# RAINBOW VERTEX-CONNECTION AND FORBIDDEN SUBGRAPHS 

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#### Abstract

A path in a vertex-colored graph is called vertex-rainbow if its internal vertices have pairwise distinct colors. A vertex-colored graph $G$ is rainbow vertex-connected if for any two distinct vertices of $G$, there is a vertexrainbow path connecting them. For a connected graph $G$, the rainbow vertexconnection number of $G$, denoted by $\operatorname{rvc}(G)$, is defined as the minimum number of colors that are required to make $G$ rainbow vertex-connected. In this paper, we find all the families $\mathcal{F}$ of connected graphs with $|\mathcal{F}| \in\{1,2\}$, for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph $G, \operatorname{rvc}(G) \leq \operatorname{diam}(G)+k_{\mathcal{F}}$, where $\operatorname{diam}(G)$ is the diameter of $G$.


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## 1. Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here.

[^0]Let $G$ be a nontrivial connected graph with an edge-coloring $c: E(G) \rightarrow$ $\{0,1, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in $G$ is called a rainbow path if no two edges of the path are colored with the same color. The graph $G$ is called rainbow connected if for any two distinct vertices of $G$, there is a rainbow path connecting them. For a connected edge-colored graph $G$, the rainbow connection number of $G$, denoted by $r c(G)$, is defined as the minimum number of colors that are needed to make $G$ rainbow connected. Observe that if $G$ has $n$ vertices, then $\operatorname{diam}(G) \leq r c(G) \leq n-1$. It is easy to verify that $r c(G)=1$ if and only if $G$ is a complete graph, and $r c(G)=n-1$ if and only if $G$ is a tree. The concept of rainbow connection of graphs was first introduced by Chartrand et al. in [3], and has been well-studied since then. For further details, we refer the reader to a survey paper [10] and a book [11].

Let $G$ be a nontrivial connected graph with a vertex-coloring $c: V(G) \rightarrow$ $\{0,1, \ldots, t\}, t \in \mathbb{N}$, where adjacent vertices may be colored with the same color. A path of $G$ is called vertex-rainbow if any two internal vertices of the path have distinct colors. The vertex-colored graph $G$ is rainbow vertex-connected if any two vertices of $G$ are connected by a vertex-rainbow path. For a connected graph $G$, the rainbow vertex-connection number of $G$, denoted by $\operatorname{rvc}(G)$, is the minimum number of colors used in a vertex-coloring of $G$ to make $G$ rainbow vertex-connected. The concept of rainbow vertex-connection of graphs was proposed by Krivelevich and Yuster in [6]. They showed that if $G$ is a connected graph with $n$ vertices and minimum degree $\delta$, then $r v c(G) \leq 11 n / \delta$. In [9], Li and Shi improved this bound. In [4], it was shown that computing the rainbow vertex-connection number of a graph is NP-hard. Recently, Li et al. in [7] proved that it is NP-complete to decide whether a given vertex-colored graph is rainbow vertex-connected even when the graph is bipartite.

For the rainbow vertex-connection number of graphs, the following observations are immediate.

Proposition 1. Let $G$ be a connected graph with $n$ vertices. Then
(i) $\operatorname{diam}(G)-1 \leq \operatorname{rvc}(G) \leq n-2$;
(ii) $\operatorname{rvc}(G)=\operatorname{diam}(G)-1$ if $\operatorname{diam}(G)=1$ or 2 , with the assumption that complete graphs have rainbow vertex-connection number 0 .

Note that the difference $\operatorname{rvc}(G)-\operatorname{diam}(G)$ can be arbitrarily large. In fact, if $G$ is a subdivision of a star $K_{1, n}$, then we have $\operatorname{rvc}(G)-\operatorname{diam}(G)=(n+1)-4=$ $n-3$, since in a rainbow vertex-connected coloring of $G$, the internal vertices must have distinct colors.

In [8], Li and Liu studied the rainbow vertex-connection number for any 2 -connected graph, and determined the precise value of the rainbow vertexconnection number of the cycle $C_{n}(n \geq 3)$.

Theorem 1 [8]. Let $C_{n}$ be a cycle of order $n(n \geq 3)$. Then

$$
\operatorname{rvc}\left(C_{n}\right)= \begin{cases}0 & \text { if } n=3 ; \\ 1 & \text { if } n=4,5 \\ 3 & \text { if } n=9 ; \\ \left\lceil\frac{n}{2}\right\rceil-1 & \text { if } n=6,7,8,10,11,12,13 \text { or } 15 \\ \left\lceil\frac{n}{2}\right\rceil & \text { if } n \geq 16 \text { or } n=14\end{cases}
$$

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph $G$ is $\mathcal{F}$-free if $G$ does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F}=\{X\}$ we say that $G$ is $X$-free, and for $\mathcal{F}=\{X, Y\}$ we say that $G$ is $(X, Y)$-free. The members of $\mathcal{F}$ will be referred to in this context as forbidden induced subgraphs, and for $|\mathcal{F}|=2$ we also say that $\mathcal{F}$ is a forbidden pair.

