# INVERSE PROBLEM ON THE STEINER WIENER INDEX 

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#### Abstract

The Wiener index $W(G)$ of a connected graph $G$, introduced by Wiener in 1947, is defined as $W(G)=\sum_{u, v \in V(G)} d_{G}(u, v)$, where $d_{G}(u, v)$ is the distance (the length a shortest path) between the vertices $u$ and $v$ in $G$. For $S \subseteq V(G)$, the Steiner distance $d(S)$ of the vertices of $S$, introduced by Chartrand et al. in 1989, is the minimum size of a connected subgraph of $G$ whose vertex set contains $S$. The $k$-th Steiner Wiener index $S W_{k}(G)$ of $G$ is defined as $S W_{k}(G)=\sum_{\substack{\subseteq \subseteq V(G) \\|S|=k}} d(S)$. We investigate the following problem: Fixed a positive integer $k$, for what kind of positive integer $w$ does there exist a connected graph $G$ (or a tree $T$ ) of order $n \geq k$ such that $S W_{k}(G)=w\left(\right.$ or $\left.S W_{k}(T)=w\right)$ ? In this paper, we give some solutions to this problem. Keywords: distance, Steiner distance, Wiener index, Steiner Wiener index. 2010 Mathematics Subject Classification: 05C05, 05C12, 05C35.


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## 1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple. We refer to [3] for graph theoretical notation and terminology not specified here. A distance is one of basic concepts of graph theory [4]. If $G$ is a connected graph and $u, v \in V(G)$, then the distance $d(u, v)=d_{G}(u, v)$ between $u$ and $v$ is the length of a shortest path connecting $u$ and $v$. For more details on this subject, see [13].

The Wiener index $W(G)$ of a connected graph $G$ is defined by

$$
W(G)=\sum_{u, v \in V(G)} d_{G}(u, v) .
$$

Mathematicians have studied this graph invariant since the 1970s in [11]; for details see the surveys [10, 33], the recent papers $[2,7,14,17,15,20]$ and the references cited therein. Information on chemical applications of the Wiener index can be found in [27, 28].

The Steiner distance of a graph, introduced by Chartrand et al. in [6] in 1989, is a natural and nice generalization of the concept of the classical graph distance. For a graph $G=(V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a subgraph $T=\left(V^{\prime}, E^{\prime}\right)$ of $G$ that is a tree with $S \subseteq V^{\prime}$. Let $G$ be a connected graph of order at least 2 and let $S$ be a nonempty set of vertices of $G$. Then the Steiner distance $d(S)$ among the vertices of $S$ (or simply the distance of $S$ ) is the minimum size of a connected subgraph whose vertex set contains $S$. Note that if $H$ is a connected subgraph of $G$ such that $S \subseteq V(H)$ and $|E(H)|=d(S)$, then $H$ is a tree. Clearly, $d(S)=\min \{|E(T)|: S \subseteq V(T)\}$, where $T$ is a subtree of $G$. Furthermore, if $S=\{u, v\}$, then $d(S)=d(u, v)$ is nothing new, but the classical distance between $u$ and $v$. Clearly, if $|S|=k$, then $d(S) \geq k-1$. For more details on Steiner distance, we refer to $[1,5,6,8,13,26]$.

In [23], we proposed a generalization of the Wiener index concept, using Steiner distance. Thus, the $k$-th Steiner Wiener index $S W_{k}(G)$ of a connected graph $G$ is defined by

$$
S W_{k}(G)=\sum_{\substack{S \subseteq V(G) \\|S|=k}} d(S)
$$

For $k=2$, the Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider $S W_{k}$ for $2 \leq k \leq n-1$, but the above definition implies $S W_{1}(G)=0$ and $S W_{n}(G)=n-1$ for a connected graph $G$ of order $n$. For more details on Steiner Wiener index, we refer to [23, 24, 25].

A chemical application of $S W_{k}$ was recently reported in [16].

It should be noted that in the 1990s, Dankelmann et al. in [8, 9] studied the average Steiner distance, which is related to our Steiner Wiener index via $S W_{k}(G) /\binom{n}{k}$.

The seemingly elementary question: "Which natural numbers are Wiener indices of graphs?" was much investigated in the past; see [12, 19, 21, 29, 31, 32]. We now consider the analogous question for Steiner Wiener indices.

