Discussiones Mathematicae
Graph Theory 38 (2018) 107–119
doi:10.7151/dmgt.1995

ON INCIDENCE COLORING OF COMPLETE MULTIPARTITE AND SEMICUBIC BIPARTITE GRAPHS [1]

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Abstract

In the paper, we show that the incidence chromatic number χ_i of a complete k-partite graph is at most $\Delta+2$ (i.e., proving the *incidence coloring conjecture* for these graphs) and it is equal to $\Delta+1$ if and only if the smallest part has only one vertex (i.e., $\Delta = n-1$). Formally, for a complete k-partite graph $G = K_{r_1,r_2,\ldots,r_k}$ with the size of the smallest part equal to $r_1 \geq 1$ we have

$$\chi_i(G) = \begin{cases} \Delta(G)+1 & \text{if } r_1 = 1, \\ \Delta(G)+2 & \text{if } r_1 > 1. \end{cases}$$

In the paper we prove that the incidence 4-coloring problem for semicubic bipartite graphs is $\mathcal{NP}$-complete, thus we prove also the $\mathcal{NP}$-completeness of $L(1,1)$-labeling problem for semicubic bipartite graphs. Moreover, we observe that the incidence 4-coloring problem is $\mathcal{NP}$-complete for cubic graphs, which was proved in the paper [12] (in terms of generalized dominating sets).

Keywords: incidence coloring, complete multipartite graphs, semicubic graphs, subcubic graphs, $\mathcal{NP}$-completeness, $L(1,1)$-labelling.

2010 Mathematics Subject Classification: 05C69, 05C05, 05C85.

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[1] This project has been partially supported by Narodowe Centrum Nauki under contract DEC-2011/02/A/ST6/00201.

1. Introduction

In the following we consider connected simple graphs only, and use standard notations in graph theory. Let n be a positive integer and $G = (V, E)$ be any n-vertex graph of the maximum degree[2] $\Delta(G) > 0$. A pair $(u, \{u, v\})$ is an *incidence* of G if and only if $u, v \in V$ and $\{u, v\} \in E$. The set of all incidences of G will be denoted by $I(G)$. To shorten the notation, we will write uv instead of $(u, \{u, v\})$. We will say that incidence uv *leads from u to v*. We will say that incidences $uv \neq wx$ are *adjacent* in G if and only if one of the following holds: (1) $u = w$; (2) $u = x$ and $v = w$; (3) ($u \neq x$ and $v = w$) or ($u = x$ and $v \neq w$), which is equivalent to $u = x$ or $u = w$ or $v = w$. Obviously, if uv is adjacent to wx, then $v \neq x$.

A function $c\colon I(G) \to \mathbb{N}$ is an *incidence coloring* of G if and only if $c(uv) \neq c(wx)$ for all adjacent incidences uv and wx. The *incidence coloring number* of G, denoted by $\chi_i(G)$, is the smallest integer k such that there is an incidence coloring c of G using exactly k colors. By the *incidence k-coloring*, we mean an incidence coloring c of G with k colors (i.e., $k = |c(I(G))|$), and by the *incidence k-coloring problem* we mean a decision problem of the existence of the incidence k-coloring in a graph G.

The notion of the incidence coloring was introduced in [3]. In [10] the author observed that the problem of incidence graph coloring is a special case of the star arboricity problem, i.e., the problem of partitioning of a set of arcs of a symmetric digraph into the smallest number of forests of directed stars. That problem was studied in [1, 2, 10].

The following bounds are well-known (see [3, 17]).

Proposition 1. *For every graph G of order $n \geq 2$ and $\Delta(G) > 0$ there is*

$$\Delta(G) + 1 \leq \chi_i(G) \leq n.$$

The upper bound $\chi_i(G) \leq 2\Delta(G)$ for every graph G has been proved in [3]. This bound has been improved in [10], where the author proved that $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$ for every graph G.

