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CORE INDEX OF PERFECT MATCHING POLYTOPE FOR A 2-CONNECTED CUBIC GRAPH

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Abstract

For a 2-connected cubic graph G, the perfect matching polytope $P(G)$ of G contains a special point $x^c = \left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$. The *core index* $\varphi(P(G))$ of the polytope $P(G)$ is the minimum number of vertices of $P(G)$ whose convex hull contains x^c. The Fulkerson's conjecture asserts that every 2-connected cubic graph G has six perfect matchings such that each edge appears in exactly two of them, namely, there are six vertices of $P(G)$ such that x^c is the convex combination of them, which implies that $\varphi(P(G)) \leq 6$. It turns out that the latter assertion in turn implies the Fan-Raspaud conjecture: In every 2-connected cubic graph G, there are three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$. In this paper we prove the Fan-Raspaud conjecture for $\varphi(P(G)) \leq 12$ with certain dimensional conditions.

Keywords: Fulkerson's conjecture, Fan-Raspaud conjecture, cubic graph, perfect matching polytope, core index.

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1. Introduction

The celebrated Fulkerson's conjecture in graph theory is the following (cf. [1, 5]).

Conjecture A (Fulkerson's conjecture)**.** *Every* 2*-connected cubic graph has six perfect matchings such that each edge appears in exactly two of them.*

We may state the polyhedral version of this conjecture as follows. Let G be a 2-connected cubic graph. Thus each edge of G is contained in a perfect matching of G. The *characteristic vector* of a perfect matching M of G is a vector $x \in \mathbb{R}^{E(G)}$ such that $x_e = 1$ if $e \in M$ and $x_e = 0$ otherwise. The *perfect* *matching polytope* of G, denoted by $P(G)$, is the convex hull of the characteristic vectors of all perfect matchings in G. Now let $x^1, x^2, \ldots, x^6$ be the characteristic vectors of the six perfect matchings in the Fulkerson's conjecture. Then $x^1+x^2+\cdots+x^6=(2,2,\ldots,2)$ and thus $\frac{1}{6}\left(x^1+x^2+\cdots+x^6\right)=\left(\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3}\right)$. That is, $x^c:=\left(\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3}\right)$ is the convex combination of $x^1, x^2, \ldots, x^6$.

We call this x^c the *core* of the perfect matching polytope $P(G)$, which lies in $P(G)$ (see Proposition 3). Furthermore, a subset $Q \subseteq P(G)$ is called a *core polytope* of $P(G)$ if it is the convex hull of k vertices of $P(G)$ such that $x^c \in Q$ and k is minimum. Meanwhile, the above minimum value k is called the *core index* of $P(G)$, denoted by $\varphi(P(G))$. In other words, the core index $\varphi(P(G))$ is the minimum number of vertices of $P(G)$ whose convex hull contains x^c. Therefore, the Fulkerson's conjecture yields the following conjecture.

Conjecture B. *For every* 2*-connected cubic graph G, $\varphi(P(G)) \leq 6$.*

The study of Conjecture B would be meaningful to cope with the Fulkerson's conjecture. In particular, the structure of the core polytope Q inside a perfect matching polytope $P(G)$ is quite mysterious. Fan and Raspaud [5] proposed the following conjecture.

Conjecture C (Fan-Raspaud conjecture)**.** *In every* 2*-connected cubic graph there exist three perfect matchings M_1, M_2, and M_3 such that $M_1 \cap M_2 \cap M_3 = \emptyset$.*

Let us see the relation of these three conjectures.

Proposition 1. *Conjecture A implies Conjecture B and Conjecture B implies Conjecture C.*

Proof. The first assertion is clear, as $x^c := \left(\frac{1}{3},\frac{1}{3},\ldots,\frac{1}{3}\right)$ is contained in the convex hull of $\{x^1, x^2, \ldots, x^6\}$. We show the second assertion. Suppose Conjecture B holds. Let $x^1, x^2, \ldots, x^6$ be six vertices of $P(G)$ whose convex hull contains x^c. Then $x^c = c_1x^1 + c_2x^2 + \cdots + c_6x^6$, where $\sum_{i=1}^{6} c_i = 1$ and $c_i \geq 0$. We may assume that c_1, c_2, c_3 are the three largest numbers among all c_i. Then $c_1+c_2+c_3 \geq \frac{1}{2}\sum_{i=1}^{6} c_i = \frac{1}{2}$. We claim that Conjecture C holds for the perfect matchings M_1, M_2, M_3 corresponding to x^1, x^2, x^3. Suppose not. Then there is an edge $e \in M_1 \cap M_2 \cap M_3$, namely, $x_e^i = 1$ for $i = 1, 2, 3$. Then $x_e^c = c_1x_e^1 + c_2x_e^2 + \cdots + c_6x_e^6 \geq c_1x_e^1 + c_2x_e^2 + c_3x_e^3 = c_1+c_2+c_3 \geq \frac{1}{2} > \frac{1}{3}$, which is a contradiction. ■

In brief, if the Fulkerson's conjecture is true, then $\varphi(P(G)) \leq 6$, and thus the Fan-Raspaud conjecture holds. However, the Fulkerson's conjecture is far from being proved at the moment. So we do not know the exact range of values $\varphi(P(G))$ for all 2-connected cubic graphs. In this circumstance we can use the parameter $\varphi(P(G))$ as a condition in proving the Fan-Raspaud conjecture.