In [5], Holub et al. considered the question: For which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph $G$ satisfies $r c(G) \leq \operatorname{diam}(G)+k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$ )? They gave a complete answer for $|\mathcal{F}| \in\{1,2\}$ in the following two results (where $N$ denotes the net, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 2 [5]. Let $X$ be a connected graph. Then there is a constant $k_{\mathcal{F}}$ such that every connected $X$-free graph $G$ satisfies $r c(G) \leq \operatorname{diam}(G)+k_{X}$ if and only if $X=P_{3}$.

Theorem 3 [5]. Let $X, Y$ be connected graphs such that $X, Y \neq P_{3}$. Then there is a constant $k_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $r c(G) \leq \operatorname{diam}(G)+k_{X Y}$ if and only if (up to symmetry) either $X=K_{1, r}(r \geq 4)$ and $Y=P_{4}$, or $X=K_{1,3}$ and $Y$ is an induced subgraph of $N$.

Naturally, we may consider an analogous question concerning the rainbow vertex-connection number of graphs. In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph $G$ being $\mathcal{F}$-free implies $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+k_{\mathcal{F}}$ ?

We give a complete answer for $|\mathcal{F}|=1$ in Section 3 , and for $|\mathcal{F}|=2$ in Section 4.

## 2. Preliminaries

In this section, we introduce some further notations and facts that will be needed for the proofs of our main results.

If $G$ is a graph and $A \subset V(G)$, then $G[A]$ denotes the subgraph of $G$ induced by the vertex set $A$, and $G-A$ the graph $G[V(G) \backslash A]$. An edge is called a
pendant edge if one of its endvertices has degree one. The subdivision of a graph $G$ is the graph obtained from $G$ by adding a vertex of degree 2 to each edge of $G$. For $x, y \in V(G)$, a path in $G$ from $x$ to $y$ will be referred to as an $(x, y)$-path, and, whenever necessary, it will be considered as oriented from $x$ to $y$. For a subpath of a path $P$ with origin $u$ and terminus $v$ (also referred to as a $(u, v)$-arc of $P$ ), we will use the notation $u P v$. If $w$ is a vertex of a path with a fixed orientation, then $w^{-}$and $w^{+}$denote the predecessor and successor of $w$, respectively.

For graphs $X$ and $G$, we write $X \subset G$ if $X$ is a subgraph of $G, X \stackrel{\text { IND }}{\subset} G$ if $X$ is an induced subgraph of $G$, and $X \simeq G$ if $X$ is isomorphic to $G$. For two vertices $x, y \in V(G)$, we use $\operatorname{dist}_{G}(x, y)$ to denote the distance between $x$ and $y$ in $G$. The diameter of $G$ is defined as the maximum of $\operatorname{dist}_{G}(x, y)$ among all pairs of vertices $x, y$ of $G$, and will be denoted by $\operatorname{diam}(G)$. A shortest path joining two vertices at distance $\operatorname{diam}(G)$ will be referred to as a diameter path. The distance between a vertex $u \in V(G)$ and a set $S \subset V(G)$ is defined as $\operatorname{dist}_{G}(u, S):=\min _{v \in S} \operatorname{dist}_{G}(u, v)$. A set $D \subset V(G)$ is called dominating if every vertex in $V(G) \backslash D$ has a neighbor in $D$. In addition, if $G[D]$ is connected, then we call $D$ a connected dominating set. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers.

For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the $k$ th-neighborhood of $S$ is the set $N_{G}^{k}(S)$ of all vertices of $G$ at distance $k$ from $S$. In the special case $k=1$, we simply write $N_{G}(S)$ for $N_{G}^{1}(S)$, and if $|S|=1$ with $x \in S$, we write $N_{G}(x)$ for $N_{G}(\{x\})$. For a set $M \subset V(G)$, we denote $N_{M}^{k}(S)=N_{G}^{k}(S) \cap M$ and $N_{M}^{k}(x)=N_{G}^{k}(x) \cap M$, and as above, we simply use $N_{M}(S)$ for $N_{M}^{1}(S)$ and $N_{M}(x)$ for $N_{M}^{1}(x)$. For a subgraph $P \subset G$, we write $N_{P}(x)$ for $N_{V(P)}(x)$. Finally, we will use $P_{k}$ to denote the path on $k$ vertices.

We end up this section with an important result that will be used in our proofs.

Theorem 4 [1]. Let $G$ be a connected $P_{5}$-free graph. Then $G$ has a dominating clique or a dominating $P_{3}$.

## 3. Families with one Forbidden Subgraph

In this section, we characterize all connected graphs $X$ such that every connected $X$-free graph $G$ satisfies $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+k_{X}$, where $k_{X}$ is a constant.

Theorem 5. Let $X$ be a connected graph. Then there is a constant $k_{X}$ such that every connected $X$-free graph $G$ satisfies rvc $(G) \leq \operatorname{diam}(G)+k_{X}$ if and only if $X=P_{3}$ or $X=P_{4}$.

Proof. We have $\operatorname{diam}(G) \leq 2$, since $G$ is $P_{4}$-free. Then it follows from Proposition 1 that $\operatorname{rvc}(G)=\operatorname{diam}(G)-1 \leq 1$.