1.1. Motivation and our results

In [3] the authors conjectured that $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph G (*incidence coloring conjecture*, shortly ICC). This was disproved by Guiduli in [10] who showed that Paley graphs have incidence coloring number at least $\Delta + \Omega(\log \Delta)$. For the following classes of graphs the incidence coloring number is at most $\Delta + 2$: trees and cycles [3], complete graphs [3] and complete bipartite

[2]We sometimes write Δ instead of $\Delta(G)$ whenever G is clear from the context.

graphs [5] and [3] (proof corrected in [17]). In fact, for all of them the exact value is equal to $\Delta + 1$ or $\Delta + 2$ and the optimal coloring is constructed in polynomial time. In [7] the authors proved that any partial 2-tree (i.e., K_4-minor free graph) admits $\Delta + 2$ incidence coloring, hence for all outerplanar graphs ICC holds. In [8] the authors proved that every planar graph with girth at least 11 or with girth at least 6 and maximum degree at least 5 has incidence coloring number at most $\Delta + 2$. Recently, the conjecture was proved for pseudo-Halin graphs [11], some powers of cycles [15] and hypercubes [16].

In the paper [19] the authors claim that the incidence coloring conjecture holds for complete multipartite graphs, but the coloring presented in the proof of Theorem 3.1 in [19] is incorrect and presented without a full proof. In Section 2 we will show that the incidence coloring number of complete k-partite graphs is at most $\Delta + 2$, and is equal to $\Delta + 1$ if and only if the size of the smallest part equals 1. We present an $O(n^2)$-time algorithm giving an optimal coloring (i.e., with the minimum number of colors).

In [6] the authors proved that ICC holds for some subclasses of cubic graphs (e.g. Hamiltonian cubic graphs). In [18] the author proved that ICC holds for cubic graphs having a Hamiltonian path and for bridgeless cubic graphs of high girth. At last, in [14] the author proved that ICC holds for subcubic graphs. In [13] the authors proved $\mathcal{NP}$-completeness of the incidence 4-colorability of *semicubic* graphs (i.e., graphs with $\Delta = 3$ and vertices of degree equal to 1 or 3). By the paper [12] we conclude that the incidence 4-coloring problem is $\mathcal{NP}$-complete for cubic graphs. The complexity of this problem was unknown for (semicubic) bipartite graphs. In Section 3 we will show that incidence 4-colorability of semicubic bipartite graphs is $\mathcal{NP}$-complete.

2. Incidence Coloring of Complete Multipartite Graphs

In this section we present an algorithm (formula) giving a coloring of a multipartite graph using at most $\Delta + 2$ colors.

In [19] the authors presented Theorem 3.1 claiming that the incidence coloring conjecture holds for complete multipartite graphs. The coloring σ presented in the proof of Theorem 3.1 is incorrect and in fact there is no proof that this coloring is a proper incidence coloring and uses at most $\Delta + 2$ colors. In the coloring definition (3.2) [19] the authors use the formula $\sum_{m=0}^{t-1}(n_m + s)$, but n_0 is undefined, so we believe it should be corrected to $m = 1$. But in this case, following the notation from [19], for $k \geq 3$ let $t = k - 1$ and $s = n_t$. Take any $j < t$, hence for $n_j \geq n_t$ we can put $i = s$, thus we get $\sigma(v_s^j, v_s^j v_s^{k-1}) = \sum_{m=1}^{k-2}(n_m + s) = \sum_{m=1}^{k-1} n_m + (k-3)n_{k-1} = \Delta + (k-3)n_{k-1} > \Delta + 2$, for $k \geq 6$ or $k = 5$ and $n_{k-1} > 1$.

In the following, we present a different coloring than the coloring σ from [19].

Let $G = K_{r_1,r_2,\ldots,r_k}$ be a complete k-partite graph with $V(G) = V_1 \cup \cdots \cup V_k$, where integer $k \geq 1$, $|V_i| = r_i$, for each $i \in \{1, \ldots, k\}$, and all V_i are independent sets and pairwise disjoint.

Theorem 2. *For any complete multipartite graph $G = K_{r_1,r_2,\ldots,r_k}$ with $k \geq 2$, there is*

$$\chi_i(G) = \Delta(G) + 1 \text{ if and only if } r_1 = 1,$$

where $r_1 = \min\{r_1, \ldots, r_k\}$.

