Discussiones Mathematicae
Graph Theory 38 (2018) 5–26
doi:10.7151/dmgt.2003

UPPER BOUNDS FOR THE STRONG CHROMATIC INDEX OF HALIN GRAPHS

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Abstract

Keywords: strong edge-coloring, strong chromatic index, Halin graphs.
2010 Mathematics Subject Classification: 05C15.

The strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is the minimum number of vertex induced matchings needed to partition the edge set of G. Let T be a tree without vertices of degree 2 and have at least one vertex of degree greater than 2. We construct a Halin graph G by drawing T on the plane and then drawing a cycle C connecting all its leaves in such a way that C forms the boundary of the unbounded face. We call T the characteristic tree of G. Let G denote a Halin graph with maximum degree Δ and characteristic tree T. We prove that $\chi'_s(G) \leqslant 2\Delta + 1$ when $\Delta \geqslant 4$. In addition, we show that if $\Delta = 4$ and G is not a wheel, then $\chi'_s(G) \leqslant \chi'_s(T) + 2$. A similar result for $\Delta = 3$ was established by Lih and Liu [21].

[1]Corresponding author. Supported in part by grants NASA MIRO NX15AQ06A and NSF DMS 1600778.

1. Introduction

Let G be a simple graph. The *distance* between two edges e and e' in G is the minimum k for which there is a sequence $e = e_0, e_1, \ldots, e_k = e'$ of distinct edges such that for $1 \leqslant i \leqslant k$, e_{i-1} and e_i share an end vertex. A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges with distance at most two must receive different colors. A *strong k-edge-coloring* is a strong edge-coloring using k colors. The *strong chromatic index* of a graph G, denoted by $\chi'_s(G)$, is the minimum k such that G admits a strong k-edge-coloring. The pre-image of each color in a strong edge-coloring is an induced matching. Thus, the strong chromatic index is also the minimum number of vertex induced matchings needed to partition the edge set of G.

Denote the maximum degree of a graph G by $\Delta(G)$ (or, simply by Δ when G is clear in the context). A trivial upper bound is that $\chi'_s(G) \leqslant 2\Delta(G)^2 - 2\Delta(G) + 1$. Fouquet and Jolivet [13] established a Brooks type upper bound $\chi'_s(G) \leqslant 2\Delta(G)^2 - 2\Delta(G)$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [24]. The following conjecture was posed by Erdős and Nešetřil [10, 11].

Conjecture 1. *For any graph G of maximum degree Δ,*

$$\chi'_s(G) \leqslant \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

For graphs with maximum degree $\Delta(G) = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [18], independently. For $\Delta(G) = 4$, while Conjecture 1 asserts that $\chi'_s(G) \leqslant 20$, Horák [17] obtained $\chi'_s(G) \leqslant 23$ and Cranston [8] proved $\chi'_s(G) \leqslant 22$. For general graphs G with maximum degree Δ, Molloy and Reed [22] showed that $\chi'_s(G) \leqslant 1.998\Delta^2$. Most recently, this bound has been improved by Bruhn and Joos [4] to $1.93\Delta^2$.

Strong edge-coloring for planar graphs has been investigated by many authors. Fouquet and Jolivet [13, 14] first studied strong edge-coloring for cubic planar graphs. Let G be a planar graph with maximum degree Δ and girth g. Faudree *et al.* [12] proved that $\chi'_s(G) \leqslant 4\Delta + 4$. Bensmail *et al.* [2] established the bound $\chi'_s(G) \leqslant 3\Delta + 1$ for $g \geqslant 6$. Hudák *et al.* [19] showed $\chi'_s(G) \leqslant 3\Delta$ if $g \geqslant 7$, and the bound is sharp for some subcubic (that is, $\Delta \leqslant 3$) planar graphs. Furthermore, Hocquard *et al.* [16] showed that $\chi'_s(G) \leqslant 9$ for subcubic planar graphs G which do not contain cycles of lengths 4 or 5. DeOrsey *et al.* [9] recently reduced this bound to $\chi'_s(G) \leqslant 5$ if $g \geqslant 30$. For planar graphs with large girth, Borodin and Ivanova [3] established a rather tight bound $\chi'_s(G) \leqslant 2\Delta - 1$ if $g \geqslant 40\lfloor \Delta/2 \rfloor + 1$; Chang *et al.* [7] further confirmed that the bound also holds if $g \geqslant 10\Delta + 46$. Clearly, the bound $\chi'_s(G) \leqslant 2\Delta - 1$ becomes sharp when G contains two adjacent vertices of maximum degree Δ.

By definition, a trivial lower bound of $\chi'_s(G)$ for a graph G would be $\sigma(G)$, where

$$\sigma(G) := \max\{\deg_G(u) + \deg_G(v) - 1 \mid uv \in E(G)\}.$$

If G has no edges, then define $\sigma(G) = 0$. It is known and easy to verify that for a tree T, we have $\chi'_s(T) = \sigma(T)$. Wu and Lin [25] proved that if $\sigma(G) \leqslant 4$ and G is not isomorphic to the graph of the 5-cycle with a chord connecting two non-adjacent vertices, then $\chi'_s(G) \leqslant 6$. Recently, Chang and Duh [5] assert that $\chi'_s(G) = \sigma(G)$ if G is a planar graph with $\sigma(G) = \sigma \geqslant 5$, $\sigma \geqslant \Delta(G)+2$, and girth $g \geqslant 5\sigma + 16$. This result implies that a planar graph with large girth behaves like a tree locally.

A *Halin graph* is a plane graph G constructed as follows. Let T be a tree with at least 4 vertices, called the *characteristic tree* of G. All vertices of T are either of degree 1, called *leaves*, or of degree at least 3. We draw T on the plane. Let C be a cycle, called the *adjoint cycle* of G, connecting all leaves of T in such a way that C forms the boundary of the unbounded face. We usually write $G = T \cup C$ to reveal the characteristic tree and the adjoint cycle. For $n \geqslant 3$, the wheel W_n with $n + 1$ vertices is a particular Halin graph whose characteristic tree is the complete bipartite graph $K_{1,n}$ (called a *star*). A graph is said to be *cubic* if the degree of every vertex is 3. For $h \geqslant 1$, a cubic Halin graph Ne_h, called a *necklace*, was introduced in [23]. Its characteristic tree T consists of the path $v_0, v_1, \ldots, v_h, v_{h+1}$ and leaves $v'_1, v'_2, \ldots, v'_h$ such that the unique neighbor of v'_i in T is v_i for $1 \leqslant i \leqslant h$ and vertices $v_0, v'_1, \ldots, v'_h, v_{h+1}$ are connected in this order to form the adjoint cycle C_{h+2}.

Lai, Lih and Tsai [20] proved the following result.

Theorem 2 [20]**.** *If a Halin graph $G = T \cup C$ is different from a certain necklace Ne_2 and any wheel W_n, $n \not\equiv 0 \pmod 3$, then $\chi'_s(G) \leqslant \chi'_s(T) + 3$.*

For cubic Halin graphs, Lih and Liu improved the above bound as follows.

Theorem 3 [21]**.** *A cubic Halin graph G different from Ne_2 or Ne_4 satisfies $\chi'_s(G) \leqslant 7$.*

The exact values of $\chi'_s(G)$ for special families of cubic Halin graphs were determined by Shiu and Tam [24] and by Chang and Liu [6].

For a Halin graph $G = T \cup C$ with maximum degree Δ, since $\chi'_s(T) \leqslant 2\Delta - 1$, the bound in Theorem 2 implies that $\chi'_s(G) \leqslant 2\Delta+2$. We improve this bound and establish a result similar to Theorem 3 for Halin graphs of maximum degree 4.