The *dimension* of a polytope P, denoted by $d(P)$, is the dimension of its affine hull (the minimal affine subspace containing P). Let G be a 2-connected cubic graph, and $P(G)$ the perfect matching polytope of G. If all vertices of $P(G)$ are affinely independent (namely, $P(G)$ is a simplex), then $d(P(G)) \leq 8$ (see [3, 9]). In our previous paper [9], we showed that Fan-Raspaud conjecture holds if $d(P(G)) \leq 9$, which implies $\varphi(P(G)) \leq 10$. In this paper, we obtain some improved results. The main results are the following:

(1) The Fan-Raspaud conjecture is true if $d(P(G)) \leq 13$ and $\varphi(P(G)) \leq 11$.

(2) The Fan-Raspaud conjecture is true if G is a cubic brick, $d(P(G)) \leq 18$, and $\varphi(P(G)) \leq 12$.

Since the dimension of a cubic brick G is $d(P(G)) = m-n = n/2$ (see Lemma 4 with $b = 1$), $d(P(G)) \leq 18$ is equivalent to $n = |V(G)| \leq 36$. Hence the above result (2) means that the Fan-Raspaud conjecture is true for cubic bricks with up to 36 vertices (provided $\varphi(P(G)) \leq 12$). Recently, in [2], the computer search shows that the Fulkerson's conjecture is true for snarks with up to 36 vertices, and so is the Fan-Raspaud conjecture. Here, a *snark* is a cyclically 4-edge connected cubic graph which cannot be 3-edge colored and has girth at least 5.

The organization of the paper is as follows. In Section 2, we present some basic properties. Section 3 is devoted to the results on 2-connected cubic graphs with $\varphi(P(G)) \leq 11$. Section 4 is concerned with cubic bricks with $\varphi(P(G)) \leq 12$.

2. PRELIMINARY ON PERFECT MATCHING POLYTOPES

The basic notions on polyhedral combinatorics can be found in [6, 8]. The well-known characterization of perfect matching polytope, due to Edmonds (1965), is the following (cf. [7]).

Lemma 2. *The perfect matching polytope of a graph G is the set of vectors $x \in \mathbb{R}^{E(G)}$ satisfying*

$$x_e \geq 0 \qquad (\textit{for all } e \in E(G)), \tag{1}$$

$$\sum_{e\in\delta(v)} x_e = 1 \qquad (\textit{for all } v \in V(G)), \tag{2}$$

$$\sum_{e\in\delta(A)} x_e \geq 1 \qquad (\textit{for all } A \subseteq V(G), |A| \textit{ is odd}), \tag{3}$$

where $\delta(v)$ stands for the set of edges incident with $v \in V(G)$, and $\delta(A)$ is the set of edges with exactly one end in A.

For a 2-connected cubic graph G, every edge is contained in a perfect matching (see Corollary 3.4.3 of [7]). A graph is *matching-covered* if every edge of this graph is contained in a perfect matching of it. Hence G is matching-covered and there are at least three different perfect matchings in G. Thus the perfect matching polytope $P(G)$ has at least three vertices. In particular, we have

Proposition 3. *Let G be a 2-connected cubic graph. Then the core $x^c = \left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$ lies in $P(G)$.*

Proof. Clearly, x^c satisfies (1) and (2) of Lemma 2. It suffices to verify (3). For a subset $A \subseteq V(G)$ with odd cardinality, the degree sum of the vertices in A is $3|A| = 2|E(G[A])| + |\delta(A)|$, where $G[A]$ is the subgraph of G induced by A. So $|\delta(A)|$ is odd. If $|\delta(A)| = 1$, then the only edge of $\delta(A)$ is a cut edge of G, contradicting the assumption that G is 2-connected. Therefore $|\delta(A)| \geq 3$ and thus $\sum_{e \in \delta(A)} x^c_e = \frac{1}{3}|\delta(A)| \geq 1$. That is, the point x^c satisfies (3), as required. ∎

The following characterization of the dimension of perfect matching polytope found by Edmonds, Lovász, and Pulleyblank (see [4] or Theorem 7.6.6 of [7]) is used in the proof of the main results.

Lemma 4. *For every matching-covered graph G, the dimension of perfect matching polytope $P(G)$ is $d(P(G)) = m - n + 1 - b$, where m, n, b are the numbers of edges, vertices, and bricks of G, respectively.*

Here, a *brick* is a 3-connected and bicritical graph, where a graph G is *bicritical* if $G - u - v$ has a perfect matching for any two distinct vertices u, v in G. Clearly, a brick is non-bipartite and matching-covered. The number of bricks of a matching-covered graph G is the number of bricks produced in a procedure of 'tight cut decomposition', see [4, 7].

With respect to the dimension, the following Carathéodory theorem is classical (Theorem 5.1 of [8]).

Lemma 5. *For any $V \subseteq \mathbb{R}^m$ and x in the convex hull of V, there exist affinely independent vectors $x^1, \ldots, x^k$ in V such that x is contained in the convex hull of $\{x^1, \ldots, x^k\}$.*

We obtain an upper bound of the core index as follows.

Proposition 6. *For every 2-connected cubic graph G, $\varphi(P(G)) \leq d(P(G)) + 1$.*

Proof. Let V be the set of vertices in $P(G)$. Then x^c is contained in the convex hull of V. By the Carathéodory theorem, x^c is contained in the convex hull of $d(P(G)) + 1$ affinely independent vectors. The assertion follows. ∎

3. Results on Core Index and Dimension

For convenience, we refer to the property specified in the Fan-Raspaud conjecture as the 3PM-*property*. We start with some simple facts.