Conversely, let $t \geq k_{X}+5$, and $G_{1}^{t}$ be the subdivision of $K_{1, t}$, and let $G_{2}^{t}$ denote the graph obtained by attaching a pendant edge to each vertex of the complete graph $K_{t}$ (see Figure 1). Since $\operatorname{rvc}\left(G_{1}^{t}\right)=t+1$ but $\operatorname{diam}\left(G_{1}^{t}\right)=4, X$ is an induced subgraph of $G_{1}^{t}$. Clearly, $\operatorname{rvc}\left(G_{2}^{t}\right)=t$ but $\operatorname{diam}\left(G_{2}^{t}\right)=3$, and $G_{2}^{t}$ is $K_{1,3}$-free and $P_{5}$-free. Hence, $X$ is an induced subgraph of $P_{4}$.

The proof is thus complete.


Figure 1. The graphs $G_{1}^{t}$ and $G_{2}^{t}$.

## 4. Families with a Pair of Forbidden Subgraphs

For $i, j, k \in \mathbb{N}$, let $S_{i, j, k}$ denote the graph obtained by identifying one endvertex from each of three vertex-disjoint paths of lengths $i, j, k$, and $N_{i, j, k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths $i, j, k$ (see Figure 2). In this context, we will also write $K_{t}^{h}$ for the graph $G_{2}^{t}$ introduced in the proof of Theorem 5.


Figure 2. The graphs $S_{i, j, k}, N_{i, j, k}$ and $G_{4}^{t}$.
The following statement, which is the main result of this section, characterizes all forbidden pairs $X, Y$ for which there is a constant $k_{X Y}$ such that $G$ being $(X, Y)$-free implies $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+k_{X Y}$. By virtue of Theorem 5, we exclude the case that one of $X, Y$ is an induced subgraph of $P_{4}$. Recall that the net is the graph $N=N_{1,1,1}$.
Theorem 6. Let $X, Y \neq P_{3}$ or $P_{4}$ be a pair of connected graphs. Then there is a constant $k_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $r v c(G) \leq$
$\operatorname{diam}(G)+k_{X Y}$ if and only if (up to symmetry) $X=P_{5}$ and $Y \stackrel{I N D}{\subset} K_{r}^{h}(r \geq 4)$, or $X \stackrel{I N D}{\subset} S_{1,2,2}$ and $Y \stackrel{I N D}{\subset} N$.

The proof of Theorem 6 will be divided into three separate results: we prove the necessity in Proposition 2, and Theorems 7 and 8 will establish the sufficiency of the forbidden pairs given in Theorem 6.
Proposition 2. Let $X, Y \neq P_{3}$ or $P_{4}$ be a pair of connected graphs for which there is a constant $k_{X Y}$ such that every connected $(X, Y)$-free graph $G$ satisfies $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+k_{X Y}$. Then (up to symmetry) $X=P_{5}$ and $Y \stackrel{I N D}{\subset} K_{r}^{h}(r \geq 4)$, or $X \stackrel{I N D}{\subset} S_{1,2,2}$ and $Y \stackrel{I N D}{\subset} N$.
Proof. Let $t \geq 2 k_{X Y}+5$, and let (see Figure 2)

- $G_{3}^{t}=N_{t-1, t-1, t-1}$;
- $G_{4}^{t}$ be the graph obtained by attaching a pendant edge to each vertex of a cycle $C_{t}$.
We will also use the graphs $G_{1}^{t}$ and $G_{2}^{t}\left(=K_{t}^{h}\right)$ shown in Figure 1.
For the graphs $G_{1}^{t}$ and $G_{2}^{t}$, we have $\operatorname{diam}\left(G_{1}^{t}\right)=4$ but $r v c\left(G_{1}^{t}\right)=t+1$, and $\operatorname{diam}\left(G_{2}^{t}\right)=3$ but $\operatorname{rvc}\left(G_{2}^{t}\right)=t$, respectively. For the graph $G_{3}^{t}$, we observe that $\operatorname{diam}\left(G_{3}^{t}\right)=2 t-1$ while $\operatorname{rvc}\left(G_{3}^{t}\right)=3(t-1)=\frac{3}{2}\left(\operatorname{diam}\left(G_{3}^{t}\right)-1\right)$, since all internal vertices must have mutually distinct colors. Analogously, for the graph $G_{4}^{t}$, we have $\operatorname{diam}\left(G_{4}^{t}\right)=\left\lfloor\frac{t}{2}\right\rfloor+2$, but $\operatorname{rvc}\left(G_{4}^{t}\right)=t \geq 2\left(\operatorname{diam}\left(G_{4}^{t}\right)-2\right)$. Thus, each of the graphs $G_{1}^{t}, G_{2}^{t}, G_{3}^{t}$ and $G_{4}^{t}$ must contain an induced subgraph isomorphic to one of the graphs $X, Y$.

Consider the graph $G_{1}^{t}$. Up to symmetry, we have that $X$ is an induced subgraph of $G_{1}^{t}$ excluding $P_{3}$ and $P_{4}$. Now we consider the graph $G_{2}^{t}$. Obviously, $G_{2}^{t}$ is $X$-free, since $G_{2}^{t}$ is $K_{1,3}$-free. Hence, $G_{2}^{t}$ contains $Y$, implying $Y \stackrel{\text { IND }}{\subset} K_{r}^{h}$ for some $r \geq 3$ (for $r \leq 2$ we get $Y \stackrel{\text { IND }}{\subset} P_{4}$, which is excluded by the assumptions).