Theorem 4. *Let G be a Halin graph with maximum degree $\Delta \geqslant 4$. Then $\chi'_s(G) \leqslant 2\Delta + 1$.*

Theorem 5. *Let $G = T \cup C$ be a Halin graph with maximum degree $\Delta = 4$, and let G be different from a wheel. Then $\chi'_s(G) \leqslant \chi'_s(T) + 2$.*

Both bounds in Theorems 4 and 5 are sharp. Consider the graph G in Figure 1. A strong edge-coloring of G must use at least 7 colors on the edges incident to u or v. Let these colors be $\{1, 2, \ldots, 7\}$. Next, since the edges w_1 and w_2 must use colors different from $\{1, 2, \ldots, 7\}$, at least 8 colors are needed. Assume that we only have 8 colors. Then w_1 and w_2 must be colored by the same new color, say color 8. This implies that the four edges e_1, e_2, e_3, e_4 shown in Figure 1 only have three admissible colors, from the set $\{5, 6, 7\}$, which is a contradiction as these edges must receive different colors. Hence $\chi'_s(G) \geqslant 9$. By coloring e_1, e_2, e_3, e_4 with colors 5, 6, 7, 9 and the last edge with color 4, it follows that $\chi'_s(G) = 9$. This example shows that both bounds in Theorems 4 and 5 are sharp.

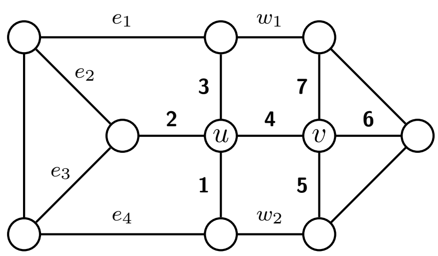


Figure 1. An example showing sharp bounds of Theorems 4 and 5.

2. Proof of Theorem 4

A *double star* is a tree with exactly two non-leaf vertices. Denote by $D_{a,b}$ a double star, where a, b are the degrees of the two non-leaf vertices and $a \leqslant b$. Prior to the proof of Theorem 4, we quote several known results as follows.

Lemma 6 [20]**.** *Let $G = T \cup C$ be a Halin graph. If $T = D_{a,b}$ is a double star with $a \leqslant b$, then*

$$\chi'_s(G) = \begin{cases} \chi'_s(T) + 4 & \text{if } a = b = 3; \\ \chi'_s(T) + 2 & \text{if } a = 3 \text{ and } b \geqslant 4; \\ \chi'_s(T) + 1 & \text{if } a \geqslant 4. \end{cases}$$

If $T = K_{1,k}$ (that is, G is a wheel W_k), then

$$\chi'_s(W_k) = \begin{cases} k + 3 & \text{if } k \equiv 0 \pmod 3; \\ k + 5 & \text{if } k = 5; \\ k + 4 & \text{otherwise.} \end{cases}$$

Lemma 7 [23]**.** *Suppose $h \geqslant 1$. Then*

$$\chi'_s(Ne_h) = \begin{cases} 6 & \text{if } h \text{ is odd;} \\ 7 & \text{if } h \geqslant 6 \text{ and } h \text{ is even;} \\ 8 & \text{if } h = 4; \\ 9 & \text{if } h = 2. \end{cases}$$

***Proof of Theorem* 4.** Let $G = T \cup C$ be a Halin graph with $\Delta(G) \geqslant 4$. If T is a star or a double star, by Lemma 6, the conclusion of Theorem 4 follows. Assume that T is neither a star nor a double star. We proceed by induction on $|C|$, the length of C. The shortest length of C is 6. Three possible graphs along with their corresponding strong edge-colorings satisfying the desired upper bounds are shown in Figure 2. So the result follows.

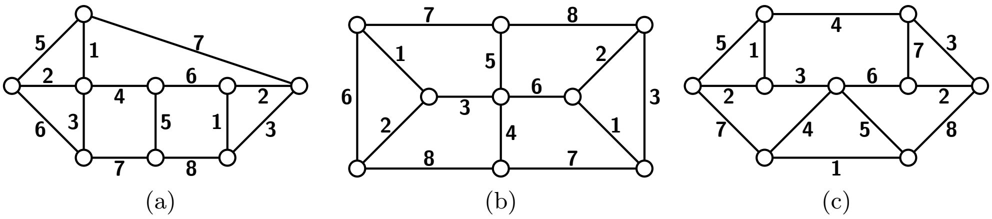


Figure 2. All Halin graphs with $|C| = 6$ and $\Delta(G) = 4$.

Assume $|C| \geqslant 7$. Let $P = u_0, u_1, \ldots, u_l$ be a longest path in T with length l. As T is neither a star nor a double star, so $l \geqslant 4$. Without loss of generality, we assume $\deg_G(u_{l-1}) \geqslant \deg_G(u_1)$.

Denote $u_1 = v$, $u_2 = u$, $u_3 = w$, and label the $k \geqslant 2$ leaf neighbors of v as $v_1, v_2, \ldots, v_k$. Since P is a longest path in T, it is easy to see that $v_1, v_2, \ldots, v_k$ must be on the adjoint cycle C. Let x_1, x_2, y_1, y_2 be vertices on C, where x_1 is adjacent to v_1 and x_2; y_1 is adjacent to v_k and y_2. Let x_3 and y_3 be vertices not on C, where x_1x_3 and y_1y_3 are edges in T (see Figure 3).

Since G is a Halin graph and u is a vertex of degree at least 3, there exists a path P' in T from u to x_1 or from u to y_1 with $P \cap P' = \{u\}$. Without loss of generality, we shall assume that P' is from u to y_1. By our assumption that P is a longest path, it must be that $|P'| \leqslant 2$. Thus, either $u = y_3$ or u is adjacent to y_3.

In the following, we denote by $G' = T' \cup C'$ the Halin graph obtained by adding some new edges to an induced subgraph of G such that $|C'| < |C|$ and $\Delta(G') \leqslant \Delta(G)$. If $\Delta(G') \geqslant 4$, then $\chi'_s(G') \leqslant 2\Delta(G)+1$ holds because T' is a star or double star (see the beginning of the proof) or by the inductive hypothesis as $|C'| < |C|$. If $\Delta(G') = 3$, then $\chi'_s(G') \leqslant 9 \leqslant 2\Delta(G) + 1$ by Theorem 2, Lemma 6, and because $\Delta(G) \geqslant 4$. In the following case analysis these steps will be repeatedly used, while may not be mentioned explicitly all the time.

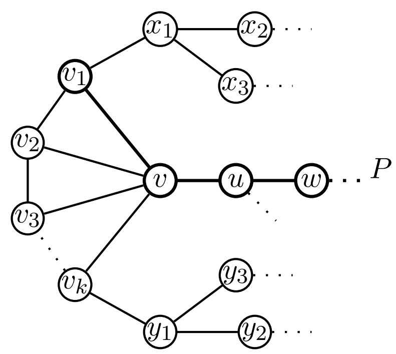


Figure 3. The neighborhood around one end of the longest path P.

We call G' a *reduction* of G. Depending on various situations, different types of G' are created. In the corresponding figures, the dashed lines represent new edges added in G', and dark vertices represent the vertices that are temporarily deleted from G.

Let ψ be a strong edge-coloring of G' using the minimum number of colors. A strong edge-coloring ϕ of G is obtained as follows. We color the edges that are in both G and G' by the same colors used in ψ, i.e., let $\phi(e) = \psi(e)$ for every $e \in E(G) \cap E(G')$. For edges in $e \in E(G) \setminus E(G')$, we develop different coloring schemes for different cases, and in each case, we give a strong edge-coloring ϕ for G with at most $2\Delta(G) + 1$ colors.

Case A. $\deg_G(v) = 3$. There are three possibilities to consider.

Case A.1. $u = y_3$. Obtain the reduction G' of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2\}$, as indicated in Figure 4. Clearly, $\Delta(G') = \Delta(G) \geqslant 4$ and $|C'| < |C|$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 4). We find admissible colors w_1, w_2, and w_3, one by one. The colors that can not be assigned to vv_1 are from $\{1, 2, t_1, t_2\}$ and the labels used by edges incident to u. Therefore, there are at most $\Delta(G) + 4$ forbidden colors for vv_1. Since $\Delta(G) \geqslant 4$, there exists an admissible color for vv_1. Color vv_1 by such an admissible color w_1.