Problem. Fixed a positive integer $k$, for what kind of positive integer $w$ does there exist a connected graph $G$ (or a tree $T$ ) of order $n \geq k$ such that $S W_{k}(G)=$ $w\left(\right.$ or $\left.S W_{k}(T)=w\right)$ ?

For $k=2$, the authors have nice results in [30, 32], completely solved a conjecture by Lepović and Gutman [22] for trees, which states that for all but 49 positive integers $w$ one can find a tree with Wiener index $w$. This is different from our problem for trees, since here we consider graphs or trees with order $n$.

## 2. The Cases $k=n$ and $k=n-1$

At first, let us consider the case $k=n$.
If $G$ is a connected graph or a tree of order $n$, then for $k=n, S W_{k}(G)=n-1$. Thus the following result is immediate.

Proposition 2.1. For a positive integer $w$, there exists a connected graph $G$ or a tree $T$ of order $n$ such that $S W_{n}(G)=w$ or $S W_{n}(T)=w$ if and only if $w=n-1$.

For the case $k=n-1$, we need the following results in [23].
Lemma 2.2 [23]. Let $T$ be a tree of order $n$, possessing $p$ pendant vertices. Then

$$
S W_{n-1}(T)=n(n-1)-p
$$

irrespective of any other structural detail of $T$.
Lemma 2.3 [23]. Let $K_{n}$ be the complete graph of order $n$, and let $k$ be an integer such that $2 \leq k \leq n$. Then

$$
S W_{k}\left(K_{n}\right)=\binom{n}{k}(k-1) .
$$

Lemma 2.4 [23]. Let $G$ be a connected graph of order $n$, and let $k$ be an integer such that $2 \leq k \leq n$. Then

$$
\binom{n}{k}(k-1) \leq S W_{k}(G) \leq(k-1)\binom{n+1}{k+1} .
$$

Moreover, the lower bound is sharp.

From the previous results, we can derive the following proposition.
Proposition 2.5. For a positive integer $w$, there exists a connected graph $G$ of order $n$ such that $S W_{n-1}(G)=w$ if and only if $n^{2}-2 n \leq w \leq n^{2}-n-2$.

Proof. By Lemma 2.4, if $G$ is a connected graph of order $n$, then

$$
n(n-2) \leq S W_{n-1}(G) \leq(n+1)(n-2)
$$

Therefore, $n^{2}-2 n \leq w \leq n^{2}-n-2$.
By Lemma 2.3, $S W_{n-1}\left(K_{n}\right)=n^{2}-2 n$.
Let $T$ be a tree of order $n$ with $p$ pendant vertices with $2 \leq p \leq n-1$. By Lemma 2.2, $S W_{n-1}(T)=n^{2}-n-p$, and thus for any integer $w$ with $n^{2}-n-(n-$ 1) $\leq w \leq n^{2}-n-2$, there exists a tree $T$ of order $n$ such that $S W_{n-1}(T)=w$.

From the proof of Proposition 2.5 the next result immediately follows.
Proposition 2.6. For a positive integer $w$, there exists a tree $T$ of order $n$ such that $S W_{n-1}(T)=w$ if and only if $n^{2}-2 n+1 \leq w \leq n^{2}-n-2$.

## 3. The Case $k=n-2$

Similarly to Lemma 2.2, we can derive the following result.
Lemma 3.1. Let $T$ be a tree of order n, possessing $p$ pendant vertices. Let $q$ be the number of vertices of degree 2 in $T$ that are adjacent to a pendant vertex. Then

$$
\begin{equation*}
S W_{n-2}(T)=\frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right) . \tag{1}
\end{equation*}
$$

Proof. For any $S \subseteq V(T)$ and $|S|=n-2$, let $\bar{S}=\{u, v\}$. If $d_{T}(u)=d_{T}(v)=1$, then $d_{T}(S)=n-3$, and this case contributes to $S W_{n-2}$ by

$$
\sum_{\substack{u, v \in \bar{S} \\ d_{T}(u)=d_{T}(v)=1}} d_{T}(S)=\binom{p}{2}(n-3) .
$$

If $d_{T}(u) \geq 2$ and $d_{T}(v) \geq 2$, then $d_{T}(S)=n-1$, and this case contributes to $S W_{n-2}$ by

$$
\sum_{\substack{u, v \in \widetilde{S} \\ d_{T}(u) \geq 2, d_{T}(v) \geq 2}} d_{T}(S)=\binom{n-p}{2}(n-1) .
$$