Now we consider the graph $G_{3}^{t}$. There are two possibilities.
(i) $Y \stackrel{\text { IND }}{\subset} G_{3}^{t}$. Then $Y \stackrel{\text { IND }}{\subset} N$. Now we consider the graph $G_{4}^{t} . G_{4}^{t}$ is $N$-free, so we get $X \stackrel{\text { IND }}{\subset} S_{1,2,2}$.
(ii) $X \stackrel{\text { IND }}{\subset} G_{3}^{t}$. Then $X=P_{5}$. As the case $X=P_{5}$ and $Y=N$ is already covered by case (i), we have that $X=P_{5}$ and $Y \stackrel{\text { IND }}{\subset} K_{r}^{h}, r \geq 4$.
This completes the proof.
It is easy to observe that if $X \stackrel{\text { IND }}{\subset} X^{\prime}$, then every $(X, Y)$-free graph is also $\left(X^{\prime}, Y\right)$-free. Thus, when proving the sufficiency of Theorem 6 , we will be always interested in maximal pairs of forbidden subgraphs, i.e., pairs $X, Y$ such that, if replacing one of $X, Y$, say $X$, with a graph $X^{\prime} \neq X$ such that $X \stackrel{\text { IND }}{\subset} X^{\prime}$, then the statement under consideration is not true for $\left(X^{\prime}, Y\right)$-free graphs.

Theorem 7. Let $G$ be a connected $\left(P_{5}, K_{r}^{h}\right)$-free graph for some $r \geq 4$. Then $r v c(G) \leq \operatorname{diam}(G)+r$.

Proof. From Theorem 4, we have that $G$ has a dominating clique or a dominating $P_{3}$.

Case 1. $G$ has a dominating $P_{3}$. We color the vertices of $P_{3}$ with colors $1,2,3$ and color the remaining vertices arbitrarily (e.g., all of them have color 1). One can easily check that this vertex-coloring can make $G$ rainbow vertex-connected. So, in this case, $r v c(G) \leq 3 \leq \operatorname{diam}(G)+r$.

Case 2. $G$ has a dominating clique, denoted by $K_{p}$. Set $W=V(G) \backslash V\left(K_{p}\right)$, $H=G \backslash E\left(K_{p}\right)$. Let $A$ be an independent set in $G[W]$ and $B \subset V\left(K_{p}\right)$ such that $H[A \cup B]=\ell K_{2}$ (that is, a matching of order $\ell$ ) and $\ell$ is maximal. Then $\ell<r$, for otherwise, $G[A \cup B]$ contains an induced $K_{r}^{h}$. Moreover, for $x \in W \backslash A$, $N_{A \cup B}(x) \neq \emptyset$, since $\ell$ is maximal. Now we define the following vertex-coloring of $G$. Use colors $1,2, \ldots, \ell$ to color each vertex in $B$, color the vertices of $A$ with color $\ell+1$, the vertices of $V\left(K_{p}\right) \backslash B$ with color $\ell+2$, and color the remaining vertices arbitrarily (e.g., all of them have color 1). Thus, pairs of vertices in $\left(A \cup V\left(K_{p}\right)\right) \times V(G)$ are rainbow vertex-connected. As for $x_{1}, x_{2} \in W \backslash A$, let $y_{1} \in N_{A \cup B}\left(x_{1}\right), y_{2} \in N_{K_{p}}\left(x_{2}\right)$. Then there is a vertex-rainbow $\left(x_{1}, x_{2}\right)$-path containing $y_{1}$ and $y_{2}$. $\operatorname{So}, r v c(G) \leq \ell+2 \leq r+1 \leq \operatorname{diam}(G)+r$.

The proof is complete.
Now let $G$ be an $\left(S_{1,2,2}, N\right)$-free graph, let $x, y \in V(G)$, and let $P: x=$ $v_{0}, v_{1}, \ldots, v_{k}=y(k \geq 3)$ be a shortest $(x, y)$-path in $G$. Let $z \in V(G) \backslash V(P)$. If $\left|N_{P}(z)\right| \geq 2$ and $\left\{v_{i}, v_{j}\right\} \subset N_{P}(z)$, then $|i-j| \leq 2$ and $\left|N_{P}(z)\right| \leq 3$, since $P$ is a shortest path. Moreover, the following facts are easily observed.

- If $\left|N_{P}(z)\right|=1$, then, since $G$ is $S_{1,2,2}$-free, $z$ is adjacent to $x, v_{1}, v_{k-1}$ or $y$.
- If $\left|N_{P}(z)\right|=3$, then the vertices of $N_{P}(z)$ must be consecutive on $P$, since $P$ is a shortest path.
This motivates the following notations:
- $A_{i}:=\left\{z \in V(G) \backslash V(P) \mid N_{P}(z)=\left\{v_{i}\right\}\right\}$ for $i=0,1, k-1, k$;
- $L_{i}:=\left\{z \in V(G) \backslash V(P) \mid N_{P}(z)=\left\{v_{i-1}, v_{i+1}\right\}\right\}$ for $1 \leq i \leq k-1$;
- $M_{i}:=\left\{z \in V(G) \backslash V(P) \mid N_{P}(z)=\left\{v_{i-1}, v_{i}\right\}\right\}$ for $1 \leq i \leq k$;
- $N_{i}:=\left\{z \in V(G) \backslash V(P) \mid N_{P}(z)=\left\{v_{i-1}, v_{i}, v_{i+1}\right\}\right\}$ for $1 \leq i \leq k-1$.

