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FIBONACCI AND TELEPHONE NUMBERS IN EXTREMAL TREES

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Abstract

In this paper we shall show applications of the Fibonacci numbers in edge-coloured trees. In particular we determine the successive extremal graphs in the class of trees with respect to the number of $(A, 2B)$-edge colourings. We show connections between these numbers and Fibonacci numbers as well as the telephone numbers.

Keywords: edge colouring, tripod, Fibonacci numbers, telephone numbers.

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1. Introduction and reliminary Results

For concepts not defined here see [3]. Let G be an undirected, connected and simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The order (number of vertices) and the size (number of edges) of G will be denoted by n and m, respectively. By $G(m)$ we mean a graph of size m. Consequently, by $P(m)$, $C(m)$, $T(m)$ and $S(m)$ we denote a path, a cycle, a tree and a star of size m, respectively. In a tree a vertex of degree at least 3 is named a *branch vertex*, a vertex of degree 1 is a *leaf*. A *tripod* is a tree with exactly three leaves. In other words, every tripod has the unique branch vertex, i.e., the vertex being the initial vertex of three elementary paths. Let $m \geq 3$, $p \geq 1$, $t \geq 1$ be integers. By $T(m, p, t)$ we mean a tripod of size m and paths of lengths p, t and $m-p-t$ with the branch vertex as the initial vertex of these paths. These paths we denote shortly by *p-path, t-path* and *$(m-p-t)$-path*, respectively.

Let consider a star with the maximum degree Δ, $\Delta \geq 3$. Let $r \geq 1$ be an integer. By $S_r(m, \Delta)$ we denote a tree of size $m, m \geq 3$, obtained from this star by inserting new vertices of degree 2 into some edges of the star such that in the resulting tree $S_r(m, \Delta)$ a longest r-path, starting from the branch vertex, has length $r, r \geq 1$. In particular, $S_1(m, \Delta)$ is isomorphic to the star $S(m)$ and $S_r(m, 3)$ is isomorphic to the tripod $T(m, r, t)$, for some $t \geq 1$.

The nth Fibonacci number is defined by the second order linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with $F_0 = F_1 = 1$. The nth Lucas number is defined by the same linear recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with different initial conditions $L_0 = 2$ and $L_1 = 1$. The Fibonacci numbers have many interesting graph interpretations, see for example trailblazing results given by Prodinger and Tichy in [5] and later results classfied in [4].

A new graph interpretation of the Fibonacci numbers was introduced recently in [1] and it is related to a special edge colouring of a graph. We recall this interpretation for the Fibonacci numbers; for the global case and more details see [1].

Let $\mathcal{C} = \{A, B\}$ be the set of two colours. A graph G is $(A, 2B)$-*edge coloured* if for every maximal B-monochromatic subgraph H of G there is a partition of H into edge disjoint paths of the length 2. Clearly, an $(A, 2B)$-edge colouring always exists, since we have no restriction on the colour A.

Let $\mathcal{L}$ be a family of all distinct $(A, 2B)$-edge coloured graphs obtained by $(A, 2B)$-colourings of a graph G. Then $\mathcal{L} = \left\{G^{(1)}, G^{(2)}, \ldots, G^{(r)}\right\}$, $r \geq 1$, where $G^{(p)}, 1 \leq p \leq r$, denotes a graph obtained by an $(A, 2B)$-edge colouring of a graph G. For an $(A, 2B)$-edge coloured graph $G^{(p)}$, $1 \leq p \leq r$, by $\theta\big(G^{(p)}\big)$ we denote the number of all partitions of B-monochromatic subgraphs of $G^{(p)}$ into edge disjoint paths of length 2. If $G^{(p)}$ is A-monochromatic then we put $\theta G^{(p)} = 1$. The number of all $(A, 2B)$-edge colourings is defined as the graph parameter as follows

$$\sigma_{(A,2B)}(G) = \sum_{p=1}^{r} \theta\left(G^{(p)}\right).$$

The parameter $\sigma_{(A,2B)}(G)$ was determined for paths and cycles. For trees the lower bound and the upper bound of it were given. We recall these results. For paths and cycles the following result was proved.

Theorem 1 [1]. *Let m be an integer. Then*

(1) $\sigma_{(A,2B)}(P_m) = F_m$, *for* $m \geq 1$,

(2) $\sigma_{(A,2B)}(C_m) = L_m$, *for* $m \geq 3$.

