Discussiones Mathematicae
Graph Theory 38 (2018) 135–142
doi:10.7151/dmgt.1998

THE DISTANCE MAGIC INDEX OF A GRAPH

Aloysius Godinho, Tarkeshwar Singh

Department of Mathematics
Birla Institute of Technology and Science Pilani
K K Birla Goa Campus, NH–17B, Zuarinagar, Goa, India

e-mail: p2014001@goa.bits-pilani.ac.in
tksingh@goa.bits-pilani.ac.in

and

S. Arumugam[1]

National Centre for Advanced Research in Discrete Mathematics
Kalasalingam University Anand Nagar
Krishnankoil–626 126, Tamil Nadu, India

e-mail: s.arumugam.klu@gmail.com

Abstract

Let G be a graph of order n and let S be a set of positive integers with $|S| = n$. Then G is said to be S-magic if there exists a bijection $\phi : V(G) \to S$ satisfying $\sum_{x\in N(u)} \phi(x) = k$ (a constant) for every $u \in V(G)$. Let $\alpha(S) = \max\{s : s \in S\}$. Let $i(G) = \min \alpha(S)$, where the minimum is taken over all sets S for which the graph G admits an S-magic labeling. Then $i(G) - n$ is called the distance magic index of the graph G. In this paper we determine the distance magic index of trees and complete bipartite graphs.

Keywords: distance magic labeling, distance magic index, S-magic graph, S-magic labeling.

2010 Mathematics Subject Classification: 05C78.

[1]Also at Department of Computer Science, Liverpool Hope University, Liverpool, UK; Department of Computer Science, Ball State University, USA.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops, multiple edges or isolated vertices. For graph theoretic terminology we refer to West [6].

Most of the graph labeling methods trace their origin to the concept of β-valuation introduced by Rosa [5]. For a general overview of recent developments on various types of graph labelings we refer to Gallian [3].

Let G be a graph of order n. Let $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ be a bijection. Then the weight of a vertex v with respect to f is defined by $\mathcal{W}_f(v) = \sum_{x \in N(v)} f(x)$. The labeling f is said to be distance magic if $\mathcal{W}_f(v) = k$, (a constant), for all vertices $v \in V(G)$. If G admits such a labeling, then G is said to be a distance magic graph and k is called the magic constant of G. Many classes of graphs are shown to be distance magic and for details one may refer to Gallian [3] and Arumugam *et al.* [1].

There are large classes of graphs which cannot be magic labelled with respect to the labeling set $\{1, 2, \ldots, n\}$. For example the graph $K_{1,3}$ is not distance magic with respect to the labeling set $\{1, 2, 3, 4\}$. However if we extend the set of labels to $\{1, 2, 3, 6\}$ or $\{1, 2, 4, 7\}$ the graph can be magic labelled with magic constants 6 or 7 respectively. Motivated by this observation we have introduced the concept of S-magic labeling in [2].

Definition 1.1. Let $G = (V, E)$ be a graph and let S be a set of positive integers with $|S| = |V|$. Then G is said to be S-magic if there exists a bijection $\phi : V \longrightarrow S$ such that $\sum_{x \in N(v)} \phi(x) = k$ for every $v \in V(G)$, where k is a constant. The constant k is called the S-magic constant.

Definition 1.2. Let $\alpha(S) = \max\{s : s \in S\}$. Let $i(G) = \min \alpha(S)$, where the minimum is taken over all S for which the graph G admits an S-magic labeling. Then $i(G) - n$ is called the distance magic index of G and is denoted by $\theta(G)$.

It follows that G is distance magic if and only if $\theta(G) = 0$. If G is not S-magic for any set of positive integers S, then we say that $\theta(G) = \infty$.

Definition 1.3. If a graph G is S-magic, then the magic spectrum of G is defined to be the set of all magic constants that can be obtained through different S-magic labelings of G and is denoted by $\mathcal{M}(G)$.

Remark 1.4. If ϕ is an S-magic labeling of G with magic constant k and $a \in \mathbb{N}$, then $\phi_1 : V \longrightarrow aS$ defined by $\phi_1(v) = a\phi(v)$ is an aS-magic labeling of G with magic constant ka. Hence if $\mathcal{M}(G)$ is non-empty, then $\mathcal{M}(G)$ is an infinite set.