Next we color vv_2 which has the forbidden colors in $\{1, 2, w_1, s\}$ and the labels used for edges incident to u. Similarly, we can find an admissible color for vv_2. Finally, the forbidden colors for v_1v_2 are in $\{1, 2, w_1, w_2, r_1, r_2, s, t_1, t_2\}$. If $s \in \{t_1, t_2\}$, then there is an admissible color for v_1v_2. Otherwise, we re-color vv_1 by s, creating an admissible color for v_1v_2.

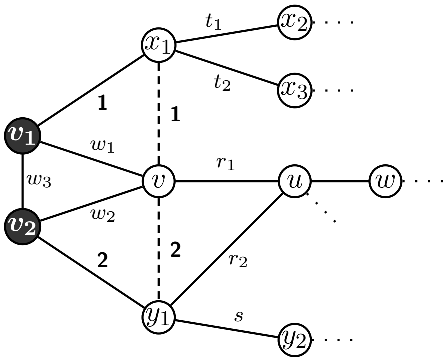


Figure 4. Case A.1.

Case A.2. *u is adjacent to* y_3*, and* $\Delta(G) \geqslant 5$. Obtain the reduction G' in the same way as in Case A.1, as indicated in Figure 5. Clearly, $\Delta(G') = \Delta(G) \geqslant 4$ and $|C'| < |C|$.

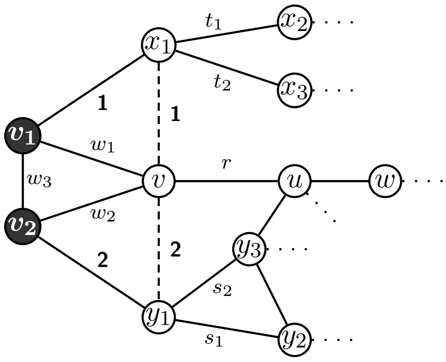
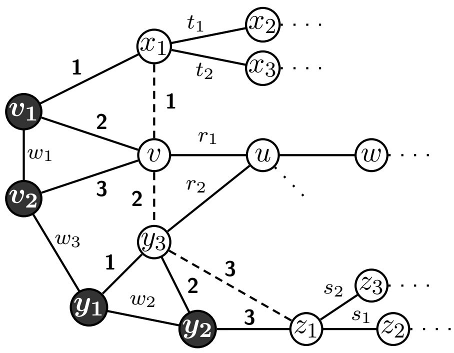


Figure 5. Case A.2.

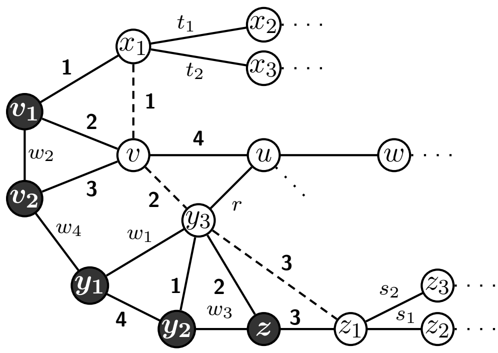
Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = 1$ and $\phi(v_2y_1) = 2$ (see Figure 5). We find admissible colors w_1, w_2, and w_3, one by one. By the same argument as in Case A.1, one can easily show that there exists an admissible color w_1. Color vv_1 by such an admissible color.

Next we color vv_2 which has the forbidden colors in $\{1, 2, w_1, s_1, s_2\}$ and the labels used for edges incident to u. Since $\Delta(G) \geqslant 5$, we can find an admissible color w_2. Finally, the forbidden colors for v_1v_2 are in $\{1, 2, w_1, w_2, r, s_1, s_2, t_1, t_2\}$. Thus, there exists an admissible color w_3.

Case A.3. *u is adjacent to* y_3*, and* $\Delta(G) = 4$. Then $\deg_G(y_3)$ is either 3 or 4. Obtain the reduction G' from G with partial labels to some vertices as indicated in Figure 6(a) and 6(b), respectively. Clearly, $\Delta(G') \leqslant \Delta(G)$ and $|C'| < |C|$. Assume that $\deg_G(y_3) = 3$. Then $\Delta(G') = \Delta(G) = 4$. We find admissible colors w_1, w_2, and w_3, one after another. For v_1v_2, the forbidden colors are in $\{1,2,3,r_1,t_1,t_2\}$. Hence there is an admissible color w_1 for v_1v_2. Next, the forbidden colors for y_1y_2 are in $\{1,2,3,w_1,r_2,s_1,s_2\}$. We can color y_1y_2 by an admissible color w_2. Finally, the forbidden colors for v_2y_1 are in $\{1,2,3,w_1,w_2,r_1,r_2\}$. Again, there exists an admissible color w_3 for v_2y_1.



(a) $\deg_G(y_3)=3$.



(b) $\deg_G(y_3)=4$.

Figure 6. Case A.3.

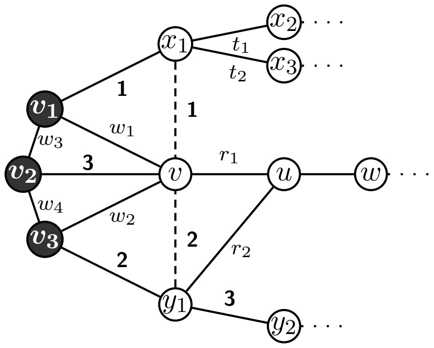
Assume $\deg_G(y_3)=4$. Note, even if $\Delta(G')=3$ or T' is a star (or double star), we can still find a strong edge coloring for G' by up to 9 colors. The forbidden colors for y_1y_3 are in $\{1,2,3\}$ and labels used on edges incident to u. Thus there are at most $\Delta(G)+3$ forbidden colors. We color y_1y_3 by an admissible color w_1. Next, the forbidden colors for v_1v_2 are $\{1,2,3,4,w_1,t_1,t_2\}$. Because $2\Delta(G)+1\geqslant 9$, we can find an admissible color w_2 for v_1v_2. The forbidden colors for y_2z are in $\{1,2,3,4,w_1,r,s_1,s_2\}$. Again, there is an admissible color w_3 for y_2z. Finally, the forbidden colors for v_2y_1 are from $\{1,2,3,4,w_1,w_2,w_3,r\}$. So there is an admissible color w_4 for v_2y_1.

Case B. $\deg_G(v)\geqslant 4$. We consider two cases separately.

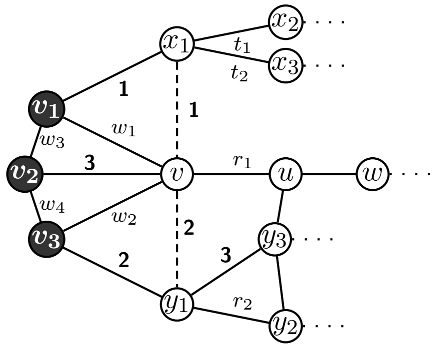
Case B.1. $\Delta(G)=4$. Then $\deg_G(v)=4$. There are two subcases.

Subcase B.1.1. $\deg_G(u)=3$. Obtain the reduction G' of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G)\setminus\{v_1,v_2,v_3\}$ as depicted in Figure 7.

Since we assumed earlier that $\deg_G(u_{l-1})\geqslant\deg_G(u_1)=\deg_G(v)=4$, we have $\Delta(G')=\Delta(G)=4$, and $|C'|<|C|$ holds. We fix colors on some edges as shown in Figure 7. Note that in Figure 7(a) we assign $\phi(y_1y_2)=\phi(vv_2)=3$ but in Figure 7(b) we assign $\phi(y_1y_3)=\phi(vv_2)=3$ and $\phi(y_1y_2)=s$. We find admissible colors w_1, w_2, w_3, and w_4.