Suppose that $d_{T}(u)=1$ and $d_{T}(v) \geq 2$. If $d_{T}(u)=1, d_{T}(v)=2$ and $u v \in E(G)$, then $d_{T}(S)=n-3$. If $d_{T}(u)=1, d_{T}(v) \geq 3$ and $u v \in E(T)$, then
$d_{T}(S)=n-2$. If $d_{T}(u)=1, d_{T}(v) \geq 2$ and $u v \notin E(T)$, then $d_{T}(S)=n-2$. Therefore, this case contributes to $S W_{n-2}$ by

$$
\begin{aligned}
\sum_{\substack{u, v \in \bar{S} \\
d_{T}(u)=1, d_{T}(v) \geq 2}} d_{T}(S) & =\sum_{\substack{u, v \in \bar{S}, u v \in E(T) \\
d_{T}(u)=1, d_{T}(v)=2}} d_{T}(S)+\sum_{\substack{u, v \in \bar{S}, u v \in E(T) \\
d_{T}(u)=1, d_{T}(v) \geq 3}} d_{T}(S)+\sum_{\substack{u, v \in \bar{S}, u v \notin E(T) \\
d_{T}(u)=1, d_{T}(v) \geq 2}} d_{T}(S) \\
& =q(n-3)+(p-q)(n-2)+p(n-p-1)(n-2) .
\end{aligned}
$$

From the above argument, we have

$$
\begin{aligned}
S W_{n-2}(T)= & \binom{p}{2}(n-3)+\binom{n-p}{2}(n-1)+q(n-3) \\
& +(p-q)(n-2)+p(n-p-1)(n-2) \\
= & \frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right)
\end{aligned}
$$

Li et al. obtained the following sharp lower and upper bounds of $S W_{k}(T)$ for a tree $T$.

Lemma 3.2 [23]. Let $T$ be a tree of order $n$, and let $k$ be an integer such that $2 \leq k \leq n$. Then

$$
\binom{n-1}{k-1}(n-1) \leq S W_{k}(T) \leq(k-1)\binom{n+1}{k+1}
$$

Moreover, among all trees of order $n$, the star $S_{n}$ minimizes the Steiner Wiener $k$-index, whereas the path $P_{n}$ maximizes the Steiner Wiener $k$-index.

For trees, we have the following result.
Theorem 3.3. For a positive integer $w$, there exists a tree $T$ of order $n(n \geq 5)$, possessing $p$ pendant vertices, such that $S W_{n-2}(T)=w$ if and only if $w=$ $\frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right)$, where $q$ is the number of vertices of degree 2 in $T$ that are adjacent to a pendant vertex, and one of the following holds:
(1) $2 \leq q \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ and $q \leq p \leq n-q-1$;
(2) $q=1$ and $3 \leq p \leq n-2$;
(3) $q=0$ and $4 \leq p \leq n-1$.

Proof. Suppose that $w=\frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right)$, where $0 \leq q \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, $q \leq p \leq n-q-1$. Let $K_{1, p-1}$ be a star of order $p$, and let $v$ be the center of $K_{1, p-1}$. Then $K_{1, p-1}^{*}$ is a graph obtained from $K_{1, p-1}$ by picking up $q-1$ edges and then replacing each of them by a path of length 2 . Note that $K_{1, p-1}^{*}$ is a
subdivision of $K_{1, p-1}$. Let $G$ be a graph obtained by $K_{1, p-1}^{*}$ and a path $P_{n-p-q+2}$ by identifying $v$ and one endvertex of the path. Clearly, $G$ is a tree of order $n$ with $p$ pendant vertices, and there are exactly $q$ vertices of degree 2 in $T$ such that each of them is adjacent to a pendant vertex. From Lemma 3.1, we have $S W_{n-2}(T)=\frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right)=w$, as desired.

Conversely, for any tree $T$ of order $n(n \geq 5)$ with $p$ pendant vertices, from Lemma 3.1, $S W_{n-2}(T)=\frac{1}{2}\left(n^{3}-2 n^{2}+n-2 n p+2 p-2 q\right)$. We now show that $p, q$ satisfy one of (1), (2), (3). Clearly, $p \geq 2,0 \leq q \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ and $q \leq p$.
Claim 1. $p+q \leq n-1$.
Proof. Assume, to the contrary, that $p+q=n$. Then $T$ is a path of order $n$. Since $n \geq 5$, it follows that there exists a vertex of degree 2 having no adjacent pendant vertex, which contradicts to $p+q=n$.

