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RAINBOW VERTEX-CONNECTION AND FORBIDDEN SUBGRAPHS

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Abstract

A path in a vertex-colored graph is called *vertex-rainbow* if its internal vertices have pairwise distinct colors. A vertex-colored graph G is *rainbow vertex-connected* if for any two distinct vertices of G, there is a vertexrainbow path connecting them. For a connected graph G, the *rainbow vertexconnection number* of G, denoted by rvc(G), is defined as the minimum number of colors that are required to make G rainbow vertex-connected. In this paper, we find all the families \mathcal{F} of connected graphs with $|\mathcal{F}| \in \{1, 2\}$, for which there is a constant $k_{\mathcal{F}}$ such that, for every connected \mathcal{F} -free graph G, $rvc(G) \leq diam(G) + k_{\mathcal{F}}$, where diam(G) is the diameter of G.

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1. INTRODUCTION

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [2] for those not defined here.

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Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow \{0, 1, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in G is called a *rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected edge-colored graph G, the *rainbow connection number* of G, denoted by rc(G), is defined as the minimum number of colors that are needed to make G rainbow connected. Observe that if G has n vertices, then $diam(G) \leq rc(G) \leq n - 1$. It is easy to verify that rc(G) = 1 if and only if G is a complete graph, and rc(G) = n - 1 if and only if G is a tree. The concept of rainbow connection of graphs was first introduced by Chartrand *et al.* in [3], and has been well-studied since then. For further details, we refer the reader to a survey paper [10] and a book [11].

Let G be a nontrivial connected graph with a vertex-coloring $c : V(G) \rightarrow \{0, 1, \ldots, t\}, t \in \mathbb{N}$, where adjacent vertices may be colored with the same color. A path of G is called vertex-rainbow if any two internal vertices of the path have distinct colors. The vertex-colored graph G is rainbow vertex-connected if any two vertices of G are connected by a vertex-rainbow path. For a connected graph G, the rainbow vertex-connection number of G, denoted by rvc(G), is the minimum number of colors used in a vertex-coloring of G to make G rainbow vertex-connected. The concept of rainbow vertex-connection of graphs was proposed by Krivelevich and Yuster in [6]. They showed that if G is a connected graph with n vertices and minimum degree δ , then $rvc(G) \leq 11n/\delta$. In [9], Li and Shi improved this bound. In [4], it was shown that computing the rainbow vertex-connection number of a graph is NP-hard. Recently, Li et al. in [7] proved that it is NP-complete to decide whether a given vertex-colored graph is rainbow vertex-connected even when the graph is bipartite.

For the rainbow vertex-connection number of graphs, the following observations are immediate.

Proposition 1. Let G be a connected graph with n vertices. Then

- (i) $diam(G) 1 \le rvc(G) \le n 2;$
- (ii) rvc(G) = diam(G) 1 if diam(G) = 1 or 2, with the assumption that complete graphs have rainbow vertex-connection number 0.

Note that the difference rvc(G) - diam(G) can be arbitrarily large. In fact, if G is a subdivision of a star $K_{1,n}$, then we have rvc(G) - diam(G) = (n+1) - 4 = n - 3, since in a rainbow vertex-connected coloring of G, the internal vertices must have distinct colors.

In [8], Li and Liu studied the rainbow vertex-connection number for any 2-connected graph, and determined the precise value of the rainbow vertex-connection number of the cycle C_n $(n \ge 3)$.

Theorem 1 [8]. Let C_n be a cycle of order $n \ (n \ge 3)$. Then

$$rvc(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13 \text{ or } 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \ge 16 \text{ or } n = 14. \end{cases}$$

Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free if G does not contain any induced subgraph isomorphic to a graph from \mathcal{F} . Specifically, for $\mathcal{F} = \{X\}$ we say that G is X-free, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)-free. The members of \mathcal{F} will be referred to in this context as forbidden induced subgraphs, and for $|\mathcal{F}| = 2$ we also say that \mathcal{F} is a forbidden pair.