(a) $u = y_3$.



(b) $u \sim y_3$.

Figure 7. Subcase B.1.1.

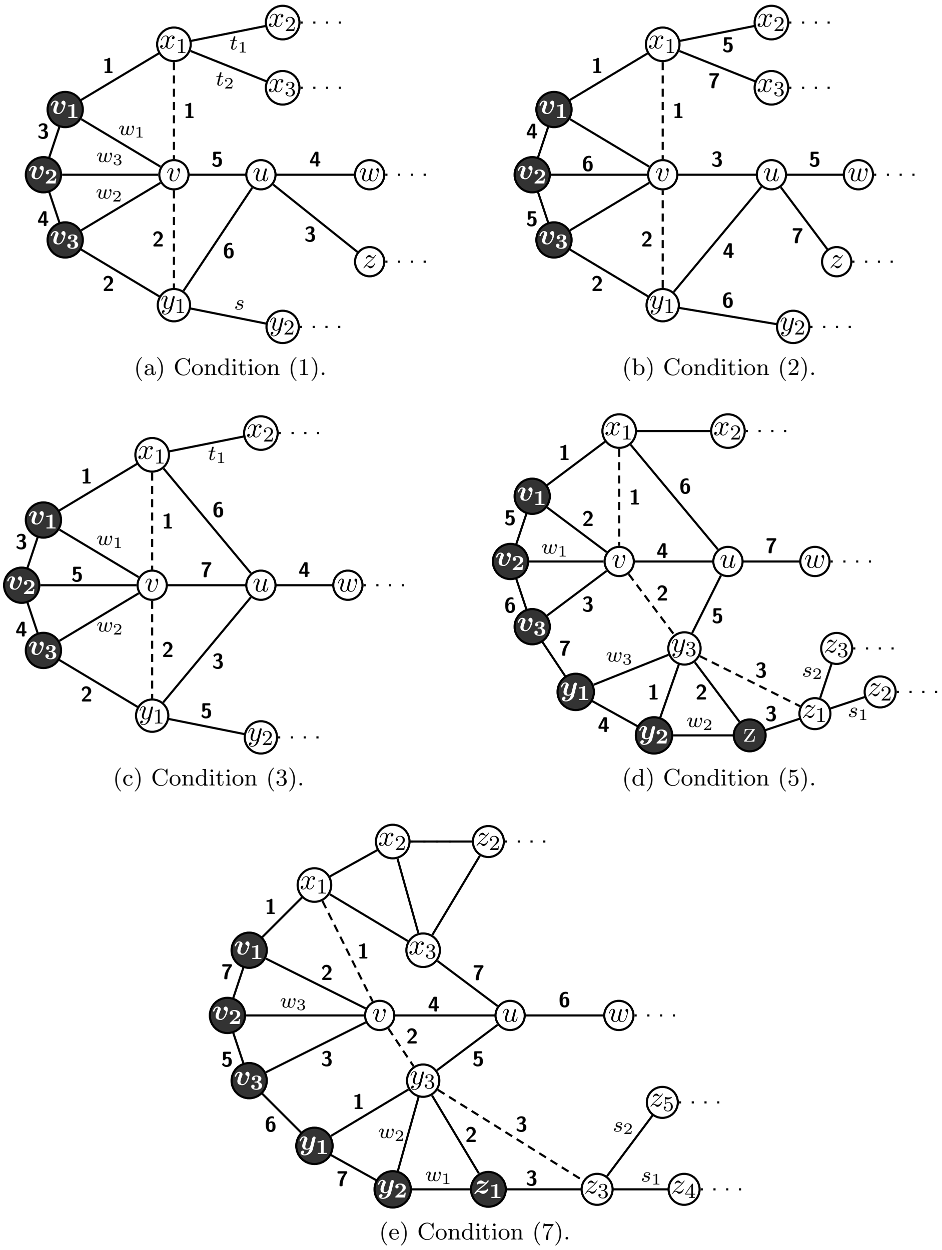
For the subcase depicted in Figure 7(a), the forbidden colors for vv_1 are in $\{1, 2, 3, t_1, t_2\}$ and the three colors used in the neighborhood of u. Thus, there are at most 8 forbidden colors, implying there is an admissible color w_1 for vv_1. Next, the forbidden colors for vv_3 are in $\{1, 2, 3, w_1\}$ and the three colors used in the neighborhood of u. There is an admissible color w_2 for vv_3. The forbidden colors for v_1v_2 are in $\{1, 2, 3, w_1, w_2, r_1, t_1, t_2\}$, so there is an admissible color w_3 for v_1v_2. Finally, the forbidden colors for v_2v_3 are in $\{1, 2, 3, w_1, w_2, w_3, r_1, r_2\}$. Therefore, there is an admissible color w_4 for v_2v_3.

For the subcase depicted in Figure 7(b), the arguments are the same as in Figure 7(a) except for vv_3, which has forbidden colors from $\{1, 2, 3, w_1, r_2\}$ and the three colors used in the neighborhood of u. So there is an admissible color w_2 for vv_3.

Subcase B.1.2. $\deg_G(u) = 4$. We distinguish several cases. In each case $\Delta(G') \leqslant \Delta(G)$ and $|C'| < |C|$ hold.

(1) $u = y_3$, u is adjacent to neither x_1 nor x_3, and $|\{\psi(uw), \psi(uz)\} \cap \{\psi(x_1x_2), \psi(x_1x_3)\}| \leqslant 1$, where z is the fourth neighbor of u, as shown in Figure 8(a). Without loss of generality, assume that $\psi(uz) \notin \{\psi(x_1x_2), \psi(x_1x_3)\}$. Let $\phi(v_1v_2) = \psi(uz) = 3$ and $\phi(v_2v_3) = \psi(uw) = 4$, as indicated in Figure 8(a). Note, $t_1, t_2 \neq 3$. The forbidden colors for vv_1 are in $\{1, 2, 3, 4, 5, 6, t_1, t_2\}$. So there is an admissible color for w_1. Next, the forbidden colors for w_2 are in $\{1, 2, 3, 4, 5, 6, w_1, s\}$. Again, there is an admissible color for w_2. The forbidden colors for w_3 are in $\{1, 2, 3, 4, 5, 6, w_1, w_2\}$, so there is an admissible color for w_3.

(2) $u = y_3$, u is adjacent to neither x_1 nor x_3, and $\{\psi(uw), \psi(uz)\} = \{\psi(x_1x_2), \psi(x_1x_3)\}$, where z is the fourth neighbor of u. Without loss of generality, we assume that $\psi(x_1x_2) = \psi(uw) = 5$ and $\psi(x_1x_3) = \psi(uz) = 7$. Let $\psi(uv) = 3$, $\phi(v_1v_2) = \psi(uy_1) = 4$, $\phi(v_2v_3) = 5$, and $\phi(vv_2) = \psi(y_1y_2) = 6$, as indicated in Figure 8(b). Clearly, the remaining edges vv_1 and vv_3 can be colored by any two colors not in the set $\{1, 2, 3, \ldots, 7\}$.



(a) Condition (1). (b) Condition (2).

(c) Condition (3). (d) Condition (5).

(e) Condition (7).

Figure 8. Subcase B.1.2.

(3) $u = y_3$ and $u = x_3$ (that is, u is adjacent to both y_1 and x_1). Let $\phi(v_1v_2) = \psi(uy_1) = 3$, $\phi(v_2v_3) = \psi(uw) = 4$ and $\phi(vv_2) = \psi(y_1y_2) = 5$ as indicated in Figure 8(c). We find admissible colors w_1 and w_2. The forbidden colors for vv_1 are in $\{1, 2, 3, 4, 5, 6, 7, t_1\}$. Hence, there is an admissible color w_1 for vv_1. Then the forbidden colors for vv_3 are in $\{1, 2, 3, 4, 5, 6, 7, w_1\}$. Thus, there is an admissible color w_2 for vv_3.

