CUBIC GRAPHS WITH TOTAL DOMATIC NUMBER AT LEAST TWO

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Abstract

Let G be a graph with no isolated vertex. A total dominating set of G is a set S of vertices of G such that every vertex is adjacent to at least one vertex in S. The total domatic number of a graph is the maximum number of total dominating sets which partition the vertex set of G. In this paper we provide a criterion under which a cubic graph has total domatic number at least two.

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1. Introduction

The total domination of graphs was introduced by Cockayne *et al.* [3]. The literature on this subject has been surveyed and detailed in the two excellent domination books by Haynes *et al.* [4, 5] who did an outstanding job of unifying results

scattered through some 1200 domination papers at that time. More recent results on this topic can be found in [6].

Throughout this paper, all graphs are simple, that is they have no loops or multiple edges. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For two positive integers m and n, C_n and $K_{m,n}$ denote the cycle of order n and the complete bipartite graph of sizes m and n, respectively.

Let G be a simple graph with no isolated vertex. The *open neighborhood* of a vertex v, denoted by $N_G(v)$, $N(v)$ for short, is the set of vertices adjacent to v in G. If for every vertex $v \in V(G)$, the degree of v in G is r, then the graph G is called *r-regular*. A 3-regular graph is called a *cubic* graph. A hypergraph H is *r-uniform* if for every hyperedge in $E(H)$ has size r, and is *r-regular* if every vertex in $V(H)$ is in exactly r hyperedges. Furthermore, a hypergraph H is *bipartite* if its vertex set can be partitioned into two sets such that every hyperedge intersects both partite sets. Equivalently, H is *bipartite* if it is 2-colorable; that is, there is a 2-coloring of the vertices with no monochromatic hyperedges.

A set $T \subseteq V(G)$ of vertices is said to be a *total dominating set* of G, if every vertex $v \in V(G)$ is adjacent to at least one vertex in T. The *total domatic number* of a graph G, denoted by $d_t(G)$, is the maximum number of total dominating sets which partition the vertex set of G. One can easily see that for a graph G with no isolated vertex, $d_t(G) \geq 1$. The main goal of this paper is to classify the family of cubic graphs for which $d_t(G) \geq 2$.

It is fairly easy to see that a graph can be partitioned into two total dominating sets if and only if its open neighborhood hypergraph is bipartite. It is an easy consequence of the renowned Lovász local lemma [8] that every r-uniform r-regular hypergraph is bipartite for all $r \geq 9$ [9]. Alon and Bregman [1] improved that result by employing algebraic methods to assert that the same holds for $r \geq 8$. Henning and Yeo [7] later adapted their technique to prove

Theorem 1 ([7]). *Every r-uniform r-regular hypergraph is bipartite for all $r \geq 4$.*

Theorem 1 immediately reveals that all r-regular graphs for $r \geq 4$ can be partitioned into two total dominating sets. It is a well-known fact that a cycle has two disjoint total dominating sets if and only if its order is divisible by 4. In the Heawood graph, which is a cubic graph, complement of no total dominating set is a total dominating set. This is derived from the fact that if F is a 3-uniform family on $n \geq 5$ points, in which each pair of points occurs in the same number of sets, then F is not 2-colorable. Therefore, one naturally seeks to classify all cubic graphs which can be partitioned into two total dominating sets. This classification, combined with Henning and Yeo's result [7], characterizes the class of regular graphs whose vertices are impossible to be decomposed into two total dominating sets. It is briefly indicated in [7] that there are infinitely many connected cubic graphs with total domatic number one, the characterization of which is an open problem.

Since coloring and partitioning are essentially the same, the total domination has also been studied in the literature of graph coloring under the name of coupon coloring [2]. A *k-coupon coloring* of a graph G is an assignment of colors from $\{1, \ldots, k\}$ to $V(G)$, such that for all $v \in V(G)$, the open neighborhood of v contains all k colors. The maximum k for which a k-coupon coloring of G exists is called the *coupon coloring number* of G. It is evident that the coupon coloring number and the total domatic number of a graph are the same.

Given a family $\mathcal{F}$ of graphs, an *$\mathcal{F}$-partitioning* of the vertices of G is a set of disjoint subgraphs of G, each isomorphic to a graph in $\mathcal{F}$, covering every vertex of G.

2.   Main Result

In this section we prove that for all cubic graphs, having no subgraph isomorphic to L (given in Figure 1), the total domatic number is at least two.
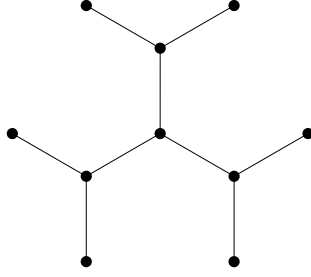

Figure 1. The graph L.