The following theorem was proved for trees.

Theorem 2 [1]. *Let $T(m)$ be a tree of size $m, m \geq 1$. Then*

$$F_m \leq \sigma_{(A,2B)}(T(m)) \leq 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

Moreover

$$T(m) = P(m), \quad \textit{if } \sigma_{(A,2B)}(P(m)) = F_m$$

and

$$T(m) = S(m), \quad \textit{if } \sigma_{(A,2B)}(S(m)) = 1 + \sum_{j\geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

From the above theorem it follows that the path $P(m)$ is the extremal graph achieving the minimum value of the parameter $\sigma_{(A,2B)}(T(m))$ and the star $S(m)$ is the extremal graph achieving the maximum value of this parameter in the class of trees of size m.

In this paper we will determine successive extremal graphs in the class of trees with respect to the parameter $\sigma_{(A,2B)}(T(m))$. We consider the number of $(A, 2B)$-edge colouring in trees $T(m)$ with the restriction that $T(m) \ncong P(m)$ and next $T(m) \ncong T(m, 2, 2)$ and also $T(m) \ncong S(m)$.

For future investigations we use the following notation. Let $e \in E(G)$ be a fixed edge. If e is coloured by A then we write $c(e) = A$. If e is coloured by B then there exists an edge coloured by B, say e', adjacent to e. Then, for indication of this fact, we will write $c(e) = 2B$. Moreover, the path $e - e'$ will be named a *$2B$-path.* Let $\sigma_{A(e)}(G)$ (respectively $\sigma_{2B(e)}(G)$) denote the number of all $(A, 2B)$-edge colourings with $c(e) = A$ (respectively $c(e) = 2B$). Then the following formula gives the basic rule for determining the parameter $\sigma_{(A,2B)}(G)$.

Let $e \in E(G)$ be a fixed edge. Then

$$\sigma_{(A,2B)}(G) = \sigma_{A(e)}(G) + \sigma_{2B(e)}(G). \tag{1}$$

The following lemma gives the rule of replacing a subgraph of a given graph G by the extremal graph with respect to the number of $(A, 2B)$-edge colourings and extends the result given in [1].

Lemma 3 [1]**.** *Let $G = H \cup T(l) \cup \{e\}$ be a connected graph, where H is a connected graph, $T(l)$ is a tree of size l, $l \geq 1$, and H and $T(l)$ are vertex disjoint. Assume that $e = uv$, where $u \in V(H)$ and $v \in V(T(l))$. Then*

$$\sigma_{(A,2B)}(H \cup P(l) \cup \{e\}) \leq \sigma_{(A,2B)}(G) \leq \sigma_{(A,2B)}(H \cup S(l) \cup \{e\}), \tag{2}$$

where the vertex v is identified with the center of the star $S(l)$. Moreover, the equality holds if $T(l) \cong P(l)$ or $T(l) \cong S(l)$.

Proof. The lower bound of the estimation (2) in the theorem was proved in [1]. Now we prove the upper bound. Let $G = H \cup T(l) \cup \{e\}$ be as in the statement of the lemma. Let $e = uv$ be the edge of the graph G such that the vertex $u \in V(H)$ and the vertex $v \in V(T(l))$. Consider two possibilities.

1. $c(e) = A$. Hence $\sigma_{A(e)}(G) = \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(T(l))$.
2. $c(e) = 2B$. Then, by the definition of an $(A, 2B)$-edge colouring, there exists an edge $e' \in E(G)$ adjacent to e such that $c(e') = 2B$ and $e' - e$ is the $2B$-path of length 2 in the B-monochromatic subgraph of G. It is clear that either $e' \in E(H)$ or $e' \in E(T(l))$. From these considerations we obtain

$$\sigma_{2B(e)}(G) = \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(T(l)) + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(T(l) \cup \{e\}).$$

Therefore,

$$\begin{aligned}\sigma_{(A,2B)}(G) &= \sigma_{A(e)}(G) + \sigma_{2B(e)}(G) = \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(T(l)) \\ &\quad + \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(T(l)) + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(T(l) \cup \{e\}).\end{aligned}$$

From Theorem 2 we have that $\sigma_{(A,2B)}(T(l)) \leq \sigma_{(A,2B)}(S(l))$, so

$$\begin{aligned}\sigma_{(A,2B)}(G) &\leq \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(S(l)) + \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(S(l)) \\ &\quad + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(S(l) \cup \{e\}) = \sigma_{(A,2B)}(H \cup S(l) \cup \{e\}),\end{aligned}$$

which ends the proof. ■

This lemma gives a tool for determining successive extremal graphs with respect to the parameter $\sigma_{(A,2B)}(T(m))$.