If $\varphi(P(G)) = k$, then there are vertices $x^1, x^2, \ldots, x^k$ of $P(G)$ whose convex hull contains the core x^c, i.e.,

$$x^c = c_1x^1 + c_2x^2 + \cdots + c_kx^k, \quad \sum_{i=1}^{k} c_i = 1,\, c_i > 0. \tag{4}$$

Let $S^c = \{c_1, c_2, \ldots, c_k\}$, which stands for the convex combination representation of the core x^c in the convex hull of $\{x^1, x^2, \ldots, x^k\}$. Meanwhile, each $c_i \in S^c$ is called a *c-element*. For a set S, by an *h-combination* X of S we mean a subset $X \subseteq S$ with $|X| = h$. Furthermore, for every edge $e \in E(G)$, we have

$$c_1x_e^1 + c_2x_e^2 + \cdots + c_kx_e^k = \frac{1}{3}\,. \tag{5}$$

Let $M_1, M_2, \ldots, M_k$ be the perfect matchings corresponding to $x^1, x^2, \ldots, x^k$ respectively, and let $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$. For an edge $e \in E(G)$, let $\mathcal{M}_e = \{M_i : x_e^i = 1,\, 1 \le i \le k\}$ $(\subseteq \mathcal{M})$, which is the set of perfect matchings containing the edge e. Then (5) is equivalent to

$$\sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}\,. \tag{6}$$

Conversely, a subset S of S^c with $\sum_{c_i \in S} c_i = \frac{1}{3}$ is not necessarily corresponding to an edge. We now give a useful definition as follows.

A subset S of S^c is called an *edge-combination* if (1) $\sum_{c_i \in S} c_i = \frac{1}{3}$; (2) there exists a 3-combination X of S such that

$$\bigcap_{c_i \in X} M_i \ne \emptyset \text{ implies } \bigcap_{c_i \in S} M_i \ne \emptyset.$$

For instance, if the 3PM-property does not hold, then any 3-combination S of S^c with $\sum_{c_i \in S} c_i = \frac{1}{3}$ is an edge-combination. Moreover, for any edge-combination S of S^c, we have $|S| \ge 3$ and $\bigcap_{c_i \in S} M_i \ne \emptyset$, and thus S indeed corresponds to an edge in $\bigcap_{c_i \in S} M_i$. This is the intention of the term "edge-combination". We call the number of all the edge-combinations in S^c the *edge-combination number* of S^c, denoted by $\mathcal{E}(S^c)$.

Lemma 7. *If $|E(G)| < \mathcal{E}(S^c)$, then the* 3PM *property holds.*

Proof. If the 3PM-property does not hold, then for each edge-combination S, we have $\bigcap_{c_i \in S} M_i \ne \emptyset$. Thus there exists an edge e contained in $\bigcap_{c_i \in S} M_i$. So each edge-combination S corresponds to an edge $e \in \bigcap_{c_i \in S} M_i$. Furthermore, we claim that an edge $e \in E(G)$ cannot correspond to two different edge-combinations. In fact, if e corresponds to two edge-combinations S and S' with $S \neq S'$, then $e \in \bigcap_{c_i \in S} M_i$ and $e \in \bigcap_{c_i \in S'} M_i$, whence $e \in \bigcap_{c_i \in S \cup S'} M_i$. By the definition of edge-combination, we have $\sum_{c_i \in S} c_i = \frac{1}{3}$ and $\sum_{c_i \in S'} c_i = \frac{1}{3}$. Hence $\sum_{M_i \in \mathcal{M}_e} c_i \geq \sum_{c_i \in S \cup S'} c_i > \sum_{c_i \in S} c_i = \frac{1}{3}$, contradicting the equation (6). In this way, we define an injection (one-to-one mapping) from the set of edge-combinations to $E(G)$. Therefore $\mathcal{E}(S^c) \leq |E(G)|$, contradicting the condition of the lemma. ■

Lemma 8. *If the* 3PM*-property does not hold, then* $S^c = \{c_1, c_2, \ldots, c_k\}$ *satisfies the following:*

(i) *For any* 3*-combination* $X \subseteq S^c$, $\sum_{c_i \in X} c_i \leq \frac{1}{3}$;
(ii) *For any* 3*-combination* $X \subseteq S^c$*, there exists an edge-combination* $S \subseteq S^c$ *such that* $X \subseteq S$;
(iii) S^c *can be partitioned into three parts* $\{S_1, S_2, S_3\}$ *such that* $\sum_{c_i \in S_k} c_i = \frac{1}{3}$ *for* $k = 1, 2, 3$.

Proof. Suppose that the 3PM-property does not hold. We show the three assertions as follows.

(i) Suppose that for $X = \{c_{i_1}, c_{i_2}, c_{i_3}\}$, $c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$. Since $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$, there is an edge $e \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$ such that $x_e^c = c_1 x_e^1 + c_2 x_e^2 + \cdots + c_k x_e^k \geq c_{i_1} x_e^{i_1} + c_{i_2} x_e^{i_2} + c_{i_3} x_e^{i_3} = c_{i_1} + c_{i_2} + c_{i_3} > \frac{1}{3}$, a contradiction.

(ii) For any 3-combination $X \subseteq S^c$, since $\bigcap_{c_i \in X} M_i \neq \emptyset$, there exists an edge e in $\bigcap_{c_i \in X} M_i$. Let $S = \{c_i : M_i \in \mathcal{M}_e\}$. Then $X \subseteq S$ and $\sum_{c_i \in S} c_i = \sum_{M_i \in \mathcal{M}_e} c_i = \frac{1}{3}$, which implies that S is an edge-combination.

(iii) We take a vertex v in G and let e_1, e_2, e_3 be the three edges incident with this vertex v in G. Since no perfect matching M_i can contain two of e_1, e_2, e_3, all perfect matchings $M_1, M_2, \ldots, M_k$ are partitioned into three disjoint sets, each of which contains one of e_1, e_2, e_3. Therefore S^c is partitioned into three parts $\{S_1, S_2, S_3\}$ such that $\sum_{c_i \in S_k} c_i = \sum_{M_i \in \mathcal{M}_{e_k}} c_i = \frac{1}{3}$ for $k = 1, 2, 3$. ■

Our previous paper [9] shows the following.

