6, n)+2 n+2$ crossings and let $D$ be such a drawing. Since the graph $P(3,1)+C_{n}$ contains $P(3,1)+P_{n}$ as a subgraph, by Theorem 11, $\operatorname{cr}_{D}\left(P(3,1)+C_{n}\right)=Z(6, n)+2 n+1$ or $Z(6, n)+2 n+2$, and by Theorem 9 , in $D$ there are at most two crossings on the edges of $C_{n}^{*}$, otherwise deleting the edges from $C_{n}^{*}$ results in an drawing of the graph $P(3,1)+n K_{1}$ fewer than $Z(6, n)+2 n$ crossings.

We claim that the edges of $C_{n}^{*}$ do not cross each other, otherwise one can modify the drawing in a sufficiently small neighborhood of the crossing point resulting in a new good drawing of $P(3,1)+C_{n}$ as shown in Figure 15, and the crossings are reduced at least one. As $P(3,1)$ is a 3 -connected graph, all vertices of $P(3,1)$ are placed in the same region in the view of the subdrawing of $C_{n}^{*}$ induced by $D$, otherwise in $D$ there are at least three crossings on the edges of $C_{n}^{*}$. The edges of $C_{n}^{*}$ are not crossed by the edges of $P(3,1)$, otherwise $c r_{D}\left(C_{n}^{*}, P(3,1)\right) \geq 2$ and $c r_{D}\left(E\left(a_{i}\right), C_{n}^{*}\right)=0$ for all $i=1,2, \ldots, 6$, and then, it follows from Lemma 8 that in $D$ there are at least $\binom{6}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor>Z(6, n)+2 n+2$ crossings. This implies that all vertices $z_{i}, i=1,2, \ldots, n$, are placed in the same region in the view of the subdrawing of $P(3,1)$ induced by $D$.

We conclude that the edges of $C_{n}^{*}$ are crossed at least once in $D$, otherwise it follows from Lemma 8 that in $D$ there are at least $\binom{6}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor>Z(6, n)+2 n+2$ crossings, a contradiction.


Figure 15. Removing the self-crossings on the edges $C_{n}^{*}$.

## Claim 13.

$$
\begin{aligned}
\operatorname{cr}_{D}(P(3,1))+\sum_{i=1}^{6} c r_{D}\left(P(3,1), E\left(a_{i}\right)\right) & =\operatorname{cr}_{D}(P(3,1))+\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \\
& \geq n+1
\end{aligned}
$$

Proof. From Remark 4, we know that all the possible subdrawings of $P(3,1)$ induced by $D$ are that shown in Figure 9. For $D_{3}$ and $D_{4}$, it is easily see that every region has at most four vertices of $P(3,1)$ on its boundary, which implies that $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 2$, for all $i=1,2, \ldots, n$, and the claim follows. For $D_{2}$, one can easily see that every region has at most five vertices of $P(3,1)$ on its boundary, which implies that $\operatorname{cr}_{D}\left(P(3,1), E\left(z_{i}\right)\right) \geq 1$, for all $i=1,2, \ldots, n$. Note that $\operatorname{cr}_{D}(P(3,1))=1$, the claim holds. For $D_{1}$, there is a region, say unbounded, with all vertices of $P(3,1)$ on its boundary, and other regions, say bounded, have at most two vertices of $P(3,1)$ on its boundary. If all vertices $z_{i}, i=1,2, \ldots, n$, are placed in some bounded region, then the claim holds. Suppose now that all vertices $z_{i}, i=1,2, \ldots, n$, are placed in this unbounded region, and there is at least one subgraph $E\left(z_{i}\right)$ which does not cross $P(3,1)$ in $D$. Note that the edges of $C_{n}^{*}$ are crossed at least once. By the similar analysis as in Case 1 of Section 4, a contradiction appears. This completes the proof of claim.