Let us consider the class of trees $T(m)$ such that $T(m) \ncong P(m)$. This means that there exists at least one branch vertex in $T(m)$. Assume that $\mathcal{T} = \{T(m,p,t); p \geq 1, t \geq 1, m \geq 3\}$ is the family of tripods. We recall some results for tripods, given in [2], which will be useful in future investigations.

Theorem 4 [2]**.** *Let $m \geq 3, p \geq 1, t \geq 1$ be integers. Then for an arbitrary $T(m,p,t) \in \mathcal{T}$ it holds*

$$\sigma_{(A,2B)}(T(m,p,t)) = F_{p+t}F_{m-t-p} + F_{m-t-p-1}\left(F_{p-1}F_t + F_pF_{t-1}\right). \tag{3}$$

From this theorem it immediately follows the next result.

Corollary 5 [2]**.** *Let $m \geq 3$, $t \geq 1$ be integers. Then*
(a) $\sigma_{(A,2B)}(T(m,1,t)) = F_{t+1}F_{m-t}$,
(b) $\sigma_{(A,2B)}(T(m,1,1)) = 2F_{m-1}$.

Using the above results, the maximum and minimum values of the parameter $\sigma_{(A,2B)}(T(m,p,t))$ were proved.

Theorem 6 [2]**.** *Let $m \geq 4$, $p \geq 1$, $t \geq 1$ be integers. Then*

$$F_{m-1} + 2F_{m-3} \leq \sigma_{(A,2B)}(T(m,p,t)) \leq 2F_{m-1}.$$

Moreover, we have $\sigma_{(A,2B)}(T(m,p,t)) = 2F_{m-1}$ if $T(m,p,t) \cong T(m,1,1)$ and $\sigma_{(A,2B)}(T(m,p,t)) = F_{m-1} + 2F_{m-3}$ if $T(m,p,t) \cong T(m,2,2)$.

In the following theorem the recurrence formula for counting the parameter $\sigma_{(A,2B)}(T(m,p,t))$ is presented.

Theorem 7 [2]**.** *Let $m \geq 3, p \geq 1, t \geq 1$ be integers. Then for an arbitrary $T(m,p,t) \in \mathcal{T}$ and $m-p-t \geq 3$ it holds*

$$\sigma_{(A,2B)}(T(m,p,t)) = \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(T(m-2,p,t)), \tag{4}$$

with the initial conditions

$$\sigma_{(A,2B)}(T(p+t+1,p,t)) = F_{p+1}F_{t+1}$$

and

$$\sigma_{(A,2B)}(T(p+t+2,p,t)) = F_{p+1}F_{t+1} + F_{p+t}.$$

2. Lower Bounds for $\sigma_{(A,2B)}(T(m))$

In this section we determine the successive minimum trees with respect to the parameter $\sigma_{(A,2B)}(T(m))$. To do this we use among others the following theorem.

Theorem 8 [2]**.** *Let $m \geq 4$, $\Delta \geq 4$, $r \geq 1$ be integers. Then for an arbitrary $T(m,p,t) \in \mathcal{T}$ it holds*

$$\sigma_{(A,2B)}\left(S_r(m,\Delta)\right) > \sigma_{(A,2B)}\left(T(m,p,t)\right). \tag{5}$$

Theorem 9. *Let $m \geq 3$ be an integer. Let $T(m) \ncong P(m)$ and $T(m) \ncong T(m,p,t)$ for all $p \geq 1, t \geq 1$. Then*

$$\sigma_{(A,2B)}(T(m)) \geq \sigma_{(A,2B)}(T(m,p,t)) \geq \sigma_{(A,2B)}(P(m)).$$

Proof. The inequality $\sigma_{(A,2B)}(T(m,p,t)) \geq \sigma_{(A,2B)}(P(m))$ immediately follows from Theorem 2. Let $T(m) \ncong P(m)$ and $T(m) \ncong T(m,p,t)$ for all $p \geq 1, t \geq 1$. We consider the following possibilities.