Proposition 9. *For a* 2*-connected cubic graph* G *with* $d(P(G)) \leq 9$*, the* 3PM*-property holds.*

Now we present several improved results. Proposition 1 says that if $\varphi(P(G)) \leq 6$, then the 3PM-property holds. The following improvement is straightforward.

Proposition 10. *For a* 2*-connected cubic graph* G *with* $\varphi(P(G)) \leq 8$*, the* 3PM*-property holds.*

Proof. Suppose that (4) holds for $k = 8$ and c_1, c_2, c_3, c_4 are the four largest elements in S^c with $c_1 \geq c_2 \geq c_3 \geq c_4$. Then $c_1+c_2+c_3+c_4 \geq \frac{1}{2}$. If $c_1+c_2+c_3 < \frac{3}{8}$, then $c_3 < \frac{1}{8}$ and $c_4 \geq \frac{1}{2} - (c_1 + c_2 + c_3) > \frac{1}{8}$, a contradiction to the assumption that $c_3 \geq c_4$. Therefore $c_1 + c_2 + c_3 \geq \frac{3}{8} > \frac{1}{3}$, and thus the assertion follows from (i) of Lemma 8. ∎

This result is independent of the dimension of the perfect matching polytope. In the results below, we have to combine the dimensional condition.

Lemma 11. *For a 2-connected cubic graph G, if $d(P(G)) = d$, then $|E(G)| \leq 6(d-1)$.*

Proof. Since G is cubic, we have $2|E(G)| = 3|V(G)|$. Moreover, as a result of the brick decompositions of graphs, G has at most $\frac{|E(G)|-|V(G)|}{2} = \frac{|E(G)|}{6}$ bricks (see Lemma 5.12 of [4]). Hence by the formula of dimension of perfect matching polytope, we have $d = m - n + 1 - b \geq m - \frac{2m}{3} + 1 - \frac{m}{6} = \frac{m}{6} + 1$, which implies that $m \leq 6(d-1)$. ∎

Theorem 12. *For a 2-connected cubic graph G with $d(P(G)) \leq 14$, if $\varphi(P(G)) \leq 10$, then the 3PM-property holds.*

Proof. By Lemma 11 and $d(P(G)) \leq 14$, we have $|E(G)| \leq 6(d-1) \leq 78$. Suppose, to the contrary, that the 3PM-property does not hold.

The case of $\varphi(P(G)) \leq 8$ has been settled in Proposition 10. We consider the case of $\varphi(P(G)) = 9$ now. In this case, the convex combination of (4) with $k = 9$ holds. Suppose, without loss of generality, that $c_1 \geq c_2 \geq \cdots \geq c_9$. By Lemma 8(i), the sum of any three c-elements is at most $\frac{1}{3}$. On the other hand, $c_1 + c_2 + c_3 \geq \frac{1}{3}\sum_{i=1}^{9} c_i = \frac{1}{3}$. It follows that $c_1 + c_2 + c_3 = \frac{1}{3}$, and so $c_4 + c_5 + c_6 = c_7 + c_8 + c_9 = \frac{1}{3}$. Therefore, $c_7 \geq \frac{1}{9}$ and $c_6 \leq \frac{1}{9}$. Since $c_6 \geq c_7$, we have $c_6 = \frac{1}{9}$, and so $c_4 = c_5 = \frac{1}{9}$. By the same way, we have $c_1 = c_2 = c_3 = \frac{1}{9}$, and $c_7 = c_8 = c_9 = \frac{1}{9}$. Hence every 3-combination of S^c is an edge-combination, and so $\mathcal{E}(S^c) = \binom{9}{3} > 78 \geq |E(G)|$. This is a contradiction to Lemma 7.

We next consider the case of $\varphi(P(G)) = 10$ with convex combination representation (4) with $k = 10$. By Lemma 8(i), the sum of any two c-elements is less than $\frac{1}{3}$. We further observe that the sum of any five c-elements is greater than $\frac{1}{3}$. This is because if there are five c-elements whose sum is less than or equal to $\frac{1}{3}$, then the sum of the remaining five c-elements is at least $\frac{2}{3}$, say $c_{i_1} + c_{i_2} + \cdots + c_{i_5} \geq \frac{2}{3}$, and thus the sum of the three largest members of $\{c_{i_1}, c_{i_2}, \ldots, c_{i_5}\}$ is greater than $\frac{1}{3}$, contradicting Lemma 8(i). Therefore, each edge of G is covered by precisely three or four of the ten perfect matchings $M_1, M_2, \ldots, M_{10}$.

By Lemma 8(iii), S^c can be partitioned into three parts each of which has sum $\frac{1}{3}$. Note that the only partition $\{k_1, k_2, k_3\}$ of integer 10 is $\{3, 3, 4\}$, where $k_1+k_2+k_3 = 10$, and $3 \leq k_i \leq 4$. Hence the above partition of the ten c-elements is a $\{3,3,4\}$-partition. Therefore, we assume, without loss of generality, that $c_1+c_2+c_3=c_4+c_5+c_6=c_7+c_8+c_9+c_{10}=\frac{1}{3}$.