In [5], Holub *et al.* considered the question: For which families \mathcal{F} of connected graphs, a connected \mathcal{F} -free graph G satisfies $rc(G) \leq diam(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on \mathcal{F})? They gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where N denotes the *net*, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 2 [5]. Let X be a connected graph. Then there is a constant $k_{\mathcal{F}}$ such that every connected X-free graph G satisfies $rc(G) \leq diam(G) + k_X$ if and only if $X = P_3$.

Theorem 3 [5]. Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rc(G) \leq diam(G) + k_{XY}$ if and only if (up to symmetry) either $X = K_{1,r}$ $(r \geq 4)$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.

Naturally, we may consider an analogous question concerning the rainbow vertex-connection number of graphs. In this paper, we will consider the following question.

For which families \mathcal{F} of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being \mathcal{F} -free implies $rvc(G) \leq diam(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and for $|\mathcal{F}| = 2$ in Section 4.

2. Preliminaries

In this section, we introduce some further notations and facts that will be needed for the proofs of our main results.

If G is a graph and $A \subset V(G)$, then G[A] denotes the subgraph of G induced by the vertex set A, and G - A the graph $G[V(G) \setminus A]$. An edge is called a pendant edge if one of its endvertices has degree one. The subdivision of a graph G is the graph obtained from G by adding a vertex of degree 2 to each edge of G. For $x, y \in V(G)$, a path in G from x to y will be referred to as an (x, y)-path, and, whenever necessary, it will be considered as oriented from x to y. For a subpath of a path P with origin u and terminus v (also referred to as a (u, v)-arc of P), we will use the notation uPv. If w is a vertex of a path with a fixed orientation, then w^- and w^+ denote the predecessor and successor of w, respectively.

For graphs X and G, we write $X \subset G$ if X is a subgraph of G, $X \subset G$ if X is an induced subgraph of G, and $X \simeq G$ if X is isomorphic to G. For two vertices $x, y \in V(G)$, we use $dist_G(x, y)$ to denote the distance between x and y in G. The diameter of G is defined as the maximum of $dist_G(x, y)$ among all pairs of vertices x, y of G, and will be denoted by diam(G). A shortest path joining two vertices at distance diam(G) will be referred to as a diameter path. The distance between a vertex $u \in V(G)$ and a set $S \subset V(G)$ is defined as $dist_G(u, S) := \min_{v \in S} dist_G(u, v)$. A set $D \subset V(G)$ is called dominating if every vertex in $V(G) \setminus D$ has a neighbor in D. In addition, if G[D] is connected, then we call D a connected dominating set. Throughout this paper, N denotes the set of all positive integers.

For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the kth-neighborhood of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case k = 1, we simply write $N_G(S)$ for $N_G^1(S)$, and if |S| = 1 with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we denote $N_M^k(S) = N_G^k(S) \cap M$ and $N_M^k(x) = N_G^k(x) \cap M$, and as above, we simply use $N_M(S)$ for $N_M^1(S)$ and $N_M(x)$ for $N_M^1(x)$. For a subgraph $P \subset G$, we write $N_P(x)$ for $N_{V(P)}(x)$. Finally, we will use P_k to denote the path on k vertices.

We end up this section with an important result that will be used in our proofs.

Theorem 4 [1]. Let G be a connected P_5 -free graph. Then G has a dominating clique or a dominating P_3 .

3. FAMILIES WITH ONE FORBIDDEN SUBGRAPH

In this section, we characterize all connected graphs X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$, where k_X is a constant.

Theorem 5. Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $rvc(G) \leq diam(G) + k_X$ if and only if $X = P_3$ or $X = P_4$.

Proof. We have $diam(G) \leq 2$, since G is P_4 -free. Then it follows from Proposition 1 that $rvc(G) = diam(G) - 1 \leq 1$.

Conversely, let $t \ge k_X + 5$, and G_1^t be the subdivision of $K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of the complete graph K_t (see Figure 1). Since $rvc(G_1^t) = t + 1$ but $diam(G_1^t) = 4$, X is an induced subgraph of G_1^t . Clearly, $rvc(G_2^t) = t$ but $diam(G_2^t) = 3$, and G_2^t is $K_{1,3}$ -free and P_5 -free. Hence, X is an induced subgraph of P_4 .