(1) $T(m)$ has a unique branch vertex, say x. Since $T(m)$ is not a tripod, so $\deg_{T(m)} x \geq 4$. Then $\Delta(T(m)) = \deg_{T(m)} x \geq 4$. Therefore, $T(m) \cong S_r(m,\Delta)$, where $r \geq 1$. Then by Theorem 8 it holds $\sigma_{(A,2B)}\left(S_r(m,\Delta)\right) > \sigma_{(A,2B)}\left(T(m,p,t)\right)$, and the inequality follows.

(2) $T(m)$ has at least two branch vertices. Let u, v be two branch vertices and let $u-v$ be the path between u, v. Consequently, $T(m) = T_1(m_1) \cup T_2(m_2) \cup \{e\}$, where $e \in E(T(m))$ is an edge belonging to the path $u-v$ and $m_1+m_2+1 = m$. Applying Lemma 3 we obtain that

$$\sigma_{(A,2B)}(T(m)) \geq \sigma_{(A,2B)}(T(m_1) \cup P(m_2) \cup \{e\}) = \sigma_{(A,2B)}(T(m_1) \cup P(m_2+1).$$

If $T(m_1) \cup P(m_2+1)$ is isomorphic to $S_r(m,\Delta)$, for some $r \geq 1$, then the result follows. If $T(m_1) \cup P(m_2+1)$ is not isomorphic to $S_r(m,\Delta)$, then we repeat this procedure until we obtain $S_r(m,\Delta)$, and the result follows. ∎

From the above theorem we deduce that if we want to determine the sequence of trees of size $m, m \geq 3$, such that the corresponding sequence of parameters $\sigma_{(A,2B)}(T(m))$ is nondecreasing, we have to determine the sequence of tripods for which the corresponding sequence of parameters $\sigma_{(A,2B)}(T(m,p,t))$ is nondecreasing. In [2] the second minimum tree of size m with respect to $\sigma_{(A,2B)}(T(m))$ was described. Consequently, the second smallest value of the parameter $\sigma_{(A,2B)}(T(m))$ was given. We recall this result.

Theorem 10 [2]. *Let $T(m) \ncong P(m)$ be a tree of size $m, m \geq 5$. Then*

$$\sigma_{(A,2B)}\left(T(m)\right) \geq F_{m-1} + 2F_{m-3}. \tag{6}$$

Moreover, $\sigma_{(A,2B)}\left(T(m)\right) = F_{m-1} + 2F_{m-3}$ if $T(m) \cong T(m,2,2)$.

Let us consider the sequence b_m defined by the linear recurrence relation $b_m = 1 + b_{m-2} + b_{m-3} - b_{m-5}$, for $m \geq 5$, with the initial conditions $b_0 = b_1 = b_2 = 0$, $b_3 = 1$ and $b_4 = 1$. This recurrence equation generates the sequence $0, 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, \ldots$, and from [8, 9] we know that b_m is the number of all nonisomorphic tripods $T(m,p,t)$ of size m.

The following table includes all nonisomorphic tripods and their parameters $\sigma_{(A,2B)}(T(m,p,t))$ for $m = 3, 4, 5, 6, 7$.

m	$T(m,p,t)$	$\sigma_{(A,2B)}(T(m,p,t))$
3	$T(3,1,1)$	4
4	$T(4,1,1)$	6
5	$T(5,1,1)$	10
	$T(5,2,2)$	9
6	$T(6,1,1)$	16
	$T(6,2,2)$	14
	$T(6,3,1)$	15
7	$T(7,1,1)$	26
	$T(7,2,2)$	23
	$T(7,3,1)$	25
	$T(7,4,2)$	24

Table 1

In the present paper we give the successive smallest values of the parameter $\sigma_{(A,2B)}(T(m))$.

Theorem 11. *Let $m \geq 7, p \geq 1, t \geq 1$ be integers and $T(m,p,t) \ncong T(m,2,2)$. Then*

$$\sigma_{(A,2B)}(T(m,p,t)) \geq 2F_{m-3} + 7F_{m-5}.$$

Moreover, the equality holds for $T(m,p,t) \cong T(m,4,2)$.