As pointed out in [2], there are graphs which are not S-magic for any set S. For example the Petersen graph, the complete graph K_n, where $n \geq 2$ and the cycle C_n, where $n \geq 5$ are not S-magic. In this paper we determine the distance magic index of trees and complete bipartite graphs.

The following theorem by Miller *et al.* [4] gives necessary and sufficient conditions for distance magic labeling of the complete bipartite and tripartite graphs.

Theorem 1.5. *Let $1 \leq a_1 \leq \cdots \leq a_p$, where $2 \leq p \leq 3$. Let $s_i = \sum_{j=1}^{i} a_j$. There exists a distance magic labeling of the complete multipartite graph $K_{a_1,a_2,\ldots,a_p}$ if and only if the following conditions hold.*

1. $a_2 \geq 2$,
2. $n(n+1) \equiv 0 \pmod{2p}$, *where* $n = s_p = |V(K_{a_1,a_2,\ldots,a_p})|$, *and*
3. $\sum_{j=1}^{s_i}(n+1-j) \geq \frac{in(n+1)}{2p}$ *for* $1 \leq i \leq p$.

We shall rewrite the third condition in Theorem 1.5 for the complete bipartite graph K_{m_1,m_2} with $1 \leq m_1 \leq m_2$ and $m_1 + m_2 = n$.

$\sum_{j=1}^{s_1}(n+1-j) = \sum_{j=1}^{m_1}(n+1-j) = m_1m_2 + \frac{m_1(m_1+1)}{2} \geq \frac{n(n+1)}{4}$.

Since

$$m_1m_2 + \frac{m_1(m_1+1)}{2} + \frac{m_2(m_2+1)}{2} = \frac{n(n+1)}{2}, \tag{1}$$

we get

$$m_1m_2 + \frac{m_1(m_1+1)}{2} \geq \frac{m_2(m_2+1)}{2}. \tag{2}$$

Conversely, suppose $m_1m_2 + \frac{m_1(m_1+1)}{2} \geq \frac{m_2(m_2+1)}{2}$. Then from (1) we obtain $m_1m_2 + \frac{m_1(m_1+1)}{2} \geq \frac{n(n+1)}{4}$.

The inequality (2) reduces to $\frac{2m_1m_2+(m_1-m_2)+(m_1^2-m_2^2)}{2} \geq 0$. From the above discussion we formulate the following theorem.

Theorem 1.6. *The complete bipartite graph K_{m_1,m_2} with $1 \leq m_1 \leq m_2$ and $m_1 + m_2 = n$ is distance magic if and only if the following conditions hold.*

1. $m_2 \geq 2$,
2. $n \equiv 0$ *or* $3 \pmod 4$, *and*
3. $\frac{2m_1m_2+(m_1-m_2)+(m_1^2-m_2^2)}{2} \geq 0$.

Thus Theorem 1.6 is just a restated version of Theorem 1.5 for the special case K_{m_1,m_2}.

2. Main Results

Theorem 2.1. *A tree T is S-magic if and only if $T = K_{1,r}$, where $r \geq 2$. Further the distance magic index of $K_{1,r}$ is $\frac{r(r-1)}{2} - 1$.*

Proof. We label the pendant vertices with the labels $\{1, 2, \ldots, r\}$ and the central vertex with the label $\frac{r(r+1)}{2}$. This gives an S-magic labeling for $K_{1,r}$. Obviously, the magic index is $\frac{r(r-1)}{2} - 1$. If T is any tree other than the star $K_{1,r}$, then T contains two support vertices s_1 and s_2 and for any labeling f, a leaf adjcent to s_1 and a leaf adjcent to s_2 have distinct weights. Hence T is not S-magic. ■

We now proceed to determine the distance magic index of some of the standard graphs.

Lemma 2.2. *If G is an S-magic graph of order n with distance magic index θ, then*

$$\frac{\delta(2(n+\theta) - \delta + 1) - \Delta(\Delta + 1)}{2} \geq 0.$$

Proof. Since the distance magic index of G is θ, there exists a set $S \subset \{1, 2, \ldots, n+\theta\}$ with $|S| = n$ and an S-magic labeling $f : V(G) \longrightarrow S$ with magic constant k. Let $v_1, v_2 \in V(G)$, $deg(v_1) = \Delta$ and $deg(v_2) = \delta$.