The proof is thus complete.



Figure 1. The graphs G_1^t and G_2^t .

4. FAMILIES WITH A PAIR OF FORBIDDEN SUBGRAPHS

For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex from each of three vertex-disjoint paths of lengths i, j, k, and $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths i, j, k (see Figure 2). In this context, we will also write K_t^h for the graph G_2^t introduced in the proof of Theorem 5.



Figure 2. The graphs $S_{i,j,k}$, $N_{i,j,k}$ and G_4^t .

The following statement, which is the main result of this section, characterizes all forbidden pairs X, Y for which there is a constant k_{XY} such that G being (X, Y)-free implies $rvc(G) \leq diam(G) + k_{XY}$. By virtue of Theorem 5, we exclude the case that one of X, Y is an induced subgraph of P_4 . Recall that the *net* is the graph $N = N_{1,1,1}$.

Theorem 6. Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq$

 $diam(G) + k_{XY}$ if and only if (up to symmetry) $X = P_5$ and $Y \stackrel{\text{IND}}{\subset} K_r^h$ $(r \ge 4)$, or $X \stackrel{\text{IND}}{\subset} S_{1,2,2}$ and $Y \stackrel{\text{IND}}{\subset} N$.

The proof of Theorem 6 will be divided into three separate results: we prove the necessity in Proposition 2, and Theorems 7 and 8 will establish the sufficiency of the forbidden pairs given in Theorem 6.

Proposition 2. Let $X, Y \neq P_3$ or P_4 be a pair of connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $rvc(G) \leq diam(G) + k_{XY}$. Then (up to symmetry) $X = P_5$ and $Y \subset K_r^h$ $(r \geq 4)$, or $X \subset S_{1,2,2}$ and $Y \subset N$.

Proof. Let $t \ge 2k_{XY} + 5$, and let (see Figure 2)

- $G_3^t = N_{t-1,t-1,t-1};$
- G_4^t be the graph obtained by attaching a pendant edge to each vertex of a cycle C_t .

We will also use the graphs G_1^t and $G_2^t (= K_t^h)$ shown in Figure 1.

For the graphs G_1^t and G_2^t , we have $diam(G_1^t) = 4$ but $rvc(G_1^t) = t + 1$, and $diam(G_2^t) = 3$ but $rvc(G_2^t) = t$, respectively. For the graph G_3^t , we observe that $diam(G_3^t) = 2t - 1$ while $rvc(G_3^t) = 3(t - 1) = \frac{3}{2}(diam(G_3^t) - 1)$, since all internal vertices must have mutually distinct colors. Analogously, for the graph G_4^t , we have $diam(G_4^t) = \lfloor \frac{t}{2} \rfloor + 2$, but $rvc(G_4^t) = t \ge 2(diam(G_4^t) - 2)$. Thus, each of the graphs G_1^t , G_2^t , G_3^t and G_4^t must contain an induced subgraph isomorphic to one of the graphs X, Y.

Consider the graph G_1^t . Up to symmetry, we have that X is an induced subgraph of G_1^t excluding P_3 and P_4 . Now we consider the graph G_2^t . Obviously, G_2^t is X-free, since G_2^t is $K_{1,3}$ -free. Hence, G_2^t contains Y, implying $Y \stackrel{\text{IND}}{\subset} K_r^h$ for some $r \ge 3$ (for $r \le 2$ we get $Y \stackrel{\text{IND}}{\subset} P_4$, which is excluded by the assumptions). Now we consider the graph G_3^t . There are two possibilities.

- (i) $Y \subset G_3^t$. Then $Y \subset N$. Now we consider the graph G_4^t . G_4^t is N-free, so we get $X \subset S_{1,2,2}$.
- (ii) $X \subset G_3^t$. Then $X = P_5$. As the case $X = P_5$ and Y = N is already covered by case (i), we have that $X = P_5$ and $Y \stackrel{\text{IND}}{\subset} K_r^h, r \ge 4$. This completes the proof.