Proof. Let m, p, t be as in the statement of the theorem. If $p = t = 1$, then by Theorem 6 it immediately follows that

$$\sigma_{(A,2B)}(T(m,1,1)) \geq \sigma_{(A,2B)}(T(m,4,2)).$$

Let $p = 2$ and $t = 1$. Then $\sigma_{(A,2B)}(T(m,2,1)) = 3F_{m-2}$ by Corollary 5. We shall show that $3F_{m-2} \geq 2F_{m-3} + 7F_{m-5}$. Using the well-known identities for Fibonacci numbers we have

$$3F_{m-2} - 2F_{m-3} - 7F_{m-5} = 3F_{m-3} + 3F_{m-4} - 2F_{m-3} - 7F_{m-5}$$

$$= F_{m-3} + 3F_{m-4} - 7F_{m-5} = F_{m-4} + F_{m-5} + 3F_{m-5} + 3F_{m-6} - 7F_{m-5}$$

$$= F_{m-5} + F_{m-6} + 3F_{m-6} - 3F_{m-5} = 4F_{m-6} - 2F_{m-5}$$

$$= 4F_{m-6} - 2F_{m-6} - 2F_{m-7} = 2F_{m-6} - 2F_{m-7} > 0.$$

Since $p \neq 2$ and $t \neq 2$ simultaneously, we assume that $p \geq 2$, $t \geq 3$. If $m = 7$ then the result follows by Table 1. Assume that $m \geq 8$. Let us consider a tripod $T(m,p,t)$ and suppose that $\sigma_{(A,2B)}(T(m,p,t)) \geq \sigma_{(A,2B)}(T(n,4,2))$ for all $n < m$. Since $m \geq 8$, we can assume, without lost of the generality, that $m-p-t \geq 3$. By Theorem 7 and by induction hypothesis we have

$$\sigma_{(A,2B)}(T(m,p,t)) = \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(T(m-2,p,t))$$

$$\geq \sigma_{(A,2B)}(T(m-1,4,2)) + \sigma_{(A,2B)}(T(m-2,4,2)) = \sigma_{(A,2B)}(T(m,4,2)),$$

and the result follows. ■

From the above theorem and by Theorem 9 we receive the third smallest value of the parameter $\sigma_{(A,2B)}(T(m))$ in the class of trees.

Corollary 12. *Let $m \geq 7$ be an integer and $T(m) \ncong P(m)$, and $T(m) \ncong T(m,2,2)$. Then*

$$\sigma_{(A,2B)}(T(m)) \geq 2F_{m-3} + 7F_{m-5}.$$

Moreover, the equality holds for $T(m) \cong T(m,4,2)$.

Using the same method as in Theorem 11 we can give successive values of the parameter $\sigma_{(A,2B)}(T(m))$.

Theorem 13. *Let $T(m) \ncong P(m)$, $T(m) \ncong T(m,2,2)$ and $T(m) \ncong T(m,4,2)$ for $m = 12, 13$. Then*

$$\sigma_{(A,2B)}(T(m)) \geq 12F_{m-6} + 3F_{m-3}.$$

Moreover, the equality holds for $T(m) \cong T(m,5,2)$.

Theorem 14. *Let $T(m) \ncong P(m)$, $T(m) \ncong T(m,2,2)$, $T(m) \ncong T(m,4,2)$ and $T(m) \ncong T(m,5,2)$ for $m = 12, 13$. Then*

$$\sigma_{(A,2B)}(T(m)) \geq 5F_{m-4} + F_{m-2}.$$

Moreover, the equality holds for $T(m) \cong T(m,3,2)$.

As it was proved in Theorem 6, the tripod $T(m,1,1)$ is extremal in the class $\mathcal{T}$ with respect to the parameter $\sigma_{(A,2B)}(T(m,p,t))$. The next theorem gives the second largest value of the parameter $\sigma_{(A,2B)}(T(m))$ if $T(m) \in \mathcal{T}$.

Theorem 15. *Let $T(m) \ncong S(m)$ and $T(m) \ncong T(m,1,1)$ for $m \geq 7$. Then*

$$\sigma_{(A,2B)}(T(m)) \geq 5F_{m-3}.$$

Moreover, the equality holds for $T(m) \cong T(m,3,1)$.

From the above considerations we have the following observation. Let $\alpha_i(m)$ be the ith minimum tree with respect to the parameter $\sigma_{(A,2B)}(T(m))$ of size m. Then $\alpha_1(m) \cong P(m)$, $\alpha_2(m) \cong T(m,2,2)$, $\alpha_3(m) \cong T(m,4,2)$, $\alpha_k(m) \cong T(m,5,2)$, $\alpha_{k+1}(m) \cong T(m,3,2)$, $\alpha_i(m) \in \mathcal{T}$ for $i = 4, \ldots, b_{m-1}$, $\alpha_{b_m}(m) \cong T(m,3,1)$, and $\alpha_{b_{m+1}}(m) \cong T(m,1,1)$.