If $q \geq 2$, then it follows from Claim 1 and $q \leq p$ that $q \leq p \leq n-q-1$. If $q=1$, then it follows from Claim 1 that $2 \leq p \leq n-2$. Furthermore, if $p=2$, then $T$ is a path of order $n$. Since $n \geq 5$, it follows that $q=2$, a contradiction. If $q=0$, then it follows from Claim 1 that $2 \leq p \leq n-1$. Furthermore, if $p=2$, then $T$ is a path of order $n$. Since $n \geq 5$, it follows that $q=2$, a contradiction. If $p=3$, then $T$ is a tree of order $n$. Since $n \geq 5$, it follows that $q \geq 1$, a contradiction.

## 4. The Case for General $k$

For trees, we have the following result.
Theorem 4.1. Let $T$ be a graph obtained from a path $P_{t}$ and a star $S_{n-t+1}$ by identifying a pendant vertex of $P_{t}$ and the center $v$ of $S_{n-t+1}$, where $1 \leq t \leq n-1$ and $k \leq n$. Then

$$
S W_{k}(T)=t\binom{n-1}{k}-\binom{t}{k+1}-\binom{n}{k+1}+\binom{n-t+1}{k+1}+(k-1)\binom{n}{k} .
$$

Proof. For any $S \subseteq V(T)$ and $|S|=k$, if $S \subseteq V\left(S_{n-t+1}\right)-v$, then $d_{G}(S)=k$. There are $\binom{n-t}{k}$ such subsets, contributing to $S W_{k}$ by $k\binom{n-t}{k}$. If $S \subseteq V\left(P_{t}\right)$, then it contributes to $S W_{k}$ by $(k-1)\binom{t+1}{k+1}$ from Lemma 3.2. Suppose that $S \cap V\left(P_{t}\right) \neq \emptyset$ and $S \cap\left(V\left(S_{n-t+1}\right)-v\right) \neq \emptyset$. Let $\left|S \cap V\left(S_{n-t+1}-v\right)\right|=i$, $\left|S \cap V\left(P_{t}\right)\right|=k-i$ and $P_{t}=u_{1} u_{2} \cdots u_{t}$, where $v=u_{1}$. Without loss of generality, let $S \cap V\left(P_{t}\right)=\left\{u_{j_{1}}, u_{j_{2}}, \ldots, u_{j_{k-i}}\right\}$, where $1 \leq j_{1}<j_{2}<\cdots<j_{k-i} \leq t$. Then $k-i \leq j_{k-i} \leq t$. Let $j_{k-i}=j$. Then $d_{G}(S)=i+j-1$, and $k-i \leq j \leq t$. Once the vertex $u_{j}$ is chosen, we have $\binom{j-2}{k-i-1}$ ways to choose $u_{j_{1}}, u_{j_{2}}, \ldots, u_{j_{k-i-1}}$. In
this case, we contribute to $S W_{k}$ by

$$
X=\sum_{i=1}^{k-1}\binom{n-t}{i}\left[\sum_{j=k-i}^{t}\binom{j-1}{k-i-1}(j+i-1)\right]
$$

Since

$$
\begin{aligned}
\binom{j-1}{k-i-1}(j+i-1) & =\binom{j-1}{k-i-1} j+\binom{j-1}{k-i-1}(i-1) \\
& =(k-i)\binom{j}{k-i}+(i-1)\binom{j-1}{k-i-1}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\sum_{j=k-i}^{t}\binom{j-1}{k-i-1}(j+i-1) & =(k-i) \sum_{j=k-i}^{t}\binom{j}{k-i}+(i-1) \sum_{j=k-i}^{t}\binom{j-1}{k-i-1} \\
& =(k-i)\binom{t+1}{k-i+1}+(i-1)\binom{t}{k-i}
\end{aligned}
$$