Suppose now that in $D$ the edges of $C_{n}^{*}$ are crossed exactly once. Without loss of generality, let $\operatorname{cr}_{D}\left(E\left(a_{1}\right), C_{n}^{*}\right)=1$. The simple modification of Lemma 8 for this case implies that $\operatorname{cr}_{D}\left(E\left(a_{1}\right), E\left(a_{i}\right)\right) \geq\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$ for $i=2,3, \ldots, 6$. And, for $i, j=2,3, \ldots, 6, i \neq j, \operatorname{cr}_{D}\left(E\left(a_{i}\right), E\left(a_{j}\right)\right) \geq\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ by Lemma 8 . Thus, by (5.2) and Claim 13, we have

$$
\begin{aligned}
\operatorname{cr}_{D}\left(P(3,1)+C_{n}\right) & =c r_{D}(P(3,1))+\sum_{i=1}^{6} c r_{D}\left(P(3,1), E\left(a_{i}\right)\right)+c r_{D}\left(\bigcup_{i=1}^{6} E\left(a_{i}\right)\right) \\
& +c r_{D}\left(P(3,1) \cup \bigcup_{i=1}^{6} E\left(a_{i}\right), C_{n}^{*}\right) \geq n+1+\binom{5}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor \\
& +5\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+1>Z(6, n)+2 n+2,
\end{aligned}
$$

a contradiction.

Finally, assume that in $D$ the edges of $C_{n}^{*}$ are crossed exactly two times. From (1.1), (1.2) and (5.1), it follows that

$$
\begin{equation*}
\operatorname{cr}_{D}(P(3,1))+\sum_{i=1}^{n} c r_{D}\left(P(3,1), E\left(z_{i}\right)\right) \leq 2 n . \tag{5.3}
\end{equation*}
$$

For the case, the similar discussion as in the proof of Theorem 11 gives a contradiction again, and the details are omitted. This completes the proof.

## Acknowledgements

The authors are indebted to two anonymous referees for their suggestion which improved the presentation and made it more readable.

## References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan Press Ltd, London, 1976).
[2] P. Erdős and R.K. Guy, Crossing number problems, Amer. Math. Monthly 80 (1973) 52-58.
doi:10.2307/2319261
[3] M.R. Garey and D.S. Johnson, Crossing number is NP-complete, SIAM J. Algebraic Discrete Methods 4 (1983) 312-316. doi:10.1137/0604033
[4] V.R. Kulli and M.H. Muddebihal, Characterization of join graphs with crossing number zero, Far East J. Appl. Math. 5 (2001) 87-97.
[5] D.J. Kleitman, The crossing number of $K_{5, n}$, J. Combin. Theory Ser. B 9 (1970) 315-323. doi:10.1016/S0021-9800(70)80087-4
[6] M. Klešč, The join of graphs and crossing numbers, Electron. Notes Discrete Math. 28 (2007) 349-355.
doi:10.1016/j.endm.2007.01.049
[7] L. Tang, J. Wang and Y.Q. Huang, The crossing number of the join of $C_{m}$ and $P_{n}$, Internat. J. Math. Com. 1 (2007) 110-116.
[8] M. Klešč and S. Schrötter, The crossing numbers of join products of paths with graphs of order four, Discuss. Math. Graph Theory 31 (2011) 321-331. doi:10.7151/dmgt. 1548
[9] M. Klešč, The crossing numbers of join of the special graph on six vertice with path and cycle, Discrete Math. 310 (2010) 1475-1481. doi:10.1016/j.disc.2009.08.018
[10] M. Klešč and S. Schrötter, The crossing numbers of join of paths and cycles with two graphs of order five, Lecture Notes in Comput. Sci. 7125 (2012) 160-167. doi:10.1007/978-3-642-28212-6_15
[11] Y.H. Peng and Y.C. Yiew, The crossing number of $P(3,1) \times P_{n}$, Discrete Math. 306 (2006) 1941-1946.
doi:10.1016/j.disc.2006.03.058
Received 17 May 2016
Revised 14 November 2016
Accepted 14 November 2016


[^0]:    ${ }^{1}$ The work was supported by the National Natural Science Foundation of China (Nos. 11301169 \& 11371133), Hunan Provincial Natural Science Foundation of China (Nos. 13JJ4110 \& 14JJ3138) and Hunan Education Department Talented Foundation (No. 16B028).