For $i = 4, \ldots, b_{m-1}$ and for $i \geq b_{m+2}$, the problem of finding trees $\alpha_i(m)$ is open. Consequently, the initial words of the nondecreasing sequence of the parameter $\sigma_{(A,2B)}(T(m))$ have the form $F_m, F_{m-1}+2F_{m-3}, 2F_{m-3}+7F_{m-5}, \ldots, 12F_{m-6}+3F_{m-3}, 5F_{m-4}+F_{m-2}, \ldots, 5F_{m-3}, 2F_{m-1}, \ldots$.

3. Telephone Numbers in Upper Bounds of $\sigma_{(A,2B)}(T(m))$

In this section we study the upper bound of the parameter $\sigma_{(A,2B)}(T(m))$. From Theorem 2 proved in [1] we have that the maximum value of the parameter $\sigma_{(A,2B)}(T(m))$ is realized in the star $S(m)$ and

$$\sigma_{(A,2B)}(S(m)) = 1 + \sum_{j\geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

In this section we shall show that the bound obtained in [1] is realized by the telephone numbers.

The telephone numbers (or involution numbers) are integers which satisfy the recurrence relation $t(n) = t(n-1) + (n-1)t(n-2)$, for $n \geq 2$, starting from $t(0) = t(1) = 1$. These numbers were studied firstly by Rothe in 1800, who introduced a recurrence equation for them. The telephone numbers have many

combinatorial interpretations. One of them counts the numbers of connections patterns in a telephone system with n subscribes, see [6]. For this reason the numbers $t(n)$ are named as the telephone numbers.

These numbers have also a graph interpretation known as the Hosoya index of the n-vertex complete graph K_n. We recall that the *Hosoya index* of the graph G is the number of all matchings of a graph G and it is usually denoted by $Z(G)$. Then $Z(K_n) = t(n)$ for $n \geq 1$, see [7].

We show that the telephone numbers have a graph interpretation related to the number of all $(A, 2B)$-edge colourings of the star.

Theorem 16. *Let $m \geq 1$ be an integer. Then*

$$\sigma_{(A,2B)}(S(m)) = t(m).$$

Proof. We use the induction on m. If $m = 1, 2$ then $S(m)$ is isomorphic to $P(m)$ and it is obvious that $\sigma_{(A,2B)}(P(1)) = 1 = t(1)$ and $\sigma_{(A,2B)}(P(2)) = 2 = t(2)$.

Let $m \geq 3$ and suppose that for $n < m$ it holds $\sigma_{(A,2B)}(S(n)) = t(n)$. We shall show that the theorem is true for m. Let $e \in E(S(m))$ be a fixed edge. We distinguish two possibilities.

1. $c(e) = A$. Then all remaining edges are coloured either by colour A or B. This means that $S(m) \setminus e \cong S(m-1)$ and by the induction hypothesis $\sigma_{A(e)}(S(m)) = t(m-1)$.

2. $c(e) = 2B$. Then there is an edge $e' \in E(S(m) \setminus e)$ such that $c(e') = 2B$. Since all edges in a star $S(m)$ are adjacent, so the edge e' can be chosen in $m-1$ ways. Moreover, $E(S(m) \setminus \{e, e'\}) \cong S(m-2)$ and using the induction hypothesis we obtain that $\sigma_{2B(e)}(S(m)) = (m-1)t(m-2)$.

By the rule (1) we have

$$\sigma_{(A,2B)}(S(m)) = t(m-1) + (m-1)t(m-2) = t(m),$$

which completes the proof. ■

From the above and by Theorem 2 we obtain the new direct formula for the telephone numbers $t(m)$ given next.

Corollary 17. *Let $m \geq 1$ be an integer. Then*

$$t(m) = 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

From the above considerations we obtain the next result as follows.

Corollary 18. *Let $m \geq 1$ be an integer. Then*

$$\sigma_{(A,2B)}(T(m)) \leq 1 + \sum_{j\geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

The telephone numbers play an important role also in finding the second largest value of the parameter $\sigma_{(A,2B)}(T(m))$.

