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A CHARACTERIZATION FOR 2-SELF-CENTERED GRAPHS

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Abstract

A graph is called 2-self-centered if its diameter and radius both equal to 2. In this paper, we begin characterizing these graphs by characterizing edge-maximal 2-self-centered graphs via their complements. Then we split characterizing edge-minimal 2-self-centered graphs into two cases. First, we characterize edge-minimal 2-self-centered graphs without triangles by introducing *specialized bi-independent covering* (SBIC) and a structure named *generalized complete bipartite graph* (GCBG). Then, we complete characterization by characterizing edge-minimal 2-self-centered graphs with some triangles. Hence, the main characterization is done since a graph is 2-self-centered if and only if it is a spanning subgraph of some edge-maximal 2-self-centered graphs and, at the same time, it is a spanning supergraph of some edge-minimal 2-self-centered graphs.

Keywords: self-centered graphs, specialized bi-independent covering (SBIC), generalized complete bipartite graphs (GCB).

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1. Introduction

Let $G = (V, E)$ be a connected finite simple graph. For $u, v \in V$ the *distance* of u and v, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest path between u and v. The *eccentricity* of a vertex v, $\mathrm{ecc}(v)$, is $\max\{d(u, v) : u \in V\}$. The maximum and minimum eccentricity of vertices of G are called *diameter* and *radius* of G and are denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. *Center* of a graph G is the subgraph induced by vertices with eccentricity $\mathrm{rad}(G)$. A graph is called *self-centered* if it is equal to its center, or equivalently, its diameter equals its radius.

A graph G is called *k-self-centered* if $\mathrm{diam}(G) = \mathrm{rad}(G) = k$. The terminology *$k$-equi-eccentric graph* is also used by some authors. For studies on these graphs see [1–4, 7] and [6].

Clearly, a graph G is 1-self-centered if and only if G is a complete graph. In this paper