and hence

$$
\begin{aligned}
X= & \sum_{i=1}^{k-1}\binom{n-t}{i}\left[\sum_{j=k-i}^{t}\binom{j-1}{k-i-1}(j+i-1)\right] \\
= & \sum_{i=1}^{k-1}\binom{n-t}{i}\left[(k-i)\binom{t+1}{k-i+1}+(i-1)\binom{t}{k-i}\right] \\
= & \sum_{i=1}^{k-1}\binom{n-t}{i}(k-i)\binom{t+1}{k-i+1}+\sum_{i=1}^{k-1}\binom{n-t}{i}(i-1)\binom{t}{k-i} \\
= & \sum_{i=1}^{k-1}(k-i)\binom{t}{k-i+1}\binom{n-t}{i}+\sum_{i=1}^{k-1}(k-i)\binom{t}{k-i}\binom{n-t}{i} \\
& +\sum_{i=1}^{k-1}(i-1)\binom{t}{k-i}\binom{n-t}{i}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{i=1}^{k-1}(k-i)\binom{t}{k-i+1}\binom{n-t}{i}+(k-1) \sum_{i=1}^{k-1}\binom{t}{k-i}\binom{n-t}{i} \\
& =\sum_{i=1}^{k-1}(k-i)\binom{t}{k-i+1}\binom{n-t}{i}+(k-1)\left[\binom{n}{k}-\binom{t}{k}-\binom{n-t}{k}\right] .
\end{aligned}
$$

Let

$$
Y=\sum_{i=1}^{k-1}(k-i)\binom{t}{k-i+1}\binom{n-t}{i}
$$

Then

$$
\begin{aligned}
Y= & \sum_{i=1}^{k-1}(k-i+1)\binom{t}{k-i+1}\binom{n-t}{i}-\sum_{i=1}^{k-1}\binom{t}{k-i+1}\binom{n-t}{i} \\
= & t \sum_{i=1}^{k-1}\binom{t-1}{k-i}\binom{n-t}{i}-\sum_{i=1}^{k-1}\binom{t}{k+1-i}\binom{n-t}{i} \\
= & t\left[\binom{n-1}{k}-\binom{t-1}{k}-\binom{n-t}{k}\right] \\
& -\left[\binom{n}{k+1}-\binom{t}{k+1}-t\binom{n-t}{k}-\binom{n-t}{k+1}\right]
\end{aligned}
$$

and hence

$$
\begin{aligned}
S W_{k}(T)= & (k-1)\binom{t+1}{k+1}+k\binom{n-t}{k}+X \\
= & (k-1)\binom{t+1}{k+1}+k\binom{n-t}{k}+Y+(k-1)\left[\binom{n}{k}-\binom{t}{k}-\binom{n-t}{k}\right] \\
= & (k-1)\binom{t+1}{k+1}+k\binom{n-t}{k}+t\left[\binom{n-1}{k}-\binom{t-1}{k}-\binom{n-t}{k}\right] \\
& -\left[\binom{n}{k+1}-\binom{t}{k+1}-t\binom{n-t}{k}-\binom{n-t}{k+1}\right] \\
& +(k-1)\left[\binom{n}{k}-\binom{t}{k}-\binom{n-t}{k}\right]
\end{aligned}
$$

$$
\begin{aligned}
= & (k-1)\binom{t}{k+1}+(k-1)\binom{t}{k}+k\binom{n-t}{k}+t\binom{n-1}{k}-t\binom{t-1}{k} \\
& -t\binom{n-t}{k}-\binom{n}{k+1}+\binom{t}{k+1}+t\binom{n-t}{k}+\binom{n-t}{k+1} \\
& +(k-1)\binom{n}{k}-(k-1)\binom{t}{k}-(k-1)\binom{n-t}{k} \\
= & (k-1)\binom{t}{k+1}+k\binom{n-t}{k}+t\binom{n-1}{k}-t\binom{t-1}{k} \\
& -\binom{n}{k+1}+\binom{t}{k+1}+\binom{n-t}{k+1}+(k-1)\binom{n}{k}-(k-1)\binom{n-t}{k} \\
= & k\binom{t}{k+1}+\binom{n-t}{k}+t\binom{n-1}{k}-t\binom{t-1}{k}-\binom{n}{k+1}+\binom{n-t}{k+1} \\
& +\left(\begin{array}{c}
k-1
\end{array}\right)\binom{n}{k} \\
= & k\binom{t}{k+1}+t\binom{n-1}{k}-t\binom{t-1}{k}-\binom{n}{k+1}+\binom{n-t+1}{k+1}+(k-1)\binom{n}{k} \\
= & t\binom{n-1}{k}-\binom{t}{k+1}-\binom{n}{k+1}+\binom{n-t+1}{k+1}+(k-1)\binom{n}{k} .
\end{aligned}
$$