We further set $S=V(P) \cup N_{G}(P)$ and $R=V(G) \backslash S$.
Lemma 1. Let $G$ be an $\left(S_{1,2,2}, N\right)$-free graph, let $x, y \in V(G)$ be such that $\operatorname{dist}_{G}(x, y) \geq 4$ and let $P: x=v_{0}, v_{1}, \ldots, v_{k}=y$, be a shortest $(x, y)$-path in $G$. Then
(i) $N_{G}\left(M_{i}\right) \subset S, i=2, \ldots, k-1$;
(ii) $N_{G}\left(N_{i}\right) \subset S, i=2, \ldots, k-2$;
(iii) $N_{G}\left(L_{i}\right) \subset S, i=1, \ldots, k-1$;
(iv) $N_{P}(R)=\emptyset$;
(v) $N_{S}(R) \subset A_{0} \cup M_{1} \cup N_{1} \cup N_{k-1} \cup M_{k} \cup A_{k}$.

Proof. If $z v \in E(G)$ for some $z \in R$ and $v \in M_{i}, 2 \leq i \leq k-1$, then we have $G\left[\left\{v_{i-2}, v_{i-1}, v_{i}, v_{i+1}, v, z\right\}\right] \simeq N$, a contradiction. Hence, (i) follows. To show (ii), we observe that if $z v \in E(G)$ for some $z \in R$ and $v \in N_{i}, 2 \leq$ $i \leq k-2$, then we have $G\left[\left\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v, z\right\}\right] \simeq S_{1,2,2}$, a contradiction. Similarly, for (iii), if $z v \in E(G)$ for some $z \in R$ and $v \in L_{i}, 1 \leq$ $i \leq k-1$, then for $i=1$ we have $G\left[\left\{v_{1}, v_{2}, v_{3}, v_{4}, v, z\right\}\right] \simeq S_{1,2,2}$, for $2 \leq$ $i \leq k-2$ we have $G\left[\left\{z, v, v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\right\}\right] \simeq S_{1,2,2}$, and for $i=k-1$, $G\left[\left\{v_{k-1}, v_{k-2}, v_{k-3}, v_{k-4}, v, z\right\}\right] \simeq S_{1,2,2}$, a contradiction. Part (iv) follows immediately from the definition of $R$, and by (i) through (iii), we have $N_{S}(R) \subset$ $A_{0} \cup A_{1} \cup M_{1} \cup N_{1} \cup N_{k-1} \cup M_{k} \cup A_{k-1} \cup A_{k}$. But if $z v \in E(G)$ for some $z \in R$ and $v \in A_{1}$, then $G\left[\left\{v_{0}, v_{1}, v_{2}, v_{3}, v, z\right\}\right] \simeq S_{1,2,2}$, a contradiction. Similarly, we have $N_{A_{k-1}}(R)=\emptyset$, implying (v).

The proof is complete.
Theorem 8. Let $G$ be a connected $\left(S_{1,2,2}, N\right)$-free graph. Then rvc $(G) \leq \operatorname{diam}(G)$ +11 .
Proof. Let $G$ be a connected $\left(S_{1,2,2}, N\right)$-free graph. If $\operatorname{diam}(G) \leq 2$, then $\operatorname{rvc}(G)$ $=\operatorname{diam}(G)-1$. Thus, for the rest of the proof we suppose that $\operatorname{diam}(G)=d \geq 3$. Let $v_{0}, v_{d} \in V(G)$ be such that $\operatorname{dist}_{G}\left(v_{0}, v_{d}\right)=d$, let $P: v_{0} v_{1} v_{2} \cdots v_{d}$ be a diameter path in $G$, and let $A_{i}, L_{i}, M_{i}, N_{i}, S, R$ be defined as above.

We distinguish three cases according to the value of $d$.
Case 1. $d=3$. First, we partition $V(G)$ into four parts $P, N_{G}(P), N_{G}^{2}(P)$ and $N_{G}^{3}(P)$ according to the distance from $P$. Then, for the vertices in $N_{G}(P)$, we can partition them into three parts $X_{1}=A_{0} \cup M_{1} \cup L_{1} \cup N_{1}, X_{2}=A_{3} \cup M_{3} \cup L_{2} \cup N_{2}$ and $X_{3}=A_{1} \cup M_{2} \cup A_{2}$. We must point out that $X_{1} \cap X_{2}=\emptyset$ and $N_{R}\left(X_{3}\right)=\emptyset$, whose proof is similar to that of Lemma 1. Then we denote $Y_{i}$ the set of vertices in $N_{G}^{2}(P)$ such that for each $v \in Y_{i}, N_{N(P)}(v) \subset X_{i}, i=1,2$, and $Y_{3}=N_{G}^{2}(P) \backslash\left(Y_{1} \cup\right.$ $Y_{2}$ ). With a similar reason as above, $N_{N_{G}^{3}(P)}\left(Y_{3}\right)=\emptyset$. So, analogously we can partition $N_{G}^{3}(P)$ into three parts $Z_{1}, Z_{2}$ and $Z_{3}$. It should be noticed that $Z_{1}=\emptyset$; otherwise there exists a vertex $z \in Z_{1}$ such that $\operatorname{dist}_{G}\left(z, v_{3}\right) \geq 4$, a contradiction. Symmetrically, we have $Z_{2}=\emptyset$.