Let $P(m_1, m - m_1 - 1)$ be a 2-palm of size $m, m \geq 5$, and diameter 3 with two support vertices $x, y \in V(P(m_1, m - m_1 - 1))$. Suppose that the support vertex x is adjacent to m_1 leaves. Then the vertex y is adjacent to $m - m_1 - 1$ leaves.

Theorem 19. *Let $m \geq 5, m_1 \geq 2$ be integers and let $m_1 \geq m - m_1 - 1$. Then*

$$\begin{aligned}\sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) &= t(m - m_1 - 1)t(m_1 + 1) \\ &\quad + (m - m_1 - 1)t(m - m_1 - 2)t(m_1).\end{aligned}$$

Proof. Let m, m_1 be as in the statement of theorem. Let $P(m_1, m - m_1 - 1)$ be a 2-palm of size m. Assume that $m_1 \geq m - m_1 - 1$. Let $e = xy \in E(P(m - 1, m - m_1 - 1))$, where x, y are support vertices adjacent to m_1 or $m - m_1 - 1$ leaves, respectively. We distinguish the following possibilities

1. $c(e) = A$. Then $\sigma_{A(e)}(P(m_1, m - m_1 - 1)) = t(m_1)t(m - m_1 - 1)$.
2. $c(e) = 2B$. Then there exists an edge $e' \in E(P(m_1, m - m_1 - 1))$ such that $\{e, e'\}$ belongs to a partition of $2B$-monochromatic subgraph of $P(m_1, m - m_1 - 1)$ and $c(e') = 2B$. Therefore $e' \in E(S(m_1))$ and it can be chosen in m_1 ways or $e' \in E(S(m - m_1 - 1))$ and can be chosen in $m - m_1 - 1$ ways. From that and Theorem 16 we obtain

$$\begin{aligned}\sigma_{2B(e)}(P(m_1, m - m_1 - 1)) &= m_1 t(m_1 - 1)t(m - m_1 - 1) \\ &\quad + (m_1 - m - 1)t(m - m_1 - 2)t(m_1).\end{aligned}$$

Using the rule (1) and the recurrence relation of telephone numbers by simple calculations we get

$$\begin{aligned}\sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) &= t(m_1)t(m - m_1 - 1) + m_1 t(m_1 - 1)t(m - m_1 - 1) \\ &\quad + (m_1 - m - 1)t(m - m_1 - 2)t(m_1) \\ &= t(m - m_1 - 1)[t(m_1) + m_1 t(m_1 - 1)] \\ &\quad + (m - m_1 - 1)t(m - m_1 - 2)t(m_1) \\ &= t(m - m_1 - 1)t(m_1 + 1) \\ &\quad + (m - m_1 - 1)t(m - m_1 - 2)t(m_1),\end{aligned}$$

which ends the proof. ∎

The next lemma shows the behavior of the parameter $\sigma_{(A,2B)}(P(m_1, m - m_1 - 1))$ after moving an edge adjacent to a support vertex to the second support vertex.

Lemma 20. *Let $m \geq 5, m_1 \geq 2$ be integers and $m_1 \geq m - m_1 - 1$. Then*

$$\sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 2)) > \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)).$$

Proof. Let m, m_1 be as in the statement of lemma. Let $P(m_1 + 1, m - m_1 - 2)$ and $P(m_1, m - m_1 - 1)$ be 2-palms of size m. We will show that

$$\sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 2)) - \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) > 0.$$

Applying Theorem 19 and the definition of the telephone numbers we obtain

$$\begin{aligned}
&\sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 2)) - \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) \\
&= t(m - m_1 - 2)t(m_1 + 2) + (m - m_1 - 2)t(m - m_1 - 3)t(m_1 + 1) \\
&- t(m - m_1 - 1)t(m_1 + 1) - (m - m_1 - 1)t(m - m_1 - 2)t(m_1) \\
&= t(m - m_1 - 2)[t(m_1 + 1) + (m_1 + 1)t(m_1)] \\
&+ t(m_1 + 1)[(m - m_1 - 2)t(m - m_1 - 3) - t(m - m_1 - 1)] \\
&- (m - m_1 - 1)t(m - m_1 - 2)t(m_1).
\end{aligned}$$