It is easy to observe that if $X \subset X'$, then every (X, Y)-free graph is also (X', Y)-free. Thus, when proving the sufficiency of Theorem 6, we will be always interested in maximal pairs of forbidden subgraphs, i.e., pairs X, Y such that, if replacing one of X, Y, say X, with a graph $X' \neq X$ such that $X \subset X'$, then the statement under consideration is not true for (X', Y)-free graphs.

Theorem 7. Let G be a connected (P_5, K_r^h) -free graph for some $r \ge 4$. Then $rvc(G) \le diam(G) + r$.

Proof. From Theorem 4, we have that G has a dominating clique or a dominating P_3 .

Case 1. G has a dominating P_3 . We color the vertices of P_3 with colors 1, 2, 3 and color the remaining vertices arbitrarily (e.g., all of them have color 1). One can easily check that this vertex-coloring can make G rainbow vertex-connected. So, in this case, $rvc(G) \leq 3 \leq diam(G) + r$.

Case 2. G has a dominating clique, denoted by K_p . Set $W = V(G) \setminus V(K_p)$, $H = G \setminus E(K_p)$. Let A be an independent set in G[W] and $B \subset V(K_p)$ such that $H[A \cup B] = \ell K_2$ (that is, a matching of order ℓ) and ℓ is maximal. Then $\ell < r$, for otherwise, $G[A \cup B]$ contains an induced K_r^h . Moreover, for $x \in W \setminus A$, $N_{A \cup B}(x) \neq \emptyset$, since ℓ is maximal. Now we define the following vertex-coloring of G. Use colors $1, 2, \ldots, \ell$ to color each vertex in B, color the vertices of A with color $\ell + 1$, the vertices of $V(K_p) \setminus B$ with color $\ell + 2$, and color the remaining vertices arbitrarily (e.g., all of them have color 1). Thus, pairs of vertices in $(A \cup V(K_p)) \times V(G)$ are rainbow vertex-connected. As for $x_1, x_2 \in W \setminus A$, let $y_1 \in N_{A \cup B}(x_1), y_2 \in N_{K_p}(x_2)$. Then there is a vertex-rainbow (x_1, x_2) -path containing y_1 and y_2 . So, $rvc(G) \leq \ell + 2 \leq r + 1 \leq diam(G) + r$.

The proof is complete.

Now let G be an $(S_{1,2,2}, N)$ -free graph, let $x, y \in V(G)$, and let $P : x = v_0, v_1, \ldots, v_k = y$ $(k \ge 3)$ be a shortest (x, y)-path in G. Let $z \in V(G) \setminus V(P)$. If $|N_P(z)| \ge 2$ and $\{v_i, v_j\} \subset N_P(z)$, then $|i - j| \le 2$ and $|N_P(z)| \le 3$, since P is a shortest path. Moreover, the following facts are easily observed.

- If $|N_P(z)| = 1$, then, since G is $S_{1,2,2}$ -free, z is adjacent to x, v_1, v_{k-1} or y.
- If $|N_P(z)| = 3$, then the vertices of $N_P(z)$ must be consecutive on P, since P is a shortest path.

This motivates the following notations:

- $A_i := \{z \in V(G) \setminus V(P) | N_P(z) = \{v_i\}\}$ for i = 0, 1, k 1, k;
- $L_i := \{z \in V(G) \setminus V(P) | N_P(z) = \{v_{i-1}, v_{i+1}\}\}$ for $1 \le i \le k-1$;
- $M_i := \{z \in V(G) \setminus V(P) | N_P(z) = \{v_{i-1}, v_i\}\}$ for $1 \le i \le k$;
- $N_i := \{z \in V(G) \setminus V(P) | N_P(z) = \{v_{i-1}, v_i, v_{i+1}\}\}$ for $1 \le i \le k-1$. We further set $S = V(P) \cup N_G(P)$ and $R = V(G) \setminus S$.