(4) u is adjacent to y_3, $u = x_3$, and $\deg_G(y_3) = 3$. (Symmetrically, u is adjacent to x_3, $u = y_3$, and $\deg_G(x_3) = 3$.) Take $P = y_1, y_3, u, w, u_4, \ldots, u_l$ as a longest path, and such a graph was discussed in Subcase A.3 (see Figure 6(b), where the positions of y_3 and v are switched).

(5) u is adjacent to y_3, $u = x_3$, and $\deg_G(y_3) = 4$. Let z be the fourth neighbor of y_3. (Symmetrically, u is adjacent to x_3, $u = y_3$, and $\deg_G(x_3) = 4$.) The reduction G' and partial labels are shown in Figure 8(d). The forbidden colors for vv_2 are in $\{1, 2, 3, 4, 5, 6, 7\}$. Hence, there is an admissible color w_1 for vv_2. The forbidden colors for y_2z are in $\{1, 2, 3, 4, 5, 7, s_1, s_2\}$. Thus, there is an admissible color w_2 for y_2z. The forbidden colors for y_1y_3 are from $\{1, 2, 3, 4, 5, 6, 7, w_2\}$, leaving an admissible color w_3 for y_1y_3.

(6) u is adjacent to both x_3 and y_3, and $\deg_G(x_3) = 3$ or $\deg_G(y_3) = 3$. Say $\deg_G(x_3) = 3$ (the other case is symmetric). Then take $P = x_1, x_3, u, w, u_4, \ldots, u_l$ as a longest path, and such case has been discussed in Case A (see Figure 6).

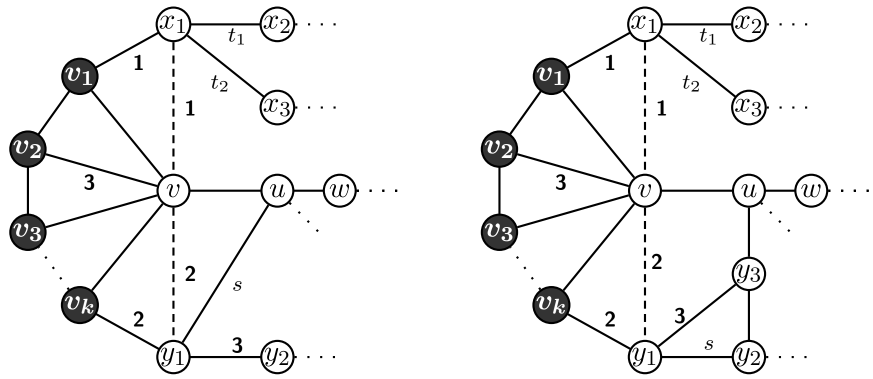
(7) u is adjacent to both x_3 and y_3, and $\deg_G(x_3) = \deg_G(y_3) = 4$. The reduction G' and partial labels are indicated in Figure 8(e). Since $\deg_G(u_{l-1}) \geqslant \deg_G(v) = 4$, we have $\Delta(G') = \Delta(G)$. The forbidden colors for y_2z_1 are from $\{1, 2, 3, 5, 6, 7, s_1, s_2\}$. Hence, there is an admissible color w_1 for y_2z_1. The forbidden colors for y_2y_3 are in $\{1, 2, 3, 4, 5, 6, 7, w_1\}$. Thus, there is an admissible color w_2 for y_2y_3. The forbidden colors for vv_2 are from $\{1, 2, 3, 4, 5, 6, 7\}$. So there is an admissible color w_3 for vv_2.

(8) u is adjacent to y_3, but not x_1 nor x_3. Then u must have another neighbor, say z, besides y_3, that is a leaf or distance one away from the adjoining cycle C. The position of z will be similar to the one in Figure 8(b) (where z might be on the cycle). We then consider the longest path $P^* = y_1, y_3, u, \ldots, u_l$, which falls in one of the cases discussed earlier.

Case B.2. $\Delta(G) \geqslant 5$. Obtain the reduction G' by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, \ldots, v_k\}$, $k \geqslant 3$, as shown in Figure 9. Since $\deg_G(u_{l-1}) \geqslant \deg_G(v)$, we have $\Delta(G) = \Delta(G')$, and $|C'| < |C|$ holds. Without loss of generality, let $\phi(v_1x_1) = \psi(vx_1) = 1$ and $\phi(v_ky_1) = \psi(vy_1) = 2$.

For $u = y_3$ (or u is adjacent to y_3, respectively), let $\phi(vv_2) = \psi(y_1y_2) = 3$ ($\phi(vv_2) = \psi(y_1y_3) = 3$, respectively) as indicated in Figure 9(a) (Figure 9(b), respectively). If $\deg_G(v) = 4$, then the coloring scheme is the same as the ones used in Subcase B.1.1.

Thus we assume $\deg_G(v) \geqslant 5$. We proceed to color the remaining edges, vv_1, $vv_3, \ldots, vv_k$ and v_jv_{j+1}, for $j = 1, 2, \ldots, k-1$.



(a) $u = y_3$. (b) $u \sim y_3$.

Figure 9. Case B.2.

For $u = y_3$ (see Figure 9(a)), the forbidden colors for vv_1 are $\{1, 2, 3, t_1, t_2\}$ and colors used in the neighborhood of u. So there are at most $\Delta(G)+5 \leqslant 2\Delta(G)$ forbidden colors. Hence, there exists an admissible color for vv_1. Next we color vv_k, which has forbidden colors $\{1, 2, 3, \phi(vv_1)\}$ and the labels used for edges incident to u. Again, there is an admissible color for vv_k. For $i = 3, 4, \ldots, k-1$, we color vv_i one after another. By direct calculation, the number of forbidden colors for vv_i is at most $\deg_G(u)+ \deg_G(v)$. Hence, we can color all vv_i by admissible colors.

Next we color v_1v_2, which has forbidden colors $\{1, t_1, t_2\}$ and colors used in the neighborhood of v. Hence there is an admissible color for v_1v_2. Next we sequentially color v_jv_{j+1} for $j = 2, 3, \ldots, k-2$. Using the assumption that $\Delta(G) \geqslant 5$, one can easily verify that there exists an admissible color at each step. Finally, the forbidden colors for $v_{k-1}v_k$ are $\{2, s, \phi(v_{k-2}v_{k-1}), \phi(v_{k-3}v_{k-2})\}$ and the labels used in the neighborhood of v. Thus we can find an admissible color for $v_{k-1}v_k$.

For the case that u is adjacent to y_3, the argument is the same except for the edge vv_k, which has forbidden colors from $\{1, 2, 3, s, \phi(vv_1)\}$ and the labels used by the edges incident to u. As $\Delta(G) \geqslant 5$, we can find an admissible color for vv_k. This completes the proof of Theorem 4. ■

3. Proof of Theorem 5

Let $G = T \cup C$ be a Halin graph with $\Delta(G) = 4$, and let G be different from a wheel. By Theorem 4, if $\chi'_s(T) = 7$, then $\chi'_s(G) \leqslant \chi'_s(T) + 2$. So Theorem 5 holds. Thus we assume $\chi'_s(T) = 6$. That is, every vertex of degree 4 is adjacent to vertices of degree 3 only. Similarly to the previous section, we proceed by induction on $|C|$, the length of C. If $|C| = 4$, then $G = W_4$ which contradicts the assumption. If $|C| = 5$, then $T = D_{3,4}$ is a double star. The result follows by Lemma 6. If $|C| = 6$, the only three possible graphs are in Figure 2(a), 2(b), and 2(c). So the result follows.