By Lemma 8(i), we have $c_1=c_2=\cdots=c_6=\frac{1}{9}$ and $c_7,\ldots,c_{10}\le\frac{1}{9}$. In this context, we choose a 3-combination $X=\{c_1,c_2,c_i\}$ where $7\le i\le 10$. If X is an edge-combination, then $c_i=\frac{1}{9}$. Otherwise, by Lemma 8(ii), X can be extended to an edge-combination S with $|S|=4$. Then there exists another c_j ($7\le j\le 10$) such that $c_i+c_j=\frac{1}{9}$. We further claim that among $c_7,\ldots,c_{10}$, there is at most one such pair with $c_i+c_j=\frac{1}{9}$. Suppose without loss of generality that $c_7+c_8=c_9+c_{10}=\frac{1}{9}$ or $c_7+c_8=c_7+c_9=\frac{1}{9}$. The former contradicts the assumption $c_7+c_8+c_9+c_{10}=\frac{1}{3}$, and the latter implies that $c_{10}>\frac{1}{9}$, also a contradiction. Therefore, we obtain that $c_1=c_2=\cdots=c_8=\frac{1}{9}$ and $c_9+c_{10}=\frac{1}{9}$.

We proceed to compute the edge-combination number $\mathcal{E}(S^c)$ as follows.

- There are $\binom{8}{3}$ 3-combinations $\{c_i,c_j,c_k\}$ chosen from $\{c_1,c_2,\ldots,c_8\}$, each of which is an edge-combination.
- There are $\binom{8}{2}$ 4-combinations $S=\{c_i,c_j,c_9,c_{10}\}$ such that $\{c_i,c_j\}$ are chosen from $\{c_1,c_2,\ldots,c_8\}$, each of which is an edge-combination.

To sum up, $\mathcal{E}(S^c)\ge\binom{8}{3}+\binom{8}{2}>78\ge|E(G)|$, contradicting Lemma 7. ■

Theorem 13. *For a* 2*-connected cubic graph* G *with* $d(P(G))\le 13$*, if* $\varphi(P(G))\le 11$*, then the* 3PM*-property holds.*

Proof. We consider the case $\varphi(P(G))=11$ and representation (4) with $k=11$. By Lemma 11, $d(P(G))\le 13$ implies $|E(G)|\le 6(d-1)\le 72$. Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), no two c-elements have sum $\frac{1}{3}$. Also, no six c-elements have sum $\frac{1}{3}$, for otherwise the remaining five c-elements would have $\sum c_i=\frac{2}{3}$, and thus there are three of them with $\sum c_i>\frac{1}{3}$, contradicting Lemma 8(i). Moreover, by Lemma 8(iii), the 11 c-elements are divided into three sets, each of which has sum $\frac{1}{3}$. Note that the only partitions $\{k_1,k_2,k_3\}$ of integer 11 are $\{3,3,5\}$ and $\{3,4,4\}$, where $k_1+k_2+k_3=11, 3\le k_i\le 5$. We distinguish two cases as follows.

Case 1. There is a $\{3,3,5\}$-partition of S^c. By Lemma 8(i), the first six c-elements are $\frac{1}{9}$ and no other c_i is greater than $\frac{1}{9}$. Therefore, we can sort all c-elements in the form that $c_1=c_2=\cdots=c_6=\frac{1}{9}\ge c_7\ge c_8\ge\cdots\ge c_{11}$ and $c_7+c_8+\cdots+c_{11}=\frac{1}{3}$. Note that there are at most two of $\{c_7,c_8,\ldots,c_{11}\}$ being $\frac{1}{9}$ (for otherwise $c_{10}=c_{11}=0$). We have the following subcases.

Subcase 1.1. $c_7=c_8=\frac{1}{9}$ and $c_9+c_{10}+c_{11}=\frac{1}{9}$. Let us see the edge-combination number.

- There are $\binom{8}{3}$ 3-combinations $\{c_i,c_j,c_k\}$ chosen from $\{c_1,c_2,\ldots,c_8\}$, each of which is an edge-combination.
- There are $\binom{8}{2}$ 5-combinations $\{c_i,c_j,c_9,c_{10},c_{11}\}$ such that $\{c_i,c_j\}$ are chosen from $\{c_1,c_2,\ldots,c_8\}$, each of which is an edge-combination.

Therefore, $\mathcal{E}(S^c) \geq \binom{8}{3} + \binom{8}{2} > 72 \geq |E(G)|$, a contradiction to Lemma 7.

Subcase 1.2. $c_7 = \frac{1}{9}$, $c_8, c_9, c_{10}, c_{11} < \frac{1}{9}$, and $c_8 + c_9 + c_{10} + c_{11} = \frac{2}{9}$. We choose a 3-combination $X = \{c_i, c_j, c_k\}$, where $1 \leq i, j \leq 7$ and $8 \leq k \leq 11$. By Lemma 8(ii), X can be extended to an edge-combination S. If $|S| = 5$, then $S \setminus X \subset \{c_8, c_9, c_{10}, c_{11}\}$ and the sum of c_k and the two elements in $S \setminus X$ is $\frac{1}{9}$. So the remaining one of c_8, c_9, c_{10}, c_{11} equals $\frac{1}{9}$, a contradiction. Therefore, $|S| = 4$ and there exists $c_{r(k)}$ $(8 \leq r(k) \leq 11)$ such that $S = \{c_i, c_j, c_k, c_{r(k)}\}$. Without loss of generality, we may assume that $9 \neq r(8)$. We obtain the following computation.

- There are $\binom{7}{3}$ 3-combinations $\{c_i, c_j, c_k\}$ chosen from $\{c_1, c_2, \ldots, c_7\}$, each of which is an edge-combination.
- There are $\binom{7}{2}$ 4-combinations $\{c_i, c_j, c_8, c_{r(8)}\}$ and $\binom{7}{2}$ 4-combinations $\{c_i, c_j, c_9, c_{r(9)}\}$ with $\{c_i, c_j\}$ chosen from $\{c_1, c_2, \ldots, c_7\}$, each of which is an edge-combination.