Theorem 2. *The vertex set of a cubic graph can be partitioned into two total dominating sets if it has no subgraph (not necessarily induced) isomorphic to the graph L. In other words, coupon coloring number of every such graph is at least two.*

In this section we prove our main result concerning the characterization of cubic graphs G with $d_t(G) \geq 2$. Before we embark on that, we state two relevant lemmas.

Lemma 3. *In a cubic graph G with no subgraph isomorphic to L, every vertex is either contained in a C_3 or a C_4.*

Proof. By contradiction, assume that there exists a vertex $v \in V(G)$ which is contained neither in a C_3 nor a C_4. Therefore, no two vertices in $N(v)$ are adjacent and moreover no two of them have a common neighbor. Thus, there are exactly 3 vertices at distance 1 and 6 vertices at distance 2 of v. Therefore, G has a subgraph isomorphic to L, a contradiction.   ∎

Figure 2 presents a family $\mathcal{F}$ of graphs which will be used in the next lemma.
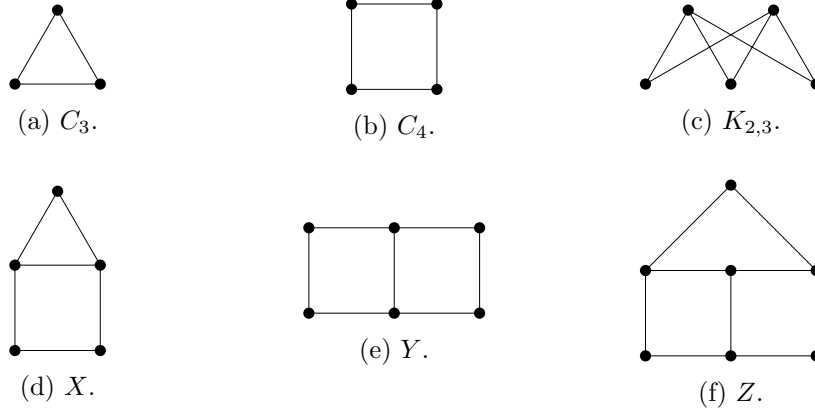

(a) C_3.   (b) C_4.   (c) $K_{2,3}$.

(d) X.   (e) Y.   (f) Z.

Figure 2. Family $\mathcal{F}$ of graphs.

Lemma 4. *Every cubic graph with no subgraph isomorphic to L has an $\mathcal{F}$-partitioning, where $\mathcal{F}$ is the family of graphs which was described in Figure 2.*

Proof. Let G be a cubic graph with no subgraph isomorphic to L. We prove this lemma by providing an algorithm. We construct an $\mathcal{F}$-partitioning of G by maintaining an $\mathcal{F}$-partitioning $\mathcal{P}$ of an induced subgraph H of G while iteratively adding some vertices to H, to obtain $H = G$.

In other words, we prove the existence of $\mathcal{P}$ by induction on size of H. We start with H as the empty subgraph and $\mathcal{P}$ as the empty set which is an $\mathcal{F}$-partitioning for H. Assume that in some step, H is an induced proper subgraph of G and $\mathcal{P}$ is an $\mathcal{F}$-partitioning of H. Choose a vertex $v \in V(G)\backslash V(H)$. We show how to add v to H and update $\mathcal{P}$ in such a way that $\mathcal{P}$ remains an $\mathcal{F}$-partitioning.

By Lemma 3, every vertex of G is either contained in a C_3 or a C_4. First we discuss the case where v is contained in a C_4, but not in a C_3. Next, we consider the other case where v is in a C_3.

First, assume that v is contained in a C_4 but not in a C_3. Let C be that 4-cycle and let u_1, u_2 and u_3 be the other vertices of C, where u_1 and u_3 are adjacent to v in C. Regarding the number of vertices among u_1, u_2 and u_3 that are already in H, one of the following holds.

Case 1. None of u_1, u_2 and u_3 lies in $V(H)$. In this case, we add v, u_1, u_2 and u_3 to H and C to $\mathcal{P}$. Clearly, H has an $\mathcal{F}$-partitioning.

Case 2. Exactly one of u_1, u_2 and u_3 is in $V(H)$. Observe that all graphs in $\mathcal{F}$ have minimum degree 2. Hence, a vertex in H with two neighbors outside H would have degree at least 4 in G, a contradiction. Thus this case never occurs.