Similarly to the proof of Theorem 4, we consider a reduction $G' = T' \cup C'$ of G with characteristic tree T' and adjoint cycle C'. If $\Delta(G') = 4$ and G' is not a wheel, then $\chi'_s(G') \leqslant \chi'_s(T') + 2 \leqslant \chi'_s(T) + 2$ follows by the induction hypothesis, since $|C'| < |C|$. If $G' = W_4$ or if G' is a cubic Halin graph different from Ne_2, then $\chi'_s(G') \leqslant 8 = \chi'_s(T) + 2$ by Theorem 3, Lemma 6, and Lemma 7. Finally, the case when $G' = Ne_2$ is considered at the end of the proof.

Assume $|C| \geqslant 7$. Let $P = u_0, u_1, \ldots, u_l$ be a longest path in T, where l is the length of P. The result holds if T is a double star by Lemma 6 (note that $b \geqslant 4$). Thus, we assume $l \geqslant 4$. Without loss of generality, we also assume that $\deg_G(u_1) \leqslant \deg_G(u_{l-1})$.

Case A. There exists a longest path P with both non-leaf ends of degree 4. That is, $\deg_G(u_1) = \deg_G(u_{l-1}) = 4$. Then $\deg_G(u_2) = 3$. Consider the following two cases.

Case A.1. *In T, u_2 has exactly one neighbor that is a leaf.*

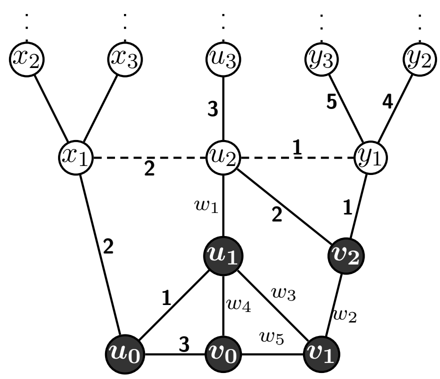


Figure 10. Case A.1.

The reduction G' along with proposed colors for some edges are depicted in Figure 10. We now find admissible colors w_1, w_2, w_3, w_4, and w_5. First we can find an admissible color w_1 for u_1u_2 that is different from $1, 2$ and the colors used in the neighborhood of u_3. Next, we can find an admissible color w_2 for v_1v_2 that is not in $\{1, 2, 3, 4, 5, w_1\}$. Finally, we find three pairwise distinct admissible colors w_3, w_4, w_5, which are not in $\{1, 2, 3, w_1, w_2\}$.

Case A.2. *In T, none of the neighbors of u_2 is a leaf.*

Without loss of generality, we assume that the colors assigned by ψ to the edges incident to u_3 are 3, 4, 5, and 6 (if u_3 has degree 3, then we only use colors 3, 4, and 5, and ignore the respective edge labeled by 6 in Figure 11). Consider two possibilities. For the graph depicted in each Figure 11(a) and 11(b) we obtain the reduction G' and complete the labeling ϕ by using only eight colors, respectively.

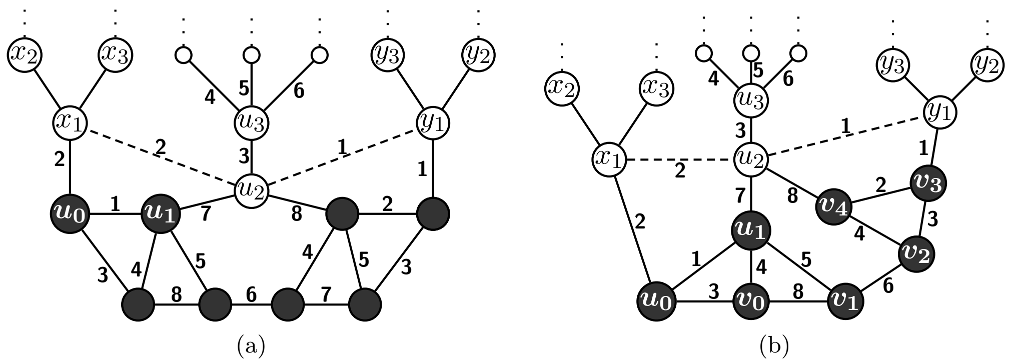


Figure 11. Case A.2.

Case B. Every longest path P has $\deg_G(u_1) = 3$. That is, at least one non-leaf end has degree 3.

Case B.1. $\deg_G(u_2) = 3$.

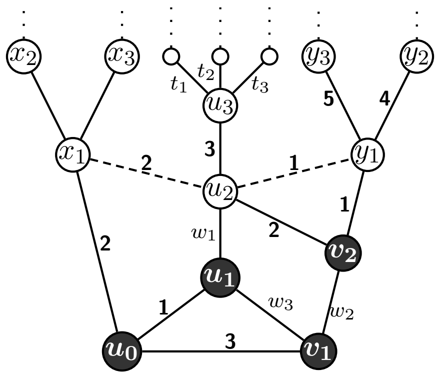


Figure 12. Subcase B.1.1.

Subcase B.1.1. In T, u_2 has exactly one neighbor that is a leaf. The reduction G' along with proposed colors for some edges are depicted in Figure 12. Note if u_3 has degree 3, we simply ignore the edge labeled by t_3 in Figure 12. We color u_1u_2 by a color w_1 not from $\{1, 2, 3, t_1, t_2, t_3\}$. Next, color v_1v_2 by a color w_2

not from $\{1,2,3,4,5,w_1\}$. Finally, color u_1v_1 by an admissible color w_3 not in $\{1,2,3,w_1,w_2\}$.

Subcase B.1.2. In T, none of the neighbors of u_2 is a leaf. Then u_2 has two neighbors, denoted as u_1 and v_4, that are distance one away from the adjoining cycle C. First consider the case that v_4 has degree 4. Then by our assumption of Case B, the degree of the other non-leaf end of the path P must have degree 3. We consider the reverse order of P, denoted as P^*, as our longest path. That is, $P^* = u_l, u_{l-1}, u_{l-2}, \ldots, u_1, u_0$, where $\deg_G(u_{l-1}) = 3$. If P^* falls again in Subcase B.1.2, $\deg_G(u_{l-2}) = 3$ and none of the neighbors of u_{l-2} is a leaf, then by the assumption of Case B, every non-leaf neighbor of v_{l-2} that is distance two away from the adjoining cycle C must be degree 3 (for otherwise, there is a longest path with both non-leaf ends of degree 4, which was discussed in Case A).

Therefore, we only need to consider the case that $\deg_G(v_4) = 3$, which is shown in Figure 13, where the reduction G' and partial labels are indicated.

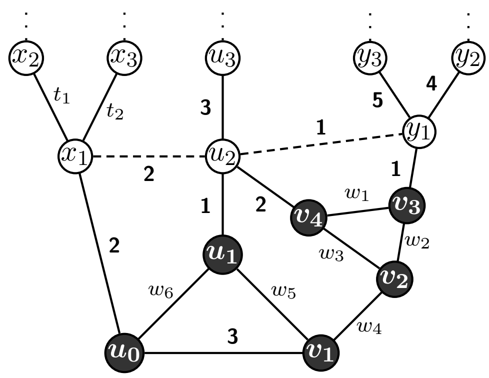


Figure 13. The second possibility of Subcase B.1.2.

We shall find colors for the remaining edges. First, color v_3v_4 and v_1v_2 by two admissible colors w_1 and w_2 different from $\{1,2,3,4,5\}$. Next, color v_2v_4 and v_1v_2 by two admissible colors w_3 and w_4 not from $\{1,2,3,w_1,w_2\}$, and assign u_1v_1 the color $w_5 = w_1$. Finally, color u_0u_1 by an admissible color w_6 different from $\{1,2,3,w_4,w_5,t_1,t_2\}$. Since we have 8 colors, this can be accomplished.

Case B.2. $\deg_G(u_2) = 4$. Then $\deg_G(u_3) = 3$.

Subcase B.2.1. In T, u_2 has exactly two neighbors that are leaves. Consider possible situations depicted in Figure 14. Figure 14(a) shows the situation that the two leaves are adjacent on C. We color v_2v_3 by a color w_1 not from the set $\{1,2,3,4,5,s_1,s_2\}$. Next, color u_2v_2 and u_1u_2 by two colors w_2 and w_3 not in $\{1,2,3,4,5,w_1\}$.