Therefore, $\mathcal{E}(S^c) \geq \binom{7}{3} + 2\binom{7}{2} > 72 \geq |E(G)|$, a contradiction to Lemma 7.

Subcase 1.3. $c_i < \frac{1}{9}$ for $7 \leq i \leq 11$. As in the previous subcase, we choose a 3-combination $X = \{c_1, c_2, c_i\}$, where $7 \leq i \leq 11$. By Lemma 8(ii), X can be extended to an edge-combination S. We can assert that $|S| = 4$. In fact, if $|S| = 5$, say $S = \{c_1, c_2, c_7, c_8, c_9\}$, then $c_7 + c_8 + c_9 = \frac{1}{9}$ and so $c_{10} + c_{11} = \frac{2}{9}$. Thus $\max\{c_{10}, c_{11}\} \geq \frac{1}{9}$, a contradiction. Hence there exists another c_j $(7 \leq j \leq 11)$ such that $S = \{c_1, c_2, c_i, c_j\}$ and $c_i + c_j = \frac{1}{9}$.

Now we consider a graph F with vertex set $\{c_7, c_8, c_9, c_{10}, c_{11}\}$, two vertices c_i and c_j being adjacent if and only if $c_i + c_j = \frac{1}{9}$. Then no two edges in F are nonadjacent, for otherwise the vertex not incident to these edges would have $c_i = \frac{1}{9}$, a contradiction. This implies that F is a star on 5 vertices. Suppose c_{11} is the center of the star. We calculate the edge-combination number as follows.

- There are $\binom{6}{3}$ 3-combinations $\{c_i, c_j, c_k\}$ chosen from $\{c_1, c_2, \ldots, c_6\}$, each of which is an edge-combination.
- There are $\binom{6}{2}\binom{4}{1}$ 4-combinations $\{c_i, c_j, c_k, c_{11}\}$ such that $\{c_i, c_j\}$ are chosen from $\{c_1, c_2, \ldots, c_6\}$ and c_k is chosen from $\{c_7, c_8, c_9, c_{10}\}$, each of which is an edge-combination.

To sum up, $\mathcal{E}(S^c) \geq \binom{6}{3} + \binom{6}{2}\binom{4}{1} > 72 \geq |E(G)|$, a contradiction to Lemma 7. This completes the proof of Case 1.

Case 2. There is a $\{3, 4, 4\}$-partition of S^c. So we may assume that

$$c_1 + c_2 + c_3 = c_4 + c_5 + c_6 + c_7 = c_8 + c_9 + c_{10} + c_{11} = \frac{1}{3}.$$

We first observe that $\min\{c_1, c_2, c_3\} \geq \max\{c_4, c_5, \ldots, c_{11}\}$. If not, say $c_4 > c_3$, then $c_1 + c_2 + c_4 > \frac{1}{3}$, contradicting Lemma 8(i). So $\max\{c_4, c_5, \ldots, c_{11}\} \leq \frac{1}{9}$. Moreover, if $\max\{c_4, c_5, \ldots, c_{11}\} = \frac{1}{9}$, then in addition to $c_1 = c_2 = c_3 = \frac{1}{9}$, each of $\{c_4, c_5, c_6, c_7\}$ and $\{c_8, c_9, c_{10}, c_{11}\}$ has at most two elements being $\frac{1}{9}$. We distinguish the following subcases.

Subcase 2.1. Each of $\{c_4, c_5, c_6, c_7\}$ and $\{c_8, c_9, c_{10}, c_{11}\}$ has exactly two elements being $\frac{1}{9}$. We may assume that $c_4 = c_5 = \frac{1}{9}$ and $c_6 + c_7 = \frac{1}{9}$, and $c_8 = c_9 = \frac{1}{9}$ and $c_{10} + c_{11} = \frac{1}{9}$. Similar to Subcase 1.2, there are $\binom{7}{3}$ edge-combinations $\{c_i, c_j, c_k\}$ chosen from $\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}$. There are $2\binom{7}{2}$ edge-combinations $\{c_i, c_j, c_6, c_{r(6)}\}$ and $\{c_i, c_j, c_{10}, c_{r(10)}\}$, where $10 \neq r(6)$, and $\{c_i, c_j\}$ are chosen from $\{c_1, c_2, c_3, c_4, c_5, c_8, c_9\}$. Therefore, $\mathcal{E}(S^c) \geq \binom{7}{3} + 2\binom{7}{2} > 72 \geq |E(G)|$, as required.

Subcase 2.2. $\{c_4, c_5, c_6, c_7\}$ has only one element being $\frac{1}{9}$, say $c_4 = \frac{1}{9}$. Then $c_5 + c_6 + c_7 = \frac{2}{9}$. Assume that $\frac{1}{9} > c_5 \geq c_6 \geq c_7$. Then $c_1 + c_2 + c_i < \frac{1}{3}$ for $5 \leq i \leq 7$. By Lemma 8(ii), $\{c_1, c_2, c_i\}$ can be extended to an edge-combination S with $|S| = 4$ (if $|S| = 5$, then we can get a $\{3, 3, 5\}$-partition of S^c, which reduces to Case 1). Note that $c_i + c_k > \frac{1}{9}$ for $5 \leq i, k \leq 7$. There is a c_j with $j > 7$ such that $S = \{c_1, c_2, c_i, c_j\}$ and $c_i + c_j = \frac{1}{9}$. Thus we can define a mapping from $\{c_i : 5 \leq i \leq 7\}$ to $\{c_j : 8 \leq j \leq 11\}$ with $c_i + c_j = \frac{1}{9}$. In this respect, we claim that it is impossible that $\{c_8, c_9, c_{10}, c_{11}\}$ has only one element being $\frac{1}{9}$. To see this, assume that $c_8 = \frac{1}{9}$ and $c_9 + c_{10} + c_{11} = \frac{2}{9}$. Then we can also define a mapping from $\{c_j : 9 \leq j \leq 11\}$ to $\{c_i : 5 \leq i \leq 7\}$ as above. Hence we obtain a bijection between $\{c_5, c_6, c_7\}$ and $\{c_9, c_{10}, c_{11}\}$. Consequently, $c_5 + c_6 + c_7 + c_9 + c_{10} + c_{11} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$ and so $c_4 + c_8 = \frac{1}{3}$, which is impossible. Therefore, $\{c_8, c_9, c_{10}, c_{11}\}$ has exactly two elements being $\frac{1}{9}$, say $c_8 = c_9 = \frac{1}{9}$, and so $c_{10} + c_{11} = \frac{1}{9}$.