Case 3. Exactly two of u_1, u_2 and u_3 are in $V(H)$. By the previous observation and noting that v is not on a C_3, these two vertices should be adjacent and they are of degree 2 in H. Furthermore, they have to be in the same element $Q \in \mathcal{P}$. The only graphs in $\mathcal{F}$ with two adjacent vertices of degree 2 are C_3, C_4, X and Y. Without loss of generality, assume that $u_2, u_3 \in V(H)$ (see Figure 3). We add v and u_1 to H and

(a) If $Q \simeq C_3$, then we replace Q with X, consisting of vertices of Q and C.

(b) If $Q \simeq C_4$, then we replace Q with Y, consisting of vertices of Q and C.

(c) If $Q \simeq X$, then we replace Q with a C_3 and a C_4, consisting of vertices of Q and C. It is clear from Figure 3(c) that the subgraph induced on $V(Q) \cup \{v, u_1\}$ contains two disjoint subgraphs isomorphic to C_3 and C_4.

(d) If $Q \simeq Y$, then we replace Q with two C_4 graphs, consisting of vertices of Q and C. It is clear from Figure 3(d) that the subgraph induced on $V(Q) \cup \{v, u_1\}$ contains two disjoint subgraphs isomorphic to C_4.
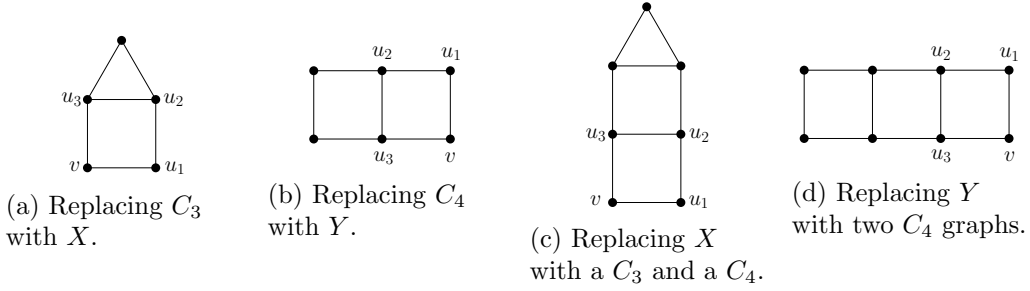

(a) Replacing C_3 with X.

(b) Replacing C_4 with Y.

(c) Replacing X with a C_3 and a C_4.

(d) Replacing Y with two C_4 graphs.

Figure 3. Updating the $\mathcal{F}$-partitioning of H.

Case 4. All u_1, u_2 and u_3 are in $V(H)$. In this case, again, u_1, u_2 and u_3 have to be in the same element Q of $\mathcal{P}$, as otherwise a vertex of degree 1 should appear in some element in $\mathcal{P}$. Thus Q is a graph with two vertices of degree 2 at distance 2 of each other. All graphs in $\mathcal{F}$ have this property, except C_3. Each of these cases is discussed below. First we add v to H. Then

(a) If $Q \simeq C_4$, we replace Q with $K_{2,3}$, consisting of v and all vertices of Q.

(b) If $Q \simeq K_{2,3}$, we replace Q with Y, consisting of v and all vertices of Q. It is clear that Figure 4(b) contains a subgraph isomorphic to Y.

(c) If $Q \simeq X$, we replace Q with Y, consisting of v and all vertices of Q. It is clear that Figure 4(c) contains a subgraph isomorphic to Y.

(d) If $Q \simeq Y$, we replace Q with Z, consisting of v and all vertices of Q.

(e) If $Q \simeq Z$, then we add v to H and replace Q with two disjoint 4-cycles.

(a) Replacing C_4 with $K_{2,3}$.   (b) Replacing $K_{2,3}$ with Y.   (c) Replacing X with Y.

(d) Replacing Y with Z.

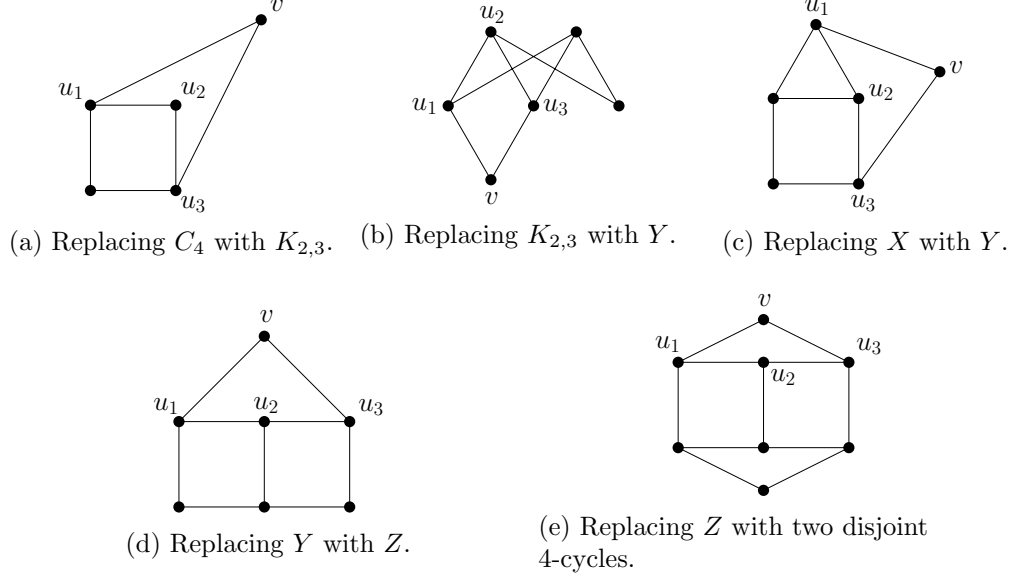(e) Replacing Z with two disjoint 4-cycles.