Proof. ($\Rightarrow$) Let c be an incidence coloring of G that uses $\Delta + 1$ colors. Suppose that $r_1 > 1$. Let $u \neq v$ be two vertices that belong to V_1. Then

- u is of degree Δ and c uses exactly $\Delta + 1$ colors, so all incidences that lead to u must have the same color, say a;
- v is of degree Δ and c uses exactly $\Delta + 1$ colors, so all incidences that lead to v must have the same color, say b;
- $a \neq b$ since u and v have identical neighborhoods;
- incidences that lead from u also lead to vertices that are adjacent to v, so colors of incidences leading from u must differ from b.

Hence c uses at least $\Delta + 2$ colors: a, b and Δ other colors on incidences that lead from u, a contradiction.

($\Leftarrow$) It follows immediately from Proposition 1, since $\Delta(G) = |V(G)| - r_1$. ■

Theorem 3. *For any complete multipartite graph $G = K_{r_1,r_2,\ldots,r_k}$ with $k \geq 2$, there is*

$$\chi_i(G) \leq \Delta(G) + 2.$$

Proof. Let us assume that $1 \leq r_1 \leq r_2 \leq \cdots \leq r_k$ and $V(G) = V_1 \cup \cdots \cup V_k$, $|V_i| = r_i$, and sets from $\{V_i\}_{i \in \{1,\ldots,k\}}$ are independent sets and pairwise disjoint. Let $n = |V(G)|$. It suffices to show that there is an incidence coloring of G that uses at most $\Delta + 2$ colors. Before we do this, we have to introduce some notations.

Let $s_i \colon V_i \to \{1, 2, \ldots, r_i\}$ be any numbering of vertices of V_i for $i \in \{1, \ldots, k\}$. Let $p \colon V(G) \to \{1, 2, \ldots, k\}$ be a function such that $p(u) = i$ if and only if $u \in V_i$. Let

$$V(G) \ni u \mapsto s(u) = r_k + 1 - s_{p(u)}(u) \in \{1, 2, \ldots, r_k\}$$

and

$$\{1, 2, \ldots, r_k\} \ni j \mapsto l(j) = \sum_{i=1}^{j-1} \left(|s^{-1}(\{i\})| - \lfloor |s^{-1}(\{i\})| / k \rfloor \right) \in \mathbb{N} \cup \{0\}.$$

It is easy to see the following properties.

(1) $u = v$ if and only if $p(u) = p(v)$ and $s(u) = s(v)$;
(2) $1 \leq |s^{-1}(\{1\})| \leq |s^{-1}(\{2\})| \leq \cdots \leq |s^{-1}(\{r_k\})| = k$;
(3) $|s^{-1}(\{i\})| = k$ if and only if $r_k \geq i \geq r_k - r_1 + 1$;
(4) $k + 1 - |s^{-1}(\{s(v)\})| \leq p(v) \leq k$.

Now we are ready to construct the required incidence coloring c. We define it in two steps. First, we define it on incidences uv such that $s(u) = s(v)$:

$$c(uv) = \begin{cases} \Delta + 2 & \text{if } p(v) = k, \\ k + 1 - p(v) + l(s(v)) & \text{if } p(u) > p(v) > k + 1 - |s^{-1}(\{s(v)\})|, \\ k - p(v) + l(s(v)) & \text{if } p(u) < p(v) < k, \\ \Delta + 1 & \text{if } p(v) = k + 1 - |s^{-1}(\{s(v)\})| \neq k. \end{cases}$$

Next, we extend it to other incidences by the formula:

$$c(uv) = \begin{cases} k + 1 - p(v) + l(s(v)) & \text{if } p(u) < p(v), \\ k - p(v) + l(s(v)) & \text{if } p(u) > p(v). \end{cases}$$

Since $p(u) \neq p(v)$ for all incidences uv, the above formula determines the value of $c(uv)$ for all incidences of G. To complete the proof, it suffices to show that $c(I(G)) \subseteq \{1, 2, \ldots, \Delta + 2\}$ and c is an incidence coloring of G.