Then

$$\mathcal{W}_f(v_1) \geq 1 + 2 + \cdots + \Delta = \frac{\Delta(\Delta+1)}{2}$$

and

$$\mathcal{W}_f(v_2) \leq (n+\theta) + (n+\theta-1) + \cdots + (n+\theta-\delta+1) = \frac{\delta(2(n+\theta) - \delta + 1)}{2}.$$

Since $\mathcal{W}_f(v_1) = \mathcal{W}_f(v_2) = k$, it follows that

$$\frac{\delta(2(n+\theta) - \delta + 1)}{2} \geq \frac{\Delta(\Delta+1)}{2}.$$

Hence

$$\frac{\delta(2(n+\theta) - \delta + 1) - \Delta(\Delta + 1)}{2} \geq 0.$$

■

Observation 2.3. *Let*

$$g(x) = \frac{\delta(2(n+x) - \delta + 1) - \Delta(\Delta + 1)}{2}.$$

Clearly $g(x)$ is a strictly increasing function of x. Hence if a is a non-negative integer satisfying

$$\frac{\delta(2(n+a) - \delta + 1) - \Delta(\Delta + 1)}{2} < 0,$$

then we can conclude that $\theta(G) > a$. *Also if* a *is the smallest non-negative integer such that* $g(a) \geq 0$, *then it follows that* $\theta(G) \geq a$. *Now*

$$g(0) = \frac{\delta(2n - \delta + 1) - \Delta(\Delta + 1)}{2}. \tag{3}$$

If $g(0) < 0$, *then the graph* G *is not distance magic.*

Lemma 2.4. *Let* G *be a graph of order* n *such that* $g(0) < 0$. *Then* $\theta(G) \geq \left\lceil \frac{|g(0)|}{\delta} \right\rceil$.

Proof. We first determine the smallest non-negative value of x for which $g(x) \geq 0$. Let $|g(0)| = q\delta + r$, $0 \leq r < \delta$.

We have

$$g(0) = \frac{\delta(2n - \delta + 1) - \Delta(\Delta + 1)}{2} = -q\delta - r. \tag{4}$$

Hence

$$\frac{\delta(2(n + q) - \delta + 1) - \Delta(\Delta + 1)}{2} = -r.$$

If $r = 0$, then it follows that q is the smallest value of x such that $g(x) \geq 0$. Hence $\theta(G) \geq q$. If $r > 0$, then $\theta(G) > q$ and

$$\frac{\delta(2(n + q) - \delta + 1) - \Delta(\Delta + 1) + 2r}{2} = 0.$$

Since $r < \delta$,

$$\frac{\delta(2(n + q) - \delta + 1) - \Delta(\Delta + 1) + 2r}{2} < \frac{\delta(2(n + q) - \delta + 1) - \Delta(\Delta + 1) + 2\delta}{2}.$$

Hence

$$\frac{\delta(2(n + q + 1) - \delta + 1) - \Delta(\Delta + 1)}{2} > 0.$$

Thus $q+1$ is the smallest non-negative value of x which satisfies the inequality $g(x) \geq 0$. Therefore we have $\theta(G) \geq q + 1$. ∎

Lemma 2.2, Observation 2.3 and Lemma 2.4 are tools that can be used to determine $\theta(G)$ for any graph G. In particular we determine $\theta(G)$ for the complete bipartite graph $G = K_{m_1,m_2}$, where $m_1 \leq m_2$. Since the case $m_1 = 1$ is covered in Theorem 2.1, we assume that $2 \leq m_1 \leq m_2$. Clearly $\delta = m_1, \Delta = m_2$ and $n = m_1 + m_2$. Substituting these values in (3), we get

$$g(0) = \frac{n(n + 1)}{2} - m_2(1 + m_2). \tag{5}$$

Theorem 2.5. *Let G be the complete bipartite graph K_{m_1,m_2}, where $2 \le m_1 \le m_2$. Then*

$$\theta(G) = \begin{cases} 0 & \text{if } n(n+1) \ge 2m_2(1+m_2) \text{ and } n \equiv 0 \text{ or } 3 \pmod 4, \\ 1 & \text{if } n(n+1) \ge 2m_2(1+m_2) \text{ and } n \equiv 1 \text{ or } 2 \pmod 4, \\ \left\lceil \frac{|n(n+1)-2m_2(1+m_2)|}{2m_1} \right\rceil & \text{if } n(n+1) < 2m_2(1+m_2). \end{cases}$$