Now, we define a vertex-coloring of $G$ that uses at most 14 colors. Color the vertices of $P$ with colors $0,1,2,3$ and color the vertices in $A_{0}, M_{1}, L_{1}, N_{1}, N_{2}$, $L_{2}, M_{3}, A_{3}, Y_{1}$ and $Y_{2}$ with colors $4,5, \ldots, 13$, respectively. Then color the remaining vertices arbitrarily (e.g., all of them have color 0 ). We can show that this vertex-coloring can make $G$ rainbow vertex-connected. We only need to
verify that for a pair of vertices $x, y \in\left(Y_{1} \times Y_{1}\right) \cup\left(Y_{2} \times Y_{2}\right)$, there exists a vertex-rainbow path connecting them. Without loss of generality, we suppose $(x, y) \in Y_{1} \times Y_{1}$. If $\operatorname{dist}_{G}(x, y) \leq 2$, then there is nothing left to do. Next we consider the case $\operatorname{dist}_{G}(x, y) \geq 3$. Let $x^{\prime}$ be an arbitrary neighbor of $x$ in $X_{1}$, and $y^{\prime}$ an arbitrary neighbor of $y$ in $X_{1}$. We claim that $x^{\prime}$ and $y^{\prime}$ cannot have the same color. Otherwise, we suppose that $x^{\prime}$ and $y^{\prime}$ are colored with the same color, i.e., they are in the same vertex-class of $X_{1}$, and let $i=\max \left\{j \mid v_{j} \in N_{P}\left(x^{\prime}\right) \cap N_{P}\left(y^{\prime}\right)\right\}$. Then we have $G\left[\left\{v_{i}, v_{i+1}, x^{\prime}, y^{\prime}, x, y\right\}\right] \simeq S_{1,2,2}$ if $x^{\prime} y^{\prime} \notin E(G)$, or $G\left[\left\{v_{i}, v_{i+1}, x^{\prime}, y^{\prime}, x, y\right\}\right] \simeq N$ if $x^{\prime} y^{\prime} \in E(G)$, respectively. So, the colors of $x^{\prime}$ and $y^{\prime}$ must be different. Then the $(x, y)$-path $P_{1}: x x^{\prime} v_{0} y^{\prime} y$ is vertex-rainbow. Hence, we have $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+11$.

Case 2. $d=4$. Similarly, with the partition and the vertex-coloring of Case 1, we can get that $\operatorname{rvc}(G) \leq 15=\operatorname{diam}(G)+11$.

Case 3. $d \geq 5$. Set $B_{c}=\left(\bigcup_{i=2}^{d-2} N_{i}\right) \cup\left(\bigcup_{i=2}^{d-1} M_{i}\right) \cup\left(\bigcup_{i=1}^{d-1} L_{i}\right) \cup A_{1} \cup A_{d-1} \cup$ $\left\{v_{1}, v_{2}, \ldots, v_{d-1}\right\}, X=A_{0} \cup M_{1} \cup N_{1} \cup N_{d-1} \cup M_{d} \cup A_{d}, X_{1}=A_{0} \cup M_{1} \cup N_{1}$, and $X_{2}=N_{d-1} \cup M_{d} \cup A_{d}$. By virtue of Lemma 1, we have $N_{G}\left(B_{c}\right) \subset S$.

Subcase 3.1. $B_{c}$ is a cut-set of $G$. We claim that $S \cup N_{G}(S)=V(G)$. Suppose, to the contrary, that $z \in R$ is at distance 2 from $S$. Then, by Lemma 1 and the assumption of Case 1 , as well as the symmetry, we can assume that $N_{S}^{2}(z) \subset X_{1}$. Let $Q$ be a shortest $\left(z, v_{d}\right)$-path, let $w$ be the first vertex of $Q$ in $B_{c}$ (it exists by the assumption of Subcase 3.1), and let $w^{-}$be the predecessor of $w$ on $Q$. By Lemma $1, \operatorname{dist}\left(w^{-}, P\right)=1$, implying $w^{-} \in X_{1}$. Then $\operatorname{dist}_{G}\left(w^{-}, v_{d}\right) \geq d-1$; otherwise, the path $v_{0} w^{-} Q v_{d}$ is a $\left(v_{0}, v_{d}\right)$-path shorter than $P$. Since $\operatorname{dist}_{G}\left(z, w^{-}\right) \geq 2$, we have $\operatorname{dist}_{G}\left(z, v_{d}\right) \geq d+1$, contradicting $\operatorname{diam}(G)=d$. Hence, we have $S \cup N_{G}(S)=V(G)$. Moreover, with a similar argument to that of Case 1, we have that for $x, y \in R$ with distance at least 3 , their neighbors $x^{\prime}$ and $y^{\prime}$ cannot be in the same vertex-class of $X$.