It is easy to see that $c \geq 1$. On the other hand, $\Delta = n - r_1 = n - |\{i \colon |s^{-1}(\{i\})| = k\}| = \sum_{i=1}^{r_k} (|s^{-1}(\{i\})| - \lfloor |s^{-1}(\{i\})|/k \rfloor) = |s^{-1}(\{r_k\})| - 1 + l(r_k) \geq |s^{-1}(\{s(v)\})| - 1 + l(s(v)) \geq k - p(v) + l(s(v))$, so $c \leq \Delta + 2$. Moreover, $k - p(v) + l(s(v)) = \Delta$ if and only if $s(v) = r_k$ and $p(v) = 1$. As an easy consequence we get that $c^{-1}(\{\Delta + 1\})$ and $c^{-1}(\{\Delta + 2\})$ are independent sets.

Suppose that c is not an incidence coloring of G. Then $c(uv) = c(wx) \leq \Delta$ for some adjacent incidences uv, wx. Without loss of generality we assume $s(x) \geq s(v)$. There are several cases to consider.

- $s(x) \geq s(v) + 2$ and $|s^{-1}(\{s(v) + 1\})| = k$.

Then $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) \geq l(s(v) + 2) \geq |s^{-1}(\{s(v)\})| + |s^{-1}(\{s(v)+1\})| - 2 + l(s(v)) \geq |s^{-1}(\{s(v)\})| + l(s(v))$. By (4) we have $|s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) \geq c(uv)$. Since $c(wx) = c(uv)$, we get $p(x) = k$ and $c(wx) = k - p(x) + l(s(x))$, a contradiction.

- $s(x) \geq s(v) + 2$ and $|s^{-1}(\{s(v) + 1\})| < k$.

Then $|s^{-1}(\{s(v)\})| < k$ and $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) \geq l(s(v) + 2) = |s^{-1}(\{s(v)\})| + |s^{-1}(\{s(v) + 1\})| + l(s(v)) > |s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) \geq c(uv)$, a contradiction.

- $s(x) = s(v) + 1$ and $c(uv) = k - p(v) + l(s(v))$.

Then $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) \geq |s^{-1}(\{s(v)\})| - 1 + l(s(v)) \geq k - p(v) + l(s(v)) = c(uv)$. These inequalities must be equalities since $c(wx) = c(uv)$. This gives $p(x) = k$ and $c(wx) = k - p(x) + l(s(x))$, a contradiction.

• $s(x) = s(v) + 1$ and $c(uv) = k + 1 - p(v) + l(s(v))$ and $|s^{-1}(\{s(v)\})| = k$.

Then $p(v) \geq 2$ and $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) = |s^{-1}(\{s(v)\})| - 1 + l(s(v)) = k - 1 + l(s(v)) \geq k + 1 - p(v) + l(s(v)) = c(uv)$. These inequalities must be equalities since $c(wx) = c(uv)$. This gives $p(x) = k$ and $c(wx) = k - p(x) + l(s(x))$, a contradiction.

• $s(x) = s(v) + 1$ and $c(uv) = k + 1 - p(v) + l(s(v))$ and $|s^{-1}(\{s(v)\})| < k$.

Then $c(wx) \geq k - p(x) + l(s(x)) \geq l(s(x)) = l(s(v) + 1) = |s^{-1}(\{s(v)\})| + l(s(v)) \geq k + 1 - p(v) + l(s(v)) = c(uv)$. These inequalities must be equalities since $c(wx) = c(uv)$. This gives $p(x) = k$ and $c(wx) = k - p(x) + l(s(x))$, a contradiction.

• $s(x) = s(v)$.

$s(x) = s(v)$ implies $p(x) \neq p(v)$. Without loss of generality we assume that $p(x) > p(v)$. Then $c(wx) \leq k + 1 - p(x) + l(s(x)) \leq k - p(v) + l(s(v)) = c(uv)$, which gives $c(wx) = k + 1 - p(x) + l(s(x))$, $c(uv) = k - p(v) + l(s(v))$ and $p(x) = p(v) + 1$. There are 4 subcases to consider.