Proof. Let U_1, U_2 be the partite sets of G such that $|U_1| = m_1$ and $|U_2| = m_2$. A labeling of G is S-magic if and only if the sum of the labels assigned to U_1 is equal to the sum of the labels assigned to U_2. Initially we label U_1 and U_2 with label sets $L_1 = \{m_2+1, m_2+2, \dots, m_1+m_2\}$ and $L_2 = \{1, 2, \dots, m_2\}$, respectively. The sum of the labels in L_1 and L_2 are $\mathcal{S}(L_1) = m_1m_2 + \frac{m_1(m_1+1)}{2}$ and $\mathcal{S}(L_2) = \frac{m_2(m_2+1)}{2}$ and it follows that $\mathcal{S}(L_1) - \mathcal{S}(L_2) = g(0)$. To make the two sums equal we either interchange labels between L_1 and L_2 or we increase some labels beyond the initial set of labels $\{1, 2, \dots, m_1+m_2\}$.

Case 1. $n(n+1) \ge 2m_2(1+m_2)$ and $n \equiv 0$ or $3 \pmod 4$. By Theorem 1.6 the graph K_{m_1,m_2} is distance magic. Hence $\theta(G) = 0$.

Case 2. $n(n+1) \ge 2m_2(1+m_2)$ and $n \equiv 1$ or $2 \pmod 4$. By Theorem 1.6 the graph K_{m_1,m_2} is not distance magic. Therefore $\theta(G) \ge 1$. For $G = K_{2,3}$, the labeling with $L_1 = \{2, 6\}$ and $L_2 = \{1, 3, 4\}$ gives an S-magic labeling and hence $\theta(G) = 1$. Now, let $m_2 \ge 4$.

We claim that $g(0) \equiv 1 \pmod 2$.

Since $n \equiv 1$ or $2 \pmod 4$, it follows that $\frac{n(n+1)}{2} \equiv 1 \pmod 2$. Furthermore, $m_2(m_2+1) \equiv 0 \pmod 2$ and hence

$$g(0) = \frac{n(n+1)}{2} - m_2(m_2+1) \equiv 1 \pmod 2.$$

Let $\mathcal{S}(L_1) - \mathcal{S}(L_2) = g(0) = 2p-1$ for some $p \ge 1$. Hence $\big(\mathcal{S}(L_1) - p + 1\big) - \big(\mathcal{S}(L_2) + p\big) = 0$. Now let $p = (m_1-1)q + r$, $0 \le r < m_1 - 1$. Therefore we have

$$\text{(6)} \qquad \Big(\mathcal{S}(L_1) - (m_1-1)q - r + 1\Big) - \Big(\mathcal{S}(L_2) + (m_1-1)q + r\Big) = 0.$$

Now from (5), since $m_1 \le m_2$, we have $g(0) = 2p - 1 \le m_1m_2$. Hence $p \le \frac{m_1m_2+1}{2}$ and so $(m_1-1)q + r \le \frac{m_1m_2+1}{2}$. Since $m_1 \ge 2$, it follows that $q < m_2$.

Now we proceed to attain equality in the sum of the labels for the two partite sets. We decrease each of the labels $m_2+1, m_2+2, \dots, m_2+m_1-1$ in L_1 by q. Hence the resulting label sets for U_1 and U_2 are $L_1' = \{m_2+1-q, m_2+2-q, \dots, m_1+m_2-1-q, m_1+m_2\}$ and $L_2' = \{1, 2, \dots, m_2-q, m_2+1, m_2+2, \dots, m_1+m_2-1\}$. It follows that $\mathcal{S}(L_1') = \mathcal{S}(L_1) - q(m_1-1)$ and