Now we define a vertex-coloring of $G$ that uses at most $d+7$ colors. Color the vertices of $P$ with colors $0,1, \ldots, d$ and color the vertices in $A_{0}, M_{1}, N_{1}, N_{d-1}$, $M_{d}$ and $A_{d}$ with colors $d+1, d+2, \ldots, d+6$, respectively. Then color the remaining vertices arbitrarily (e.g., all of them have color 0 ). We can show that this vertex-coloring can make $G$ rainbow vertex-connected. For any pair of vertices in $S \times(S \cup R)$, we can easily find a vertex-rainbow path connecting them. For a pair $(x, y) \in R \times R$, if $\operatorname{dist}_{G}(x, y) \leq 2$, then there is nothing left to do. Next we consider $\operatorname{dist}_{G}(x, y) \geq 3$. From above, we know that their neighbors $x^{\prime}$ and $y^{\prime}$ in $X$ are colored differently. So, the $(x, y)$-path containing $x^{\prime}$ and $y^{\prime}$ is vertex-rainbow. Consequently, we have $\operatorname{rvc}(G) \leq \operatorname{diam}(G)+7$.

Subcase 3.2. $B_{c}$ is not a cut-set of $G$. Set $H=G-B_{c}$. Let $P^{\prime}: v_{d} v_{d+1} \cdots$ $v_{d+\ell-1} v_{d+\ell}=v_{0}$ be a shortest $\left(v_{d}, v_{0}\right)$-path in $H$. Since $P$ is a diameter path,
$\ell \geq d \geq 5$. If $v_{d+1}$ is adjacent to $v_{d-2}$, then $G\left[\left\{v_{d}, v_{d+1}, v_{d-2}, v_{d-3}, v_{d+2}, v_{d+3}\right\}\right] \simeq$ $S_{1,2,2}$, a contradiction. So, $v_{d+1} \in A_{d} \cup M_{d}$. Similarly, we have $v_{d+\ell-1} \in A_{0} \cup M_{1}$.

Set $P^{d}: v_{d-1} v_{d} v_{d+1}$ if $v_{d-1} v_{d+1} \notin E(G)$, or $P^{d}: v_{d-1} v_{d+1}$ if $v_{d-1} v_{d+1} \in E(G)$, respectively. Similarly, set $P^{0}: v_{d+\ell-1} v_{0} v_{1}$ if $v_{d+\ell-1} v_{1} \notin E(G)$, or $P^{d}: v_{d+\ell-1} v_{1}$ if $v_{d+\ell-1} v_{1} \in E(G)$, respectively. Finally, set $C: v_{1} P v_{d-1} P^{d} v_{d+1} P^{\prime} v_{d+\ell-1} P^{0} v_{1}$. Then $C$ is a cycle of length at least $2 d-2$.

Claim 1. The cycle $C$ is chordless.
Proof. This proof can be found in [5]. But for the sake of completeness, we provide the proof here. Suppose, to the contrary, that $v_{i} v_{j} \in E(G)$ is a chord in $C$. Since both $P$ and $P^{\prime}$ are chordless, we can choose the notation such that $1 \leq i \leq d-1$ and $d+1 \leq j \leq d+\ell-1$. Since $v_{j} \in V\left(P^{\prime}\right)$, we have $v_{j} \notin B_{c}$ by the definition of $P^{\prime}$, implying $i=d-1$ and $v_{j} \in M_{d}$, or, symmetrically, $i=1$ and $v_{j} \in M_{1}$. This implies that in the first case $v_{j}=v_{d+1}$; in the second case $v_{j}=v_{d+\ell-1}$; and in both cases $v_{i} v_{j} \in E(C)$ by the definition of $C$. Thus, $C$ is chordless.

Claim 2. $\ell \leq d+2$.
Proof. Assume that $\ell \geq d+3$, and let $Q$ be a shortest $\left(v_{0}, v_{d+2}\right)$-path in $G$. Then $|E(Q)| \leq d$ (since $\operatorname{diam}(G)=d$ ). Since $\ell \geq d+3$ and $P^{\prime}$ is shortest in $H=G-B_{c}$, we have $\operatorname{dist}_{H}\left(v_{0}, v_{d+2}\right) \geq d+1$. So, $Q$ must contain a vertex from $B_{c}$. Let $w$ be the last vertex of $Q$ in $B_{c}$, and let $w^{-}$and $w^{+}$be its predecessor and successor on $Q$, respectively (they exist since $v_{d+2} \notin B_{c}$ by the definition of $P^{\prime}$ ). By Lemma $1, w^{+}$ is at distance at most 1 from $P$. Since clearly $w^{+} \notin\left\{v_{0}, v_{d}\right\}$, either $w^{+} v_{0} \in E(G)$ or $w^{+} v_{d} \in E(G)$. If $w^{+} v_{0} \in E(G)$, then $v_{0} w^{+} Q v_{d+2}$ is a ( $v_{0}, v_{d+2}$ )-path shorter than $Q$, a contradiction. Thus, $w^{+} v_{d} \in E(G)$. Now, $w^{+} \neq v_{d+2}$ since $P^{\prime}$ is chordless, implying $\operatorname{dist}_{G}\left(v_{0}, w^{+}\right) \leq d-1$. On the other hand, $\operatorname{dist}_{G}\left(v_{0}, w^{+}\right) \geq$ $d-1$; otherwise, $v_{0} Q w^{+} v_{d}$ is a $\left(v_{0}, v_{d}\right)$-path of length at most $d-1$, contradicting the fact that $P$ is a diameter path. Hence, $\operatorname{dist}_{G}\left(v_{0}, w^{+}\right)=d-1$, implying that $\operatorname{dist}_{G}\left(v_{0}, w\right)=d-2$ and $w^{+} v_{d+2} \in E(Q)$. Since $v_{d+2}, v_{d+3} \in R$, we have $G\left[\left\{v_{d+3}, v_{d+2}, v_{d}, w^{+}, w, w^{-}\right\}\right] \simeq S_{1,2,2}$, a contradiction. Hence, $\ell \leq d+2$.