(a) $s(u) = s(v)$, $p(u) < p(v)$ and $s(w) = s(x)$, $p(w) > p(x)$. Then $p(w) > p(x) > p(v) > p(u)$, which shows that $u \neq x$, $u \neq w$ and $v \neq w$, a contradiction.

(b) $s(u) = s(v)$, $p(u) < p(v)$ and $s(w) \neq s(x)$, $p(w) < p(x)$. Then $p(u) < p(x)$, $s(u) \neq s(w)$ and $s(v) \neq s(w)$, which shows that $u \neq x$, $u \neq w$ and $v \neq w$, a contradiction.

(c) $s(u) \neq s(v)$, $p(u) > p(v)$ and $s(w) = s(x)$, $p(w) > p(x)$. Then $s(u) \neq s(x)$, $s(u) \neq s(w)$ and $p(w) > p(v)$, which shows that $u \neq x$, $u \neq w$ and $v \neq w$, a contradiction.

(d) $s(u) \neq s(v)$, $p(u) > p(v)$ and $s(w) \neq s(x)$, $p(w) < p(x)$. Then $s(u) \neq s(x)$ which shows that $u \neq x$. If $u = w$, then $p(v) < p(u) = p(w) < p(x) = p(v) + 1$, which is impossible. Then $v = w$ and $s(x) \neq s(w) = s(v) = s(x)$, a contradiction. ■

Corollary 4. *Let $G = K_{r_1,r_2,\ldots,r_k}$ be a complete k-partite graph with $k \geq 2$ and let $r_1 = \min\{r_1, \ldots, r_k\}$. Then*

$$\chi_i(G) = \begin{cases} \Delta(G) + 1 & \text{if } r_1 = 1, \\ \Delta(G) + 2 & \text{if } r_1 > 1. \end{cases}$$

3. $\mathcal{NP}$-Completeness of the Incidence 4-Coloring of Semicubic Bipartite Graphs

In this section we discuss the complexity results of the incidence 4-coloring problems and prove that the incidence 4-coloring problem for semicubic bipartite graphs is $\mathcal{NP}$-complete. By semicubic graphs we mean graphs with $\Delta = 3$ and vertices of degree equal to 1 or 3.

Theorem 5 [13]**.** *The incidence* 4*-coloring problem for semicubic graphs is* $\mathcal{NP}$*-complete.*

In fact, the authors observed in [13] that for semicubic graphs the problem of 2-distance coloring (i.e., a proper vertex coloring such that all vertices having a common neighbour are of distinct colors) is equivalent to the incidence 4-coloring problem. Indeed, for any incidence 4-coloring of a semicubic graph, the colors of incidences leading to a common vertex (say v) are equal (say a), hence we can assign the color a to the vertex v. Thus from the definition of adjacent incidences we get a proper 2-distance vertex coloring.

Proposition 6 [13]**.** *For semicubic graphs the incidence* 4*-coloring problem is equivalent to the* 2*-distance* 4*-coloring problem.*

By an $L(p, q)$*-labelling* [4] we mean an assignment of nonnegative integers to the vertices of a graph such that adjacent vertices are labelled using colors at least p apart, and vertices having a common neighbour are labelled using colors at least q apart. By [4] any 2-distance vertex coloring of a graph is the same as its $L(1, 1)$-labelling, thus we have the following result.

Proposition 7. *For semicubic graphs the incidence* 4*-coloring problem is equivalent to the* $L(1, 1)$*-labelling problem with* 4 *colors.*

In [12] the authors introduced the concept of generalized dominating sets as follows. For a given graph $G = (V, E)$ and two subsets σ and ρ of nonnegative integers, by a (σ, ρ)-set we mean any subset $S \subset V$ such that for any $v \in S$ we have $|N(v) \cap S| \in \sigma$ and for any $v \notin S$ there is $|N(v) \cap S| \in \rho$. By a (k, σ, ρ)-partition of V we mean a partition $V_1 \cup \cdots \cup V_k = V$ such that each V_i is the (σ, ρ)-set, for $i = 1, 2, \ldots, k$. In [4] the author observed that any $(k, \{0\}, \{0, 1\})$-partition is equivalent to an $L(1, 1)$-labelling with k colors, thus we get the following.