As stated in Subcase 1.3, we may define a graph F with vertex set $\{c_5, c_6, c_7, c_{10}, c_{11}\}$, two vertices c_i and c_j being adjacent if and only if $c_i + c_j = \frac{1}{9}$. Then F is a star on 5 vertices. By the same calculation as in Subcase 1.3, we obtain $\mathcal{E}(S^c) \geq \binom{6}{3} + \binom{6}{2}\binom{4}{1} > 72 \geq |E(G)|$, a contradiction to Lemma 7.

Subcase 2.3. $\max\{c_4, c_5, \ldots, c_{11}\} < \frac{1}{9}$. For this, we cannot assure $\{c_1, c_2, c_3\}$ has some element being $\frac{1}{9}$. We may assume that $c_1 \geq c_2 \geq c_3 \geq c_4 \geq c_5 \geq c_6 \geq c_7$, $c_8 \geq c_9 \geq c_{10} \geq c_{11}$, and $c_4 \geq c_8$. We show that $c_3 = c_4$. In fact, if $c_3 > c_4$, then $c_1 + c_2 + c_4 < \frac{1}{3}$, and so there is an i with $5 \leq i \leq 11$ such that $c_1 + c_2 + c_4 + c_i = \frac{1}{3}$ (by Lemma 8(ii)). Noting $c_3 \leq \frac{1}{9}$, we have $c_1 + c_2 \geq \frac{2}{9}$. This implies that $c_4 + c_i \leq \frac{1}{9}$, and thus $c_8 + c_i \leq \frac{1}{9}$. Consequently, there would be c_j and c_k ($4 \leq j, k \leq 11$) such that $c_j + c_k \geq \frac{2}{9}$. Then one of them is at least $\frac{1}{9}$, contradicting our assumption. By the same argument, we can show that $c_4 = c_5$, and further $c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = \frac{1}{12}$.

Furthermore, we claim that $c_2 = c_3$. In fact, if $c_1 \geq c_2 > c_3$, then $c_2 \leq \frac{1}{2}\left(\frac{1}{3} - \frac{1}{12}\right) = \frac{1}{8}$, and so $c_2 + c_3 + c_4 \leq \frac{1}{8} + \frac{1}{6} = \frac{7}{24} < \frac{1}{3}$. By Lemma 8(ii), there is an i ($i \geq 5$) such that $c_2 + c_3 + c_4 + c_i = \frac{1}{3}$. But this contradicts the fact that $c_2 + c_3 + c_4 + c_i > \frac{4}{12} = \frac{1}{3}$. Therefore, $c_2 = c_3 = \cdots = c_{11} = \frac{1}{12}$ which implies $c_1 = \frac{1}{6}$.

From this, we consider the edge-combination number $\mathcal{E}(S^c)$ as follows.

- There are $\binom{10}{3}$ 3-combinations $\{c_i, c_j, c_k\}$ with $c_i = c_j = c_k = \frac{1}{12}$, apart from $c_1 = \frac{1}{6}$, at most $\binom{4}{3}$ of which correspond to a common edge-combination $\left\{\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right\}$.
- There are $\binom{10}{2}$ 3-combinations $\{c_1, c_i, c_j\}$ with $c_1 = \frac{1}{6}$ and $c_i = c_j = \frac{1}{12}$, each of which corresponds to an edge-combination $\{\frac{1}{6}, \frac{1}{12}, \frac{1}{12}\}$.

To summarize, $\mathcal{E}(S^c) \geq \binom{10}{3}/4 + \binom{10}{2} > 72 \geq |E(G)|$, a contradiction to Lemma 7. This completes the proof. ■

As a consequence, if $d(P(G)) \leq 10$ (whence $\varphi(P(G)) \leq 11$), then the 3PM-property holds. This includes our previous result (Proposition 9).

4. Bricks

Recall that a brick is 3-connected and bicritical. Relative to Theorem 12 and 13, we can get better results for cubic bricks.

Lemma 14. *For a cubic brick G, if $d(P(G)) = d$, then $|E(G)| = 3d$.*

Proof. Since G is cubic, $2|E(G)| = 3|V(G)|$. Since G is a brick, $b = 1$. By Lemma 4, $d = m - n + 1 - b = m - \frac{2m}{3} = \frac{m}{3}$, which implies $|E(G)| = 3d$. ■

Theorem 15. *For a cubic brick G with $d(P(G)) \leq 24$, if $\varphi(P(G)) \leq 11$, then the* 3PM*-property holds.*

Proof. Recalling the proofs of Theorems 12 and 13, we note that when $\varphi(P(G)) \leq 11$, as long as $|E(G)| \leq 72$, all these proofs are valid. Now for a brick G, if $d(P(G)) \leq 24$, then $|E(G)| = 3d \leq 72$, as we needed in the proofs. ■

Theorem 16. *For a cubic brick G with $d(P(G)) \leq 18$, if $\varphi(P(G)) \leq 12$, then the* 3PM*-property holds.*

Proof. By Lemma 14, $|E(G)| = 3d \leq 54$. By Theorem 15, we need only consider the case $\varphi(P(G)) = 12$. Then the equations (4) and (5) hold for $k = 12$. Suppose, to the contrary, that the 3PM-property does not hold.