Lemma 1. Let G be an $(S_{1,2,2}, N)$ -free graph, let $x, y \in V(G)$ be such that $dist_G(x, y) \ge 4$ and let $P : x = v_0, v_1, \ldots, v_k = y$, be a shortest (x, y)-path in G. Then

(i) $N_G(M_i) \subset S, i = 2, ..., k - 1;$

(ii) $N_G(N_i) \subset S, i = 2, ..., k - 2;$ (iii) $N_G(L_i) \subset S, i = 1, ..., k - 1;$ (iv) $N_P(R) = \emptyset;$ (v) $N_S(R) \subset A_0 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_k.$

Proof. If $zv \in E(G)$ for some $z \in R$ and $v \in M_i$, $2 \leq i \leq k-1$, then we have $G[\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v, z\}] \simeq N$, a contradiction. Hence, (i) follows. To show (ii), we observe that if $zv \in E(G)$ for some $z \in R$ and $v \in N_i$, $2 \leq i \leq k-2$, then we have $G[\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v, z\}] \simeq S_{1,2,2}$, a contradiction. Similarly, for (iii), if $zv \in E(G)$ for some $z \in R$ and $v \in L_i$, $1 \leq i \leq k-1$, then for i = 1 we have $G[\{v_1, v_2, v_3, v_4, v, z\}] \simeq S_{1,2,2}$, for $2 \leq i \leq k-2$ we have $G[\{z, v, v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\}] \simeq S_{1,2,2}$, and for i = k-1, $G[\{v_{k-1}, v_{k-2}, v_{k-3}, v_{k-4}, v, z\}] \simeq S_{1,2,2}$, a contradiction. Part (iv) follows immediately from the definition of R, and by (i) through (iii), we have $N_S(R) \subset A_0 \cup A_1 \cup M_1 \cup N_1 \cup N_{k-1} \cup M_k \cup A_{k-1} \cup A_k$. But if $zv \in E(G)$ for some $z \in R$ and $v \in A_1$, then $G[\{v_0, v_1, v_2, v_3, v, z\}] \simeq S_{1,2,2}$, a contradiction. Similarly, we have $N_{A_{k-1}}(R) = \emptyset$, implying (v).

The proof is complete.

Theorem 8. Let G be a connected $(S_{1,2,2}, N)$ -free graph. Then $rvc(G) \leq diam(G) + 11$.

Proof. Let G be a connected $(S_{1,2,2}, N)$ -free graph. If $diam(G) \leq 2$, then rvc(G) = diam(G) - 1. Thus, for the rest of the proof we suppose that $diam(G) = d \geq 3$. Let $v_0, v_d \in V(G)$ be such that $dist_G(v_0, v_d) = d$, let $P : v_0v_1v_2\cdots v_d$ be a diameter path in G, and let A_i, L_i, M_i, N_i, S, R be defined as above.

We distinguish three cases according to the value of d.

Case 1. d = 3. First, we partition V(G) into four parts $P, N_G(P), N_G^2(P)$ and $N_G^3(P)$ according to the distance from P. Then, for the vertices in $N_G(P)$, we can partition them into three parts $X_1 = A_0 \cup M_1 \cup L_1 \cup N_1$, $X_2 = A_3 \cup M_3 \cup L_2 \cup N_2$ and $X_3 = A_1 \cup M_2 \cup A_2$. We must point out that $X_1 \cap X_2 = \emptyset$ and $N_R(X_3) = \emptyset$, whose proof is similar to that of Lemma 1. Then we denote Y_i the set of vertices in $N_G^2(P)$ such that for each $v \in Y_i$, $N_{N(P)}(v) \subset X_i$, i = 1, 2, and $Y_3 = N_G^2(P) \setminus (Y_1 \cup Y_2)$. With a similar reason as above, $N_{N_G^3(P)}(Y_3) = \emptyset$. So, analogously we can partition $N_G^3(P)$ into three parts Z_1, Z_2 and Z_3 . It should be noticed that $Z_1 = \emptyset$; otherwise there exists a vertex $z \in Z_1$ such that $dist_G(z, v_3) \ge 4$, a contradiction. Symmetrically, we have $Z_2 = \emptyset$.