Proposition 8. *For semicubic graphs the incidence* 4*-coloring problem is equivalent to the* $(4, \{0\}, \{0, 1\})$*-partition problem.*

In [12] the authors proved that the $(4, \{0\}, \{0, 1\})$-partition problem is $\mathcal{NP}$-complete for cubic graphs, thus by Proposition 8 we have the following theorem.

Theorem 9. *The incidence* 4*-coloring problem of cubic graphs is* $\mathcal{NP}$*-complete.*

In the following, we use the $\overline{\mathsf{X3C}}$ problem, which is $\mathcal{NP}$-complete [9].

$\overline{\mathsf{X3C}}$	
Instance:	A subcubic bipartite graph $G = (V \cup M, E)$ without pendant vertices, such that $\|V\| = 3q$ and for every vertex $m \in M$ we have $\deg(m) = 3$ and m is adjacent to three vertices from V.
Question:	Is there a subset $M' \subset M$ of cardinality $\|M'\| = q$ dominating all vertices in V?

Theorem 10. *The incidence* 4*-coloring problem for semicubic bipartite graphs is* $\mathcal{NP}$*-complete.*

Proof. The proof proceeds by the reduction from the problem $\overline{\mathsf{X3C}}$. Let $G = (V \cup M, E)$ be a subcubic bipartite graph such that $|V| = 3q$ and for every vertex $m \in M$ we have $\deg(m) = 3$ and m is adjacent to exactly three vertices from V. We construct a semicubic bipartite graph G^* such that there is a subset $M' \subset M$ of cardinality $|M'| = q$ dominating all vertices in V if and only if there is a 2-distance 4-coloring of graph G^*, which by Proposition 6 is equivalent to the existence of an incidence 4-coloring of graph G^*.

Let n_2 and n_3 be the number of vertices in V of degree 2 and 3, respectively. Let us consider graphs H and H_i (for $i = 2, 3, \ldots$), shown in Figures 1, 2 and 3. Let H be a graph shown in Figure 1 (on the left-hand side) consisting of white vertices only (i.e., without vertices x and y) and edges between them.

Figure 1. An auxiliary graph H $(x, y \notin V(H))$.

Let H_2 be a graph shown in Figure 2 (on the left-hand side) consisting of two isomorphic and disjoint copies of graph H with attached two white vertices, i.e., vertex y and its pendant neighbour. We assume that two vertices x_1 and x_2 do not belong to H_2. Let H_2^* be a graph shown in Figure 4, i.e., the graph H_2 with attached two vertices x_1 and x_2.

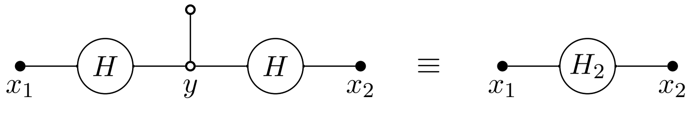

Figure 2. An auxiliary graph H_2 $(x_1, x_2 \notin V(H_2))$.

For each integer $i \geq 2$, let H_{i+1} be a graph shown in Figure 3 and constructed as follows: Take an isomorphic copy of graph H_i^*, i.e., the graph H_i with attached pendant vertices $x_1', \ldots, x_i'$ (shown on the left-hand side in Figure 3) and add two disjoint isomorphic copies of graph H_2 with attached two pendant vertices to each of them (in the manner as shown in Figure 2). Further, as shown in Figure 3, identify the vertex x_i' with two joined pendant vertices, and label by x_i and x_{i+1} the two others. Then, relabel x_k' with x_k for each $k \in \{1, \ldots, i-1\}$. We assume that $x_1, \ldots, x_{i+1} \notin V(H_{i+1})$.

Let H_{i+1}^* be a graph obtained from the graph H_{i+1} by attaching pendant vertices $x_1, \ldots, x_{i+1}$, as shown in Figure 3 (on the right-hand side). For each integer $i \geq 2$, the graph H_i^* is bipartite and the vertices $x_1, \ldots, x_i$ are in the same partition. Moreover, the graph H_i^* is semicubic and 2-distance 4-colorable.