By Lemma 8(i), each edge-combination contains from three to six c-elements of $c_1, c_2, \ldots, c_{12}$. Moreover, by Lemma 8(iii), the 12 c-elements are divided into three sets, each of which has sum $\frac{1}{3}$. Note that the only partitions $\{k_1, k_2, k_3\}$ of integer 12, where $k_1 + k_2 + k_3 = 12$, $3 \leq k_i \leq 6$, are $\{3, 3, 6\}, \{3, 4, 5\}$, and $\{4, 4, 4\}$. So we consider three cases as follows.

Case 1. There is a $\{3, 3, 6\}$-partition of S^c. By Lemma 8(i), there are six c-elements being $\frac{1}{9}$ and no other c_i is greater than $\frac{1}{9}$. So we may assume that

$c_1 = c_2 = \cdots = c_6 = \frac{1}{9}$, $c_7 + c_8 + \cdots + c_{12} = \frac{1}{3}$, and $\frac{1}{9} \geq c_7 \geq \cdots \geq c_{12}$. Similarly to Case 1 in the proof of Theorem 13, we have the following subcases.

Subcase 1.1. $c_7 = c_8 = \frac{1}{9}$ and $c_9 + c_{10} + c_{11} + c_{12} = \frac{1}{9}$. Since each 3-combination $\{c_i, c_j, c_k\}$ taken from $\{c_1, c_2, \ldots, c_8\}$ is an edge-combination, we have $\mathcal{E}(S^c) \geq \binom{8}{3} > 54 \geq |E(G)|$, contradicting Lemma 7.

Subcase 1.2. $c_7 = \frac{1}{9}$ and $c_8 + c_9 + c_{10} + c_{11} + c_{12} = \frac{2}{9}$. Then there are $\binom{7}{3}$ 3-combinations $\{c_i, c_j, c_k\}$ taken from $\{c_1, c_2, \ldots, c_7\}$, each of which is an edge-combination. For each $\{c_i, c_j\}$ chosen from $\{c_1, c_2, \ldots, c_7\}$, $c_i + c_j + c_8 < \frac{1}{3}$. By Lemma 8(ii), $\{c_i, c_j, c_8\}$ can be extended to an edge-combination. Therefore, $\mathcal{E}(S^c) \geq \binom{7}{3} + \binom{7}{2} > 54 \geq |E(G)|$, as required.

Subcase 1.3. $c_i < \frac{1}{9}$ for $7 \leq i \leq 12$ and $c_7 \geq c_8 \geq \cdots \geq c_{12}$. For a pair of given i, j $(1 \leq i, j \leq 6)$, $c_i + c_j + c_k < \frac{1}{3}$ for any k with $7 \leq k \leq 12$, and so $\{c_i, c_j, c_k\}$ can be extended to an edge-combination S_k. Note that $S_k \setminus \{c_i, c_j\} \subseteq \{c_7, c_8, \ldots, c_{12}\}$. It is impossible that $|S_k| = 6$, for otherwise there would be a $c_l \geq \frac{1}{9}$ for $7 \leq l \leq 12$. So $4 \leq |S_k| \leq 5$.

If there is an S_k with $|S_k| = 5$, then the sum of the three c-elements of $S_k \setminus \{c_i, c_j\}$ is $\frac{1}{9}$ and the sum of the remaining three c-elements in $\{c_7, c_8, \ldots, c_{12}\}$ is $\frac{2}{9}$, say $c_{k_1} + c_{k_2} + c_{k_3} = \frac{2}{9}$. Since the sum of any two c-elements of $c_{k_1}, c_{k_2}, c_{k_3}$ is more than $\frac{1}{9}$, S_{k_1}, S_{k_2}, and S_{k_3} are different. If each S_k has exactly four c-elements, say $S_k = \{c_i, c_j, c_k, c_{r(k)}\}$, then there are at least three different S_k's.

Now we evaluate the number of edge-combinations. There are $\binom{6}{3}$ 3-combinations taken from $\{c_1, c_2, \ldots, c_6\}$, each of which is an edge-combination. Moreover, each 2-combination $\{c_i, c_j\}$ taken from $\{c_1, c_2, \ldots, c_6\}$ corresponds to at least three edge-combinations. Therefore, we have $\mathcal{E}(S^c) \geq \binom{6}{3} + 3\binom{6}{2} > 54 \geq |E(G)|$, as needed.

Case 2. There is a $\{4, 4, 4\}$-partition of S^c. Then every edge-combination has cardinality 4. In this context, there are $\binom{12}{3}$ 3-combinations $\{c_i, c_j, c_k\}$, at most $\binom{4}{3}$ of which correspond to a common edge-combination. Therefore, $\mathcal{E}(S^c) \geq \binom{12}{3}/4 > 54 \geq |E(G)|$, as needed.

Case 3. Every 3-partition of S^c is a $\{3, 4, 5\}$-partition. Then for any vertex v of G, the three edges of G incident with v lie in exactly three, four, and five perfect matchings of $M_1, M_2, \ldots, M_{12}$, respectively. Let N_1, N_2, N_3 be the sets of edges which lie in exactly three, four, and five perfect matchings of $M_1, M_2, \ldots, M_{12}$, respectively. Then we see that N_1, N_2, and N_3 are three pairwise disjoint perfect matchings of G, a contradiction to the assumption that 3PM-property fails. This completes the proof. ■

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