Now, we define a vertex-coloring of G that uses at most 14 colors. Color the vertices of P with colors 0, 1, 2, 3 and color the vertices in $A_0, M_1, L_1, N_1, N_2,$ L_2, M_3, A_3, Y_1 and Y_2 with colors $4, 5, \ldots, 13$, respectively. Then color the remaining vertices arbitrarily (e.g., all of them have color 0). We can show that this vertex-coloring can make G rainbow vertex-connected. We only need to verify that for a pair of vertices $x, y \in (Y_1 \times Y_1) \cup (Y_2 \times Y_2)$, there exists a vertex-rainbow path connecting them. Without loss of generality, we suppose $(x, y) \in Y_1 \times Y_1$. If $dist_G(x, y) \leq 2$, then there is nothing left to do. Next we consider the case $dist_G(x, y) \geq 3$. Let x' be an arbitrary neighbor of x in X_1 , and y' an arbitrary neighbor of y in X_1 . We claim that x' and y' cannot have the same color. Otherwise, we suppose that x' and y' are colored with the same color, i.e., they are in the same vertex-class of X_1 , and let $i = \max\{j \mid v_j \in N_P(x') \cap N_P(y')\}$. Then we have $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq S_{1,2,2}$ if $x'y' \notin E(G)$, or $G[\{v_i, v_{i+1}, x', y', x, y\}] \simeq N$ if $x'y' \in E(G)$, respectively. So, the colors of x' and y' must be different. Then the (x, y)-path $P_1 : xx'v_0y'y$ is vertex-rainbow. Hence, we have $rvc(G) \leq diam(G) + 11$.

Case 2. d = 4. Similarly, with the partition and the vertex-coloring of Case 1, we can get that $rvc(G) \le 15 = diam(G) + 11$.

Case 3. $d \ge 5$. Set $B_c = \left(\bigcup_{i=2}^{d-2} N_i\right) \cup \left(\bigcup_{i=2}^{d-1} M_i\right) \cup \left(\bigcup_{i=1}^{d-1} L_i\right) \cup A_1 \cup A_{d-1} \cup \{v_1, v_2, \dots, v_{d-1}\}, X = A_0 \cup M_1 \cup N_1 \cup N_{d-1} \cup M_d \cup A_d, X_1 = A_0 \cup M_1 \cup N_1, \text{ and } X_2 = N_{d-1} \cup M_d \cup A_d.$ By virtue of Lemma 1, we have $N_G(B_c) \subset S$.

Subcase 3.1. B_c is a cut-set of G. We claim that $S \cup N_G(S) = V(G)$. Suppose, to the contrary, that $z \in R$ is at distance 2 from S. Then, by Lemma 1 and the assumption of Case 1, as well as the symmetry, we can assume that $N_S^2(z) \subset X_1$. Let Q be a shortest (z, v_d) -path, let w be the first vertex of Q in B_c (it exists by the assumption of Subcase 3.1), and let w^- be the predecessor of w on Q. By Lemma 1, $dist(w^-, P) = 1$, implying $w^- \in X_1$. Then $dist_G(w^-, v_d) \ge d - 1$; otherwise, the path $v_0w^-Qv_d$ is a (v_0, v_d) -path shorter than P. Since $dist_G(z, w^-) \ge 2$, we have $dist_G(z, v_d) \ge d + 1$, contradicting diam(G) = d. Hence, we have $S \cup N_G(S) = V(G)$. Moreover, with a similar argument to that of Case 1, we have that for $x, y \in R$ with distance at least 3, their neighbors x' and y' cannot be in the same vertex-class of X.

Now we define a vertex-coloring of G that uses at most d + 7 colors. Color the vertices of P with colors $0, 1, \ldots, d$ and color the vertices in $A_0, M_1, N_1, N_{d-1}, M_d$ and A_d with colors $d + 1, d + 2, \ldots, d + 6$, respectively. Then color the remaining vertices arbitrarily (e.g., all of them have color 0). We can show that this vertex-coloring can make G rainbow vertex-connected. For any pair of vertices in $S \times (S \cup R)$, we can easily find a vertex-rainbow path connecting them. For a pair $(x, y) \in R \times R$, if $dist_G(x, y) \leq 2$, then there is nothing left to do. Next we consider $dist_G(x, y) \geq 3$. From above, we know that their neighbors x'and y' in X are colored differently. So, the (x, y)-path containing x' and y' is vertex-rainbow. Consequently, we have $rvc(G) \leq diam(G) + 7$.