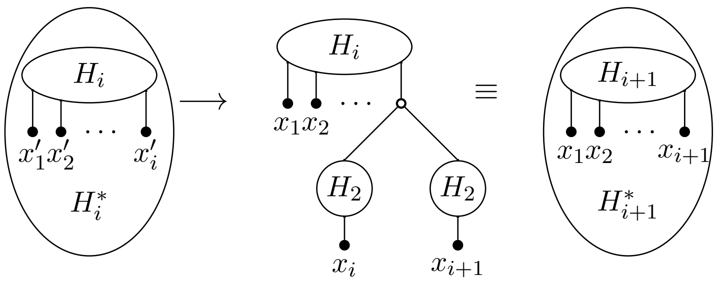

Figure 3. The iterative construction of auxiliary graphs H_{i+1} and H_{i+1}^* (for $i = 2, 3, \ldots$).

Observation 11. *For every graph H_i^* $(i \geq 2)$, in every 2-distance 4-coloring of graph H_i^* the colors assigned to vertices $x_1, \ldots, x_i$ are equal.*

Proof. Let $i = 2$ and let c be any 2-distance 4-coloring of graph H_2^*. The graph H_2^* contains as a subgraph two copies of graph H. By a simple analysis, we leave it to the reader, we can prove that $c(x_1) = c(y)$ and analogously $c(y) = c(x_2)$. By induction, the thesis follows for every $i \geq 2$. □

Observation 12. *For every graph H_i^* $(i \geq 2)$, if we precolor vertices $x_1, \ldots, x_i$ with one color, say 1, and the neighbors of $x_1, \ldots, x_i$ with arbitrary colors from the set $\{2, 3, 4\}$, then we can extend this precoloring to a 2-distance 4-coloring of the whole graph H_i^*.*

Proof. Let $i = 2$ and let v_1 and v_2 be vertices neighboring in the graph H_2^* with vertices x_1 and x_2, respectively. Let w_1 be a neighbor of the *interior* vertex y (see Figure 2) that is at distance 2 from v_1, and, analogously, let w_2 be a neighbor of y at distance 2 from w_2, which is shown in Figure 4.

Now, without loss of generality, let us assume that we precolor vertices x_1 and x_2 with color 1, and v_1 with color 2, and v_2 with color p, that may equal either 2 or 3. In both cases, we color the vertex w_1 with 3 and the vertex w_2 with color 4, what is extendible to the whole graph H_2^*, which we leave to the reader. By induction, we have the thesis for every integer $i \geq 2$. □

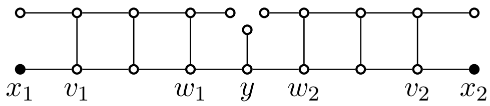

Figure 4. Graph H_2^*.

Let us consider graphs A_2 and A_3, shown in Figures 5 and 6. By a detailed (but simple) analysis of graphs A_2 and A_3 we have the following results.

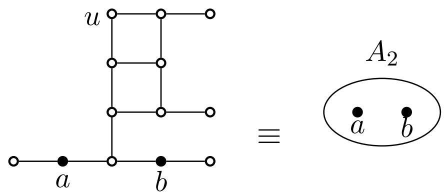

Figure 5. Graph A_2.

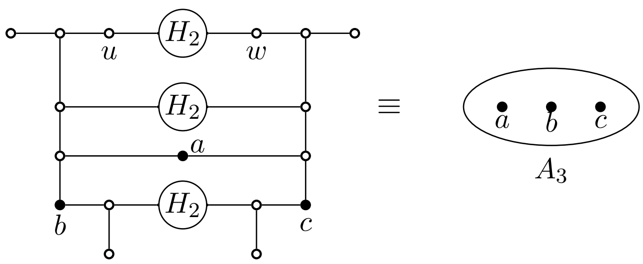

Figure 6. Graph A_3.

Observation 13. (i) *In every* 2*-distance* 4*-coloring of the graph* A_2 *the colors assigned to vertices* a *and* b *are different and one of them is equal to the color of vertex* u.