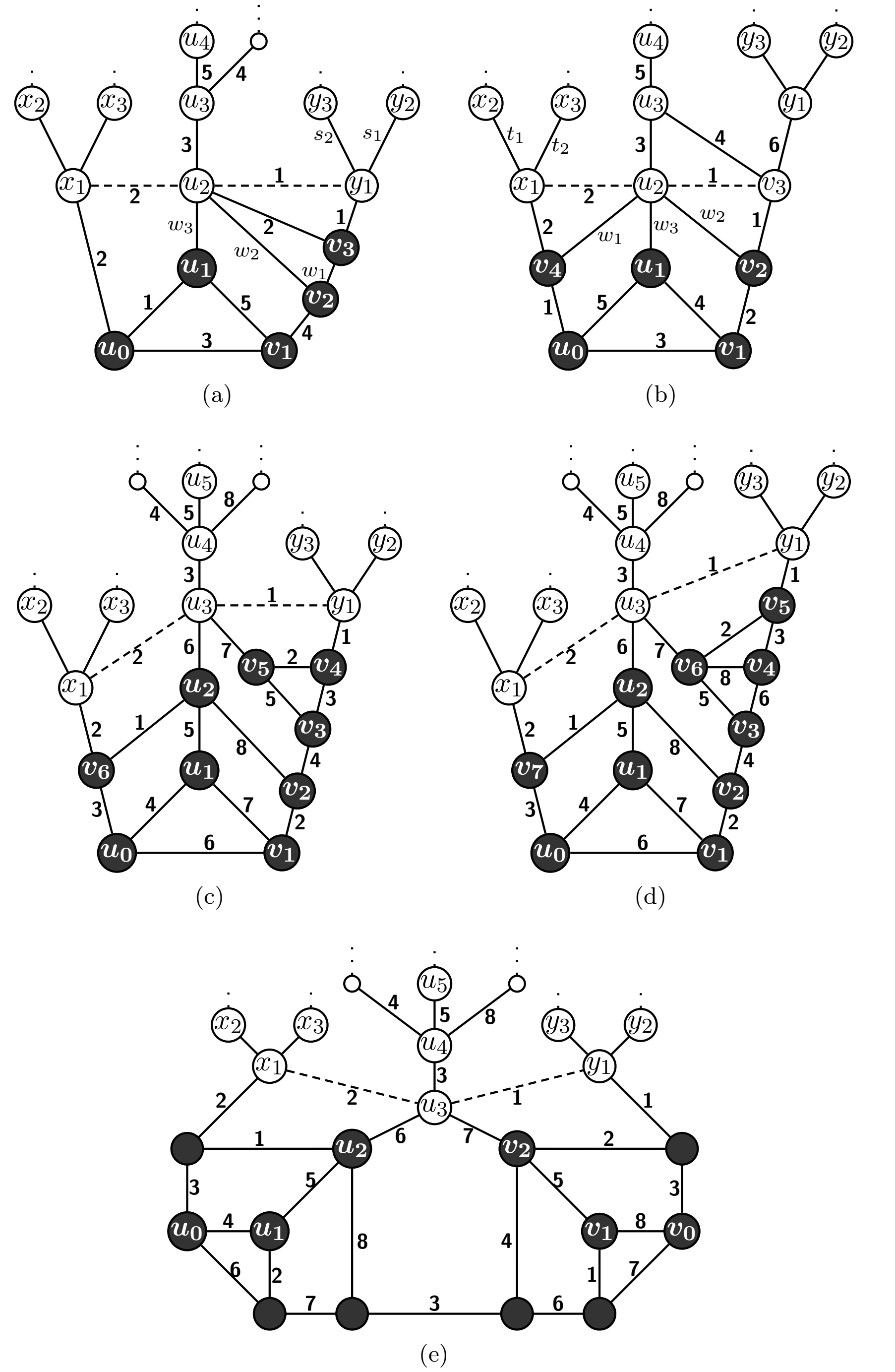


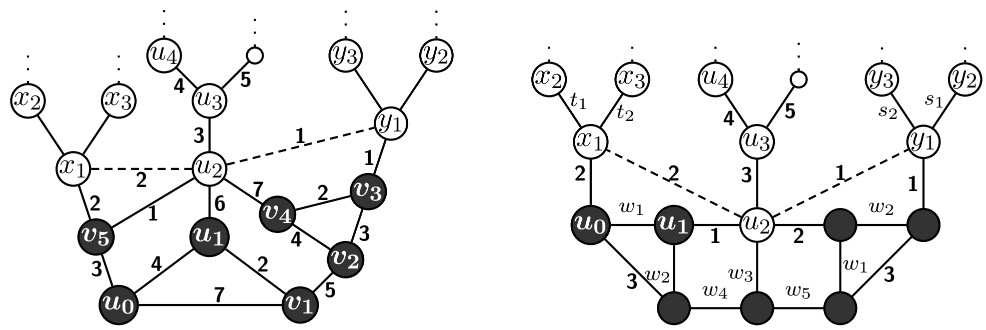
Figure 14. Five possibilities of Subcase B.2.1.

Now assume that the two leaves are not adjacent on C. The length of a longest path from u_3 to the adjoint cycle C on one side of v_1 is at most three, as P is a longest path. Suppose the length is one. Then there is only one possibility which is shown in Figure 14(b). Color u_2v_4 by a color w_1 not in $\{1, 2, 3, 4, 5, t_1, t_2\}$. Color u_2v_2 by a color w_2 not in $\{1, 2, 3, 4, 5, 6, w_1\}$. Finally, color u_1u_2 by a color w_3 not in $\{1, 2, 3, 4, 5, w_1, w_2\}$.

If there is a path of length two from u_3 to the adjoint cycle C, then there are two possibilities as shown in Figure 14(c) and Figure 14(d). Assume that the colors used in the neighborhood of u_4 are from the set $\{3, 4, 5, 8\}$. We directly color the remaining edges as depicted on those two figures.

Assume that there is a path of length three from u_3 to the adjoint cycle C which intersects P only at u_3. Let u_3, v_2, v_1, v_0 be such a path from u_3 to C. Then there is another longest path in T, $P' = u_l, u_{l-1}, \ldots, u_3, v_2, v_1, v_0$. Assume $\deg_G(v_1) = 4$. By our assumption that every longest path has at least one non-leaf end of degree 3, it must be that $\deg_G(u_{l-1}) = 3$. We then consider P^*, the reverse ordering of P, namely, $P^* = u_l, u_{l-1}, \ldots, u_1, u_0$. Observe that the same situation will not occur to P^*, since if $\deg_G(u_{l-2}) = 4$, $\deg_G(u_{l-3}) = 3$, there is a path of length three from u_{l-3} to C (denoted as $u_{l-3}, v_2', v_1', v_0'$), and $\deg_G(v_1') = 4$, then we obtain a longest path $v_0', v_1', v_2', u_{l-3}, \ldots, u_0$ with both non-leaf ends of degree 4, which has been discussed in Case A.

Thus, assume $\deg_G(v_1) = 3$. By symmetry of considering P and P', the only possibility is drawn in Figure 14(e), in which an extended strong edge-coloring is shown using 8 colors.



(a) (b)

Figure 15. Two possibilities of Subcase B.2.2.

Subcase B.2.2. In T, u_2 has exactly one neighbor that is a leaf. There are two possible situations as shown in Figure 15. In Figure 15(a), a strong edge-coloring is given on the extended edges of G'. In Figure 15(b), we color the edges by the following sequence: Color the two edges labeled as w_1 by an admissible color not from $\{1, 2, 3, t_1, t_2\}$. Color the two edges labeled as w_2 by an admissible color not from $\{1, 2, 3, w_1, s_1, s_2\}$. Color the edge labeled as w_3 by an admissible color not from $\{1, 2, 3, 4, 5, w_1, w_2\}$. Finally, color the remaining two edges labeled as w_4 and w_5 by two different admissible colors not from $\{1, 2, 3, w_1, w_2, w_3\}$.