Subcase 3.2. B_c is not a cut-set of G. Set $H = G - B_c$. Let $P' : v_d v_{d+1} \cdots v_{d+\ell-1} v_{d+\ell} = v_0$ be a shortest (v_d, v_0) -path in H. Since P is a diameter path,

 $\ell \geq d \geq 5$. If v_{d+1} is adjacent to v_{d-2} , then $G[\{v_d, v_{d+1}, v_{d-2}, v_{d-3}, v_{d+2}, v_{d+3}\}] \simeq S_{1,2,2}$, a contradiction. So, $v_{d+1} \in A_d \cup M_d$. Similarly, we have $v_{d+\ell-1} \in A_0 \cup M_1$.

Set $P^d: v_{d-1}v_dv_{d+1}$ if $v_{d-1}v_{d+1} \notin E(G)$, or $P^d: v_{d-1}v_{d+1}$ if $v_{d-1}v_{d+1} \in E(G)$, respectively. Similarly, set $P^0: v_{d+\ell-1}v_0v_1$ if $v_{d+\ell-1}v_1 \notin E(G)$, or $P^d: v_{d+\ell-1}v_1$ if $v_{d+\ell-1}v_1 \in E(G)$, respectively. Finally, set $C: v_1Pv_{d-1}P^dv_{d+1}P'v_{d+\ell-1}P^0v_1$. Then C is a cycle of length at least 2d-2.

Claim 1. The cycle C is chordless.

Proof. This proof can be found in [5]. But for the sake of completeness, we provide the proof here. Suppose, to the contrary, that $v_i v_j \in E(G)$ is a chord in C. Since both P and P' are chordless, we can choose the notation such that $1 \leq i \leq d-1$ and $d+1 \leq j \leq d+\ell-1$. Since $v_j \in V(P')$, we have $v_j \notin B_c$ by the definition of P', implying i = d-1 and $v_j \in M_d$, or, symmetrically, i = 1 and $v_j \in M_1$. This implies that in the first case $v_j = v_{d+1}$; in the second case $v_j = v_{d+\ell-1}$; and in both cases $v_i v_j \in E(C)$ by the definition of C. Thus, C is chordless.

Claim 2. $\ell \leq d+2$.

Proof. Assume that $\ell \geq d+3$, and let Q be a shortest (v_0, v_{d+2}) -path in G. Then $|E(Q)| \leq d$ (since diam(G) = d). Since $\ell \geq d+3$ and P' is shortest in $H = G - B_c$, we have $dist_H(v_0, v_{d+2}) \geq d+1$. So, Q must contain a vertex from B_c . Let w be the last vertex of Q in B_c , and let w^- and w^+ be its predecessor and successor on Q, respectively (they exist since $v_{d+2} \notin B_c$ by the definition of P'). By Lemma 1, w^+ is at distance at most 1 from P. Since clearly $w^+ \notin \{v_0, v_d\}$, either $w^+v_0 \in E(G)$ or $w^+v_d \in E(G)$. If $w^+v_0 \in E(G)$, then $v_0w^+Qv_{d+2}$ is a (v_0, v_{d+2}) -path shorter than Q, a contradiction. Thus, $w^+v_d \in E(G)$. Now, $w^+ \neq v_{d+2}$ since P' is chordless, implying $dist_G(v_0, w^+) \leq d-1$. On the other hand, $dist_G(v_0, w^+) \geq d-1$; otherwise, $v_0Qw^+v_d$ is a (v_0, v_d) -path of length at most d-1, contradicting the fact that P is a diameter path. Hence, $dist_G(v_0, w^+) = d-1$, implying that $dist_G(v_0, w) = d-2$ and $w^+v_{d+2} \in E(Q)$. Since $v_{d+2}, v_{d+3} \in R$, we have $G[\{v_{d+3}, v_{d+2}, v_d, w^+, w, w^-\}] \simeq S_{1,2,2}$, a contradiction. Hence, $\ell \leq d+2$.