$\mathcal{S}(L_2') = \mathcal{S}(L_2)+q(m_1-1)$. We now replace the label $m_1+m_2 \in L_1'$ with the label m_1+m_2+1, leaving L_2' unchanged. The resulting sets of labels are $L_1'' = \{m_2+1-q, m_2+2-q, \ldots, m_1+m_2-1-q, m_1+m_2+1\}$, $L_2'' = L_2'$ and we have the relations $\mathcal{S}(L_1'') = \mathcal{S}(L_1')+1 = \mathcal{S}(L_1)-(m_1-1)q+1$, $\mathcal{S}(L_2'') = \mathcal{S}(L_2') = \mathcal{S}(L_2)+(m_1-1)q$. If $r = 0$, we label the vertices of U_1 and U_2 with the labels L_1'' and L_2''. Using the relations for $\mathcal{S}(L_1'')$ and $\mathcal{S}(L_2'')$ and equation (6), we obtain $\mathcal{S}(L_1'') = \mathcal{S}(L_2'')$. Hence this labeling is S-magic. If $r > 0$, then we interchange the label $m_2 - q + r \in L_1'$ with the label $\big(m_2 - q + r\big) - r = m_2 - q \in L_2'$. The resulting label sets are $L_1''' = \{m_2-q, m_2+1-q, \ldots, m_2-q+r-1, m_2-q+r+1, \ldots, m_2+m_1-q-1, m_1+m_2+1\}$, $L_2''' = \{1, 2, \ldots, m_2-q-1, m_2-q+r, m_2+1, m_2+2, \ldots, m_1+m_2-1\}$ and $\mathcal{S}(L_1''') = \mathcal{S}(L_1'')-r = \mathcal{S}(L_1)-q(m_1-1)-r+1$. Similarly, $\mathcal{S}(L_2''') = \mathcal{S}(L_2'')+r = \mathcal{S}(L_2)+q(m_1-1)+r$. Using these relations in (6) we get $\mathcal{S}(L_1''') = \mathcal{S}(L_2''')$. Now we label the partite sets U_1 and U_2 with labels L_1''' and L_2''', respectively. This labeling is S-magic. Since the highest used label is $n+1$, we have $\theta(G) = 1$. Note that since $q < m_2$, the above steps are valid.

Case 3. $n(n+1) < 2m_2(1+m_2)$. Let $|g(0)| = m_1q + r,\ 1 \le r < m_1$. Then $\mathcal{S}(L_1) - \mathcal{S}(L_2) = g(0) = -m_1q - r$. Therefore,

$$\Bigg(\mathcal{S}(L_1) + m_1q + r\Bigg) - \mathcal{S}(L_2) = 0. \tag{7}$$

Applying Lemma 2.4 we obtain, $\theta(G) \ge \left\lceil \frac{|g(0)|}{\delta} \right\rceil$. We claim that $\theta(G) = \left\lceil \frac{|g(0)|}{\delta} \right\rceil$. We start with the initial labelling as described in the beginning of the proof. In the first step we increase each label in L_1 by q leaving L_2 unchanged. Hence the resulting label sets are $L_1' = \{m_2+1+q, m_2+2+q, \ldots, m_2+m_1+q\}$, $L_2' = L_2$ and $\mathcal{S}(L_1') = \mathcal{S}(L_1) + m_1q$. If $r = 0$, then we label the vertices of U_1 and U_2 with the labels L_1' and L_2', respectively. Using the relations $\mathcal{S}(L_1') = \mathcal{S}(L_1) + m_1q$, $\mathcal{S}(L_2') = \mathcal{S}(L_2)$ and (7), we obtain $\mathcal{S}(L_1') = \mathcal{S}(L_2')$. Hence the labeling is S-magic. Since the highest label used is m_1+m_2+q, we have $\theta(G) = q$.

If $r > 0$ we replace the label $m_1 + m_2 + q - r + 1 \in L_1'$ with the label $m_1+m_2+q-(r-1)+r = m_1+m_2+q+1$. Hence the resulting sets of labels are $L_1'' = \{m_2+1+q, m_2+2+q, \ldots, m_1+m_2+q-r, m_1+m_2+q-r+2, \ldots, m_2+m_1+q, m_1+m_2+q+1\}$ and $L_2'' = L_2$. Now $\mathcal{S}(L_1'') = \mathcal{S}(L_1')+r = \mathcal{S}(L_1)+m_1q+r$ and $\mathcal{S}(L_2'') = \mathcal{S}(L_2)$. Therefore we label the vertices of U_1 and U_2 with the labels L_1'' and L_2'', respectively. We have $\mathcal{S}(L_1'') = \mathcal{S}(L_2'')$. Hence the labeling is S-magic. Since the highest label used is $n+q+1$, we have $\theta(G) = q+1$. This completes the proof. ∎

3. Conclusion and Scope

In this paper we have introduced the concept of distance magic index of graphs and determined the same for complete bipartite graphs. Determining the distance magic index for other families of graphs and obtaining tight bounds are directions for further research.

Acknowledgment

The first and second authors are thankful to Department of Science and Technology, Govt. of India for supporting the research activities under the project SR/S4/MS/734/11. The authors are thankful to the referees for their helpful suggestions which led to substantial improvement in the presentation.

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Received 28 June 2016
Revised 20 October 2016
Accepted 20 October 2016