Claim 3. $C \cup N_{G}(C)=V(G)$, and every vertex in $V(G) \backslash V(C)$ has at least 2 neighbors in $C$.
Proof. Suppose that a vertex $x \in V(G) \backslash V(C)$ at distance 1 from $C$ has exactly one neighbor in $C$, and set $N_{C}(x)=\{y\}$. Let $z_{1}, z_{2} \in N_{C}^{2}(x)$, and let $z_{1}^{\prime}, z_{2}^{\prime} \in$ $N_{C}^{3}(x)$. Then we have $G\left[\left\{x, y, z_{1}, z_{2}, z_{1}^{\prime}, z_{2}^{\prime}\right\}\right] \simeq S_{1,2,2}$, a contradiction.

Secondly, suppose, to the contrary, that $z \in V(G)$ is at distance 2 from $C$, and $y$ is a neighbor of $z$ at distance 1 from $C$. Then $\operatorname{dist}_{G}(z, P) \geq 2$; otherwise, $y=v_{0}$ or $y=v_{d}$, without loss of generality, we assume $y=v_{0}$. Then $v_{1}$ must be adjacent to $v_{d+\ell-1}$, and thus, $G\left[\left\{z, y, v_{1}, v_{2}, v_{d+\ell-1}, v_{d+\ell-2}\right\}\right] \simeq N$, a contradiction. Hence, $z \in R$. If $y \in R$, then $y$ is not adjacent to any of $v_{1}, v_{2}$
and $v_{3}$. If $y \notin R$, then we have $y \in X$. Without loss of generality, we assume $y \in X_{2}$. Then $y$ is not adjacent to any of $v_{1}, v_{2}$ and $v_{3}$. Moreover, from above we know that $y$ has at least 2 neighbors in $C$. Let $x_{1}, x_{2} \in N_{C}(y)$ be the vertices closest to $v_{1}$ and $v_{3}$, respectively. Let $x_{1}^{\prime}$ and $x_{2}^{\prime}$ be their neighbors that are closer to $v_{1}$ and $v_{3}$ in $C$, respectively. Then $G\left[\left\{y, z, x_{1}, x_{2}, x_{1}^{\prime}, x_{2}^{\prime}\right\}\right] \simeq S_{1,2,2}$ if $x_{1} x_{2} \notin E(G)$, or $G\left[\left\{y, z, x_{1}, x_{2}, x_{1}^{\prime}, x_{2}^{\prime}\right\}\right] \simeq N$ if $x_{1} x_{2} \in E(G)$, respectively. Thus, $C$ is a dominating set of $G$.

By Claims 1 and 2, we know that $C$ is a chordless cycle of length at most $d+\ell \leq 2 d+2$. Now, we define a vertex-coloring of $G$ that uses at most $d+1$ colors. Relabel $C$ : $x_{1} x_{2} \cdots x_{k} x_{k+1}\left(=x_{1}\right), 8 \leq 2 d-2 \leq k \leq 2 d+2$. Then we assign color $i$ to the vertex $x_{i}$ if $1 \leq i \leq\left\lceil\frac{k}{2}\right\rceil$ and assign color $i-\left\lceil\frac{k}{2}\right\rceil$ to $x_{i}$ if $\left\lceil\frac{k}{2}\right\rceil<i \leq k$. We color the remaining vertices arbitrarily. We can show that this vertex-coloring can make $G$ rainbow vertex-connected.

From Theorem 1 and Claim 3, we know that under this vertex-coloring, pairs in $C \times V(G)$ are rainbow vertex-connected. For each vertex $z \in N_{G}(C)$, we may strengthen the result of Claim 3 that $z$ has at least two neighbors colored differently in $C$. Otherwise, we suppose that $z_{1}$ and $z_{2}$ are the only two neighbors of $z$ having the same color in $C$. From the vertex-coloring, we know that $\operatorname{dist}_{C}\left(z_{1}, z_{2}\right)=\left\lfloor\frac{k}{2}\right\rfloor \geq 4$. Then we can easily find an induced $S_{1,2,2}$, a contradiction. So, for a pair $(x, y) \in N_{G}(C) \times N_{G}(C)$, we can find a vertex $x^{\prime} \in N_{C}(x)$ and a vertex $y^{\prime} \in N_{C}(y)$ such that $x^{\prime}$ and $y^{\prime}$ are colored differently. Since there exists a vertex-rainbow path $P$ connecting $x^{\prime}$ and $y^{\prime}$ and the internal vertices of $P$ are colored differently from $x^{\prime}$ and $y^{\prime}$, the path $x x^{\prime} P y^{\prime} y$ is vertex-rainbow and connects $x$ and $y$. Hence, $\operatorname{rvc}(G) \leq d+1$.

The proof of Theorem 8 is complete.
Combining Proposition 2 with Theorems 7 and 8, we have proved Theorem 6.

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