Claim 3. $C \cup N_G(C) = V(G)$, and every vertex in $V(G) \setminus V(C)$ has at least 2 neighbors in C.

Proof. Suppose that a vertex $x \in V(G) \setminus V(C)$ at distance 1 from C has exactly one neighbor in C, and set $N_C(x) = \{y\}$. Let $z_1, z_2 \in N_C^2(x)$, and let $z'_1, z'_2 \in N_C^3(x)$. Then we have $G[\{x, y, z_1, z_2, z'_1, z'_2\}] \simeq S_{1,2,2}$, a contradiction.

Secondly, suppose, to the contrary, that $z \in V(G)$ is at distance 2 from C, and y is a neighbor of z at distance 1 from C. Then $dist_G(z, P) \ge 2$; otherwise, $y = v_0$ or $y = v_d$, without loss of generality, we assume $y = v_0$. Then v_1 must be adjacent to $v_{d+\ell-1}$, and thus, $G[\{z, y, v_1, v_2, v_{d+\ell-1}, v_{d+\ell-2}\}] \simeq N$, a contradiction. Hence, $z \in R$. If $y \in R$, then y is not adjacent to any of v_1, v_2 and v_3 . If $y \notin R$, then we have $y \in X$. Without loss of generality, we assume $y \in X_2$. Then y is not adjacent to any of v_1, v_2 and v_3 . Moreover, from above we know that y has at least 2 neighbors in C. Let $x_1, x_2 \in N_C(y)$ be the vertices closest to v_1 and v_3 , respectively. Let x'_1 and x'_2 be their neighbors that are closer to v_1 and v_3 in C, respectively. Then $G[\{y, z, x_1, x_2, x'_1, x'_2\}] \simeq S_{1,2,2}$ if $x_1x_2 \notin E(G)$, or $G[\{y, z, x_1, x_2, x'_1, x'_2\}] \simeq N$ if $x_1x_2 \in E(G)$, respectively. Thus, C is a dominating set of G.

By Claims 1 and 2, we know that C is a chordless cycle of length at most $d + \ell \leq 2d + 2$. Now, we define a vertex-coloring of G that uses at most d + 1 colors. Relabel $C : x_1 x_2 \cdots x_k x_{k+1} (= x_1), 8 \leq 2d - 2 \leq k \leq 2d + 2$. Then we assign color i to the vertex x_i if $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ and assign color $i - \lfloor \frac{k}{2} \rfloor$ to x_i if $\lfloor \frac{k}{2} \rfloor < i \leq k$. We color the remaining vertices arbitrarily. We can show that this vertex-coloring can make G rainbow vertex-connected.

From Theorem 1 and Claim 3, we know that under this vertex-coloring, pairs in $C \times V(G)$ are rainbow vertex-connected. For each vertex $z \in N_G(C)$, we may strengthen the result of Claim 3 that z has at least two neighbors colored differently in C. Otherwise, we suppose that z_1 and z_2 are the only two neighbors of z having the same color in C. From the vertex-coloring, we know that $dist_C(z_1, z_2) = \lfloor \frac{k}{2} \rfloor \geq 4$. Then we can easily find an induced $S_{1,2,2}$, a contradiction. So, for a pair $(x, y) \in N_G(C) \times N_G(C)$, we can find a vertex $x' \in N_C(x)$ and a vertex $y' \in N_C(y)$ such that x' and y' are colored differently. Since there exists a vertex-rainbow path P connecting x' and y' and the internal vertices of P are colored differently from x' and y', the path xx'Py'y is vertex-rainbow and connects x and y. Hence, $rvc(G) \leq d + 1$.

The proof of Theorem 8 is complete.

Combining Proposition 2 with Theorems 7 and 8, we have proved Theorem 6.

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