Figure 4. Updating $\mathcal{F}$-partitioning of H.

Figure 4 illustrates that $\mathcal{P}$ will in the end be an $\mathcal{F}$-partitioning of H.

Next, assume that v is contained in a 3-cycle C, with vertex set $\{v, w_1, w_2\}$. Again, by observing the degrees of the graphs in $\mathcal{F}$, it can be seen that either both w_1 and w_2 are in H or none of them is in H. If neither of w_1 and w_2 are in H, then we add v, w_1 and w_2 to H and add C to $\mathcal{P}$. If both of w_1 and w_2 are in H, then they have to be in the same element $Q \in \mathcal{P}$. Thus Q is a subgraph with two adjacent vertices of degree 2. The only graphs in $\mathcal{F}$ with this property are C_3, C_4, X and Y. We add all vertices of C to H.

1. If $Q \simeq C_3$, then we replace Q with C_4, consisting of v and vertices of Q.
2. If $Q \simeq C_4$, then we replace Q with X, consisting of v and vertices of Q.
3. If $Q \simeq X$, then we replace Q with two C_3 graphs, consisting of v and vertices of Q.
4. If $Q \simeq Y$, then we replace Q with C_3 and a C_4, consisting of v and vertices of Q.

Note that the case where v is in both a C_3 and a C_4 is included in this case. The proof is complete.   ∎

Using Lemma 4, we can now state the proof of Theorem 2.

***Proof of Theorem* 2.** To prove the theorem, we show that the vertices of any cubic graph G, with no subgraph isomorphic to L, can be colored by black and white such that each vertex is adjacent to at least one vertex of each color. By

Lemma 4, G has an $\mathcal{F}$-partitioning $\mathcal{P}$. It is easy to verify that C_4, $K_{2,3}$, X and Y each have total domatic number at least 2. Choose an arbitrary 2-coupon coloring for each element $Q \in \mathcal{P}$ isomorphic to C_4, $K_{2,3}$, X or Y. It now remains to color the vertices of C_3 and Z subgraphs. Figure 5 shows two colorings of C_3 and Z, respectively. In this figure, a filled square represents a black vertex and an empty square represents a white vertex. To color the vertices of C_3 and Z subgraphs, we repeatedly choose an arbitrary C_3 subgraph or Z subgraph $Q \in \mathcal{P}$ and an arbitrary vertex v in $N_G(Q)$. If v is already colored at this step, we color the vertices of Q with respect to the color of v as in Figure 5. However, it is possible for v to be uncolored. This is the case when v belongs to another C_3 subgraph or Z subgraph Q'. In this case we color v black, and then choose the colors of vertices of Q and Q' based on the colorings given in Figure 5.
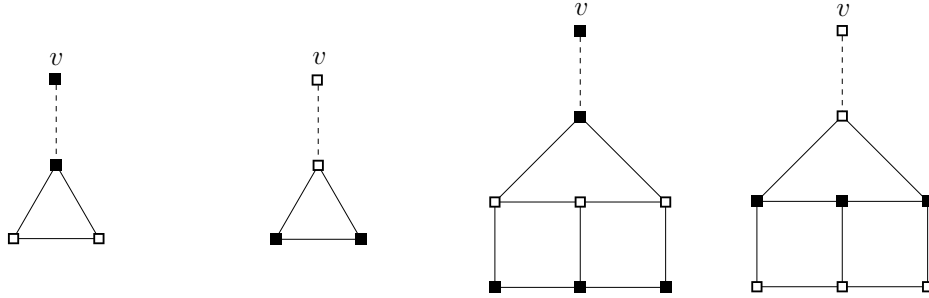

Figure 5. Colorings of C_3 and Z.

When every vertex is colored, each color class is a total dominating set, since any vertex of G has at least one neighbor of each color. Thus we have partitioned the graph into two total dominating sets. ∎

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