(ii) *Any precoloring of vertices* $\{a, b, u\}$ *of the graph* A_2*, where the colors assigned to vertices* a *and* b *are different, and either* a *or* b *has the same color as* u*, we can extend to a* 2*-distance* 4*-coloring of the graph* A_2.

Observation 14. (i) *In every* 2*-distance* 4*-coloring of the graph* A_3 *the colors assigned to vertices* a*,* b *and* c *are different and one of them is equal to the color of vertex* u.

(ii) *Any precoloring of vertices* $\{a, b, c, u\}$ *of the graph* A_3*, where the colors assigned to vertices* a, b, c *are different, and either* a*,* b *or* c *has the same color as* u*, we can extend to a* 2*-distance* 4*-coloring of graph* A_3.

We will transform (in polynomial time) the graph G into G^* in four steps:

1. each vertex $v \in V$ of degree 2 and neighbors $m_1, m_2 \in M$ replace with a graph $A_2(v)$ (isomorphic to A_2) and add two edges $\{m_1, a\}$ and $\{m_2, b\}$,
2. each vertex $v \in V$ of degree 3 and neighbors $m_1, m_2, m_3 \in M$ replace with a graph $A_3(v)$ (isomorphic to A_3) and add three edges $\{m_1, a\}, \{m_2, b\}$ and $\{m_3, c\}$; graphs of both types ($A_2(v)$ or $A_3(v)$) we call further A-graphs,
3. each vertex $m \in M$ replace with a graph $H_3^*(m)$ (isomorphic to H_3^*) and identify three neighbors of m (in an A-graph) with vertices x_1, x_2, $x_3 \in V(H_3^*(m))$,
4. attach a graph H_p^*, where $p = 2n_3 + n_2$ and uniquely identify the pendant vertices $x_1, \ldots, x_p \in V(H_p^*)$ with vertices u and w in all A-graphs.

It is easy to observe that the graph G^* is a semicubic bipartite graph. By Observation 11 and Observations 13(i) and 14(i) we have the following.

Observation 15. *In every* 2*-distance* 4*-coloring of the graph* G^* *the same color* (*say* 1) *is assigned to vertices* u *and* w *in all* A*-graphs, and in every* A*-graph there is exactly one vertex of* a, b, c *colored with* 1.

($\Rightarrow$) Suppose, $M' \subset M$ dominates all vertices in V and $|M'| = q$. We construct a 2-distance 4-coloring of graph the G^* as follows: (1) for every $m \in M'$ color with 1 vertices x_1, x_2, x_3 from the graph $H_3^*(m)$, (2) color with 1 vertices u and w in all A-graphs. Let us notice that after removing set of vertices M' from the graph G, each vertex from V in the result graph is of degree 1 or 2, thus (3) for every $m \in M \setminus M'$ we can color vertices x_1, x_2, x_3 from the graph $H_3^*(m)$ with either 2, 3 or 4 (by Brooks theorem). By Observation 12 and Observations 13(ii) and 14(ii) we can extend this precoloring to the 2-distance 4-coloring of the whole graph G^*.

($\Leftarrow$) Let c be any 2-distance 4-coloring of the graph G^*. By Observation 15 the colors assigned to vertices u and w in all A-graphs are equal (say 1). Moreover, by Observation 15 there is exactly one vertex from $\{a, b\}$ in every graph $A_2(v)$ and exactly one vertex from $\{a, b, c\}$ in every graph $A_3(v)$ colored with 1, thus the set of all vertices $m \in M$ such that the corresponding graph $H_3^*(m)$ has vertices x_1, x_2, x_3 colored with 1, is the solution to the $\overline{\mathsf{X3C}}$ problem. ■

By Proposition 7 we have the following.

Corollary 16. *The* $L(1, 1)$*-labelling problem with* 4 *colors for bipartite semicubic graphs is* $\mathcal{NP}$*-complete.*

The complexity of the incidence 4-coloring problem (and equivalently, the $L(1,1)$-labelling problem with 4 colors) for cubic bipartite graphs remains unknown.

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Received 8 March 2016
Revised 7 October 2016
Accepted 14 October 2016