The following corollary is immediate from Theorem 4.1.
Corollary 4.2. For a positive integer $w$, there exists a tree $T$ of order $n$ such that $S W_{k}(T)=w$ if

$$
w=t\binom{n-1}{k}-\binom{t}{k+1}-\binom{n}{k+1}+\binom{n-t+1}{k+1}+(k-1)\binom{n}{k}
$$

where $1 \leq t \leq n-1$ and $k \leq n$.
For general graphs, we have the following.
Theorem 4.3. Let $G$ be a graph obtained from a clique $K_{n-r}$ and a star $S_{r+1}$ by identifying a vertex of $K_{n-r}$ and the center $v$ of $S_{r+1}$. For $k \leq r \leq n-1-k$,

$$
S W_{k}(G)=(n-1)\binom{n-1}{k-1}-\binom{n-r-1}{k}
$$

Proof. For any $S \subseteq V(G)$ and $|S|=k$, if $S \subseteq V\left(K_{n-r}\right)$, then $d_{G}(S)=k-1$. There are $\binom{n-r}{k}$ such subsets, contributing to $S W_{k}$ by $(k-1)\binom{n-r}{k}$. If $S \subseteq$ $V\left(S_{r+1}\right)-v$, then $d_{G}(S)=k$. There are $\binom{r}{k}$ such subsets, contributing to $S W_{k}$ by $k\binom{r}{k}$. Suppose that $S \cap V\left(K_{n-r}\right) \neq \emptyset$ and $S \cap\left(V\left(S_{r+1}\right)-v\right) \neq \emptyset$. If $v \in S$, then $d_{G}(S)=k-1$. There are $\binom{n-r-1}{k-x-1}\binom{r}{x}$ such subsets, contributing to $S W_{k}$ by $(k-1) \sum_{x=1}^{k-1}\binom{n-r-1}{k-x-1}\binom{r}{x}$. If $v \notin S$, then $d_{G}(S)=k$. There are $\binom{n-r-1}{k-x}\binom{r}{x}$ such subsets, contributing to $S W_{k}$ by $k \sum_{x=1}^{k-1}\binom{n-r-1}{k-x}\binom{r}{x}$. Then

$$
\begin{aligned}
& S W_{k}(G) \\
&=(k-1)\binom{n-r}{k}+k\binom{r}{k}+(k-1) \sum_{x=1}^{k-1}\binom{n-r-1}{k-x-1}\binom{r}{x} \\
&+k \sum_{x=1}^{k-1}\binom{n-r-1}{k-x}\binom{r}{x} \\
&=(k-1)\binom{n-r}{k}+k\binom{r}{k}+(k-1)\left[\binom{n-1}{k-1}-\binom{n-1-r}{k-1}\right] \\
&=\left.(k-1)\binom{n-r}{k}+\left(\begin{array}{c}
n-1)\left[\binom{n-1}{k-1}-\binom{n-1-r}{k-1}\right] \\
k-r \\
k-1
\end{array}\right)\right] \\
&+k\left[\binom{n-1}{k}-\binom{n-1-r}{k}\right] \\
&=(k-1)\binom{n-r}{k}+(n-1)\binom{n-1}{k-1}-(k-1)\binom{n-1-r}{k-1}-k\binom{n-1-r}{k} \\
&=(n-1)\binom{n-1}{k-1}+(k-1)\binom{n-r-1}{k}-k\binom{n-1-r}{k} \\
&=(n-1)\binom{n-1}{k-1}-\binom{n-1-r}{k},
\end{aligned}
$$

as desired.
The following corollary is immediate from Theorems 4.1 and 4.3.
Corollary 4.4. For a positive integer $w$, there exists a connected graph $G$ of order $n$ such that $S W_{k}(G)=w$ if $w$ satisfies one of the following conditions.
(1) $w=t\binom{n-1}{k}-\binom{t}{k+1}-\binom{n}{k+1}+\binom{n-t+1}{k+1}+(k-1)\binom{n}{k}$, where $1 \leq t \leq n-1$ and $k \leq n$.
(2) $w=(n-1)\binom{n-1}{k-1}-\binom{n-r-1}{k}$, where $k \leq r \leq n-1-k$ and $k \leq n$.

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