Subcase B.2.3. In T, none of the neighbors of u_2 is a leaf. The reduction G' and the completion of ϕ using eight colors are demonstrated in Figure 16. This completes all cases.

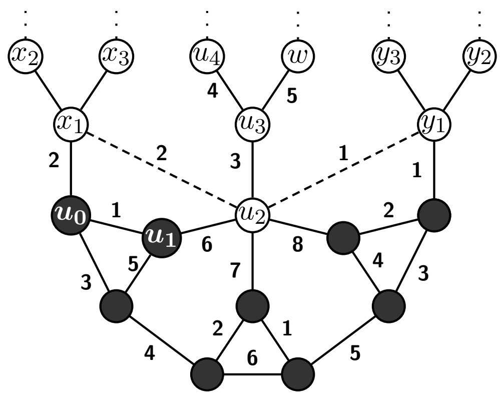


Figure 16. Subcase B.2.3.

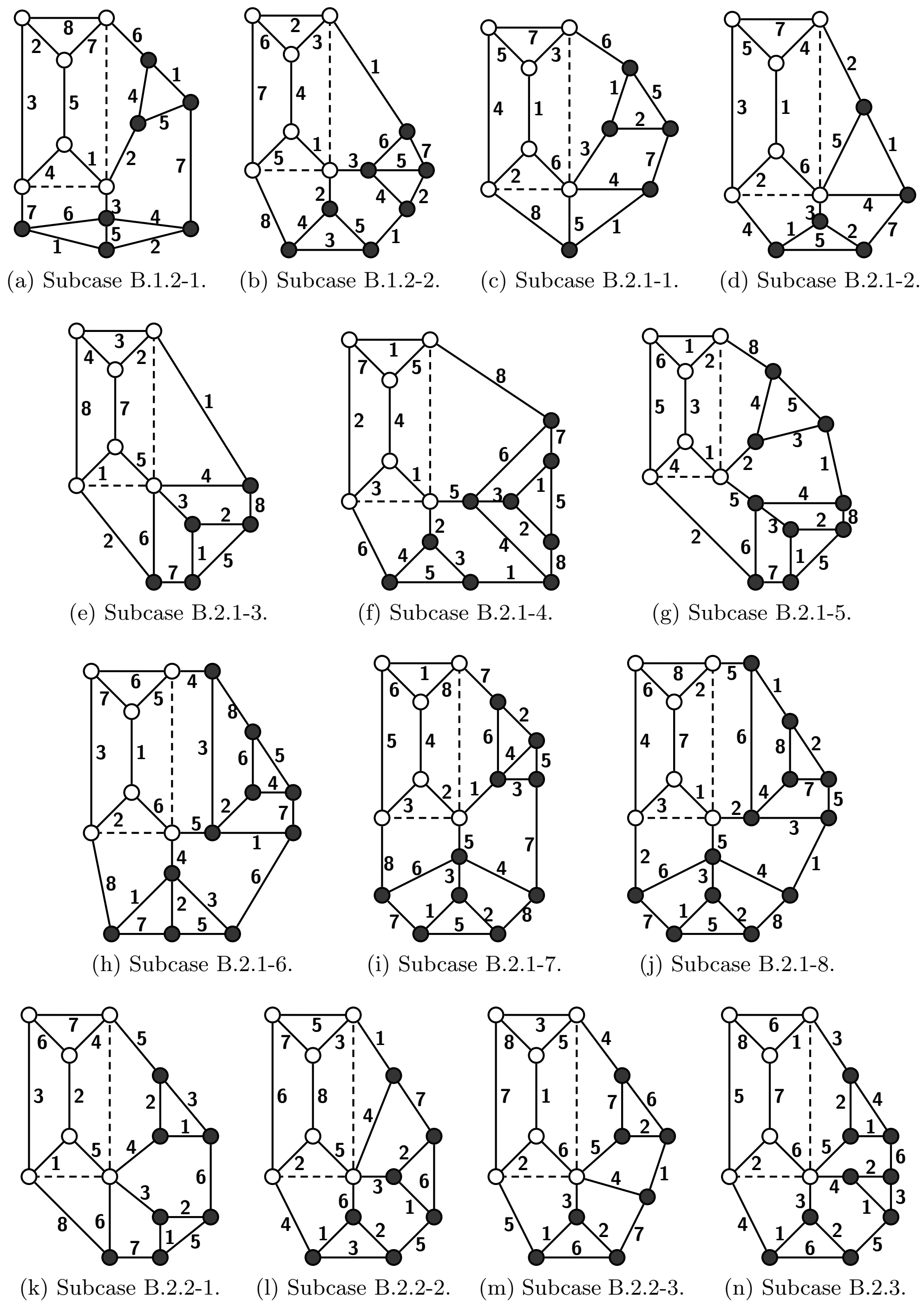
We now discuss the situation that the reduction graph G' is Ne_2. Notice that this does not occur in Case A. For Subcase B.1.1, if $G' = Ne_2$, then G is a cubic graph, contradicting our assumption that $\chi'_s(T) = 6$. Similarly, for the second possibility in Subcase B.1.2, G' is not Ne_2.

These leave a total of fourteen possible situations from the first possibility (Figure 11(b)) of Subcase B.1.2, as well as Subcases B.2.1, B.2.2 and B.2.3, when the reduction graph G' is Ne_2. These fourteen situations are depicted in Figure 17, where a strong edge coloring using at most eight colors is given in each situation. This completes the proof of Theorem 5.

For a Halin graph $G = T \cup C$ with maximum degree at most 4 and G is not a wheel, Ne_2, nor Ne_4, it has been shown that $\chi'_s(G) \leqslant \chi'_s(T) + 2$, and the bound is sharp (cf. [21] and Theorem 5). We propose

Conjecture 8. *If $G = T \cup C$ is a Halin graph other than a wheel, Ne_2, or Ne_4, then $\chi'_s(G) \leqslant \chi'_s(T) + 2$.*

If the answer to Conjecture 8 is affirmative, then the bound is sharp for infinitely many graphs besides the ones mentioned in Lemmas 6 and 7. Let a, b, c be positive integers, $b \geqslant 4$. A tree T is a *triple star*, denoted by $T = S_{a,b,c}$, if it has exactly three non-leaf vertices which have degrees a, b, and c (in this order on a longest path), respectively. We draw T on the plane by fixing a longest path of



(a) Subcase B.1.2-1. (b) Subcase B.1.2-2. (c) Subcase B.2.1-1. (d) Subcase B.2.1-2.

(e) Subcase B.2.1-3. (f) Subcase B.2.1-4. (g) Subcase B.2.1-5.

(h) Subcase B.2.1-6. (i) Subcase B.2.1-7. (j) Subcase B.2.1-8.

(k) Subcase B.2.2-1. (l) Subcase B.2.2-2. (m) Subcase B.2.2-3. (n) Subcase B.2.3.

Figure 17. Fourteen special graphs.

length four horizontally, $u_0 - u - v - w - w_0$ (where u,v, w are non-leaf vertices), and draw at least one pendant edge of v towards each of the up and down sides of the path. For instance, Figure 1 shows $T = S_{3,4,4}$. Let $k \geqslant 4$ be a positive integer. Similar to the argument for Figure 1, one can show that if $T = S_{3,k,3}$, then $\chi'_s(G) = \chi'_s(T) + 2$.

Acknowledgment

The authors are grateful of the three anonymous referees for careful reading of the manuscript and for helpful constructive comments. Part of the work was done when Daphne Liu was visiting the National Center for Theoretical Sciences, Taiwan. She is very much thankful for the Center's great hospitality.

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Received 17 September 2015
Revised 30 August 2016
Accepted 30 August 2016