After some calculations and applying once again the definition of the telephone numbers we get

$$\begin{aligned}
&\sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 2)) - \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) \\
&= t(m_1 + 1)[t(m - m_1 - 2) + (m - m_1 - 2)t(m - m_1 - 3) - t(m - m_1 - 1)] \\
&+ t(m_1)t(m - m_1 - 2)[m_1 + 1 - m + m_1 + 1] \\
&= t(m_1 + 1)[t(m - m_1 - 1) - t(m - m_1 - 1)] + t(m_1)t(m - m_1 - 2)[2m_1 + 2 - m].
\end{aligned}$$

Observe that $t(m_1)t(m - m_1 - 2)[2m_1 + 2 - m] > 0$, by the assumption that $2m_1 \geq m - 1$. Hence this completes the proof. ■

Now we can give the second largest value of the parameter $\sigma_{(A,2B)}(T(m))$.

Theorem 21. *Let $T(m)$ be a tree of size $m, m \geq 5$, and $T(m) \ncong S(m)$. Then*

$$\sigma_{(A,2B)}(T(m)) \leq \sigma_{(A,2B)}(P(m - 2, 1)).$$

Proof. Let $T(m)$ be a tree of size $m, m \geq 5$, such that $T(m) \ncong S(m)$. Since $T(m)$ is not isomorphic to the star $S(m)$, there exist at least two vertices $x, y \in V(T(m))$ which are not leaves. Let $e = xy \in E(T(m))$ and x, y are not leaves. Then

$T(m) = T(m_1) \cup \{e\} \cup T(m_2)$, where $T_i(m_i)$, for $i = 1, 2$, are trees of size m_i and $x \in V(T(m_1))$, $y \in V(T(m_2))$.

Applying Lemma 3 we obtain

$$\sigma_{(A,2B)}(T(m)) = \sigma_{(A,2B)}(T(m_1) \cup \{e\} \cup T(m_2)) \leq \sigma_{(A,2B)}(S(m_1) \cup \{e\} \cup S(m_2)),$$

where the vertex x is the center of the star $S(m_1)$ and the vertex y is the center of the star $S(m_2)$.

If $m_1 = 1$ or $m_2 = 1$, then $S(m_1) \cup \{e\} \cup S(m_2) \cong P(m-2, 1)$ and the theorem is proved.

Let $m_1 > 1$ and $m_2 > 1$ and, without loos of the generality, suppose that $m_1 \geq m_2$. Then $S(m_1) \cup \{e\} \cup S(m_2)$ is isomorphic to the 2-palm $P(m_1, m_2) \cong P(m_1, m - m_1 - 1)$. Applying Lemma 20 we obtain

$$\sigma_{(A,2B)}(T(m)) = \sigma_{(A,2B)}(P(m_1, m - m_1 - 1)) < \sigma_{(A,2B)}(P(m_1 + 1, m - m_1 - 2)).$$

If $m - m_1 - 2 = 1$, then $P(m_1 + 1, m - m_1 - 2)$ is isomorphic to $P(m - 2, 1)$ and the result follows.

If $P(m_1 + 1, m - m_1 - 2)$ is not isomorphic to $P(m - 2, 1)$, then we apply Lemma 20 until we obtain the 2-palm $P(m - 2, 1)$, which ends the proof. ■

Corollary 22. *Let $m \geq 5$ be an integer and $T(m) \ncong S(m)$. Then*

$$\sigma_{(A,2B)}(T(m)) \leq t(m-1) + t(m-2),$$

with the initial conditions $t(3) = 4, t(4) = 10$.

Proof. By Theorem 21 it sufficies to determine the number $\sigma_{(A,2B)}(P(m-2, 1))$. Using Theorem 19 and the definition of the telephone numbers we obtain that

$$\sigma_{(A,2B)}(P(m-2, 1)) = t(1)t(m-1) + t(0)t(m-2) = t(m-1) + t(m-2). \quad ■$$

Let $\beta_i(m)$ be the ith maximum tree of size m with respect to the parameter $\sigma_{(A,2B)}(T(m))$. Then $\beta_1(m) \cong S(m), \beta_2(m) \cong P(m-2, 1)$ and $\beta_3(m) \cong P(m-3, 2), \ldots$.

For $i \geq 4$ the problem of finding $\beta_i(m)$ is open. Consequently, the initial words of nonincreasing sequence of the parameter $\sigma_{(A,2B)}(T(m))$ using the telephone numbers have the form $t(m), t(m-1)+t(m-2), 2t(m-2)+2t(m-3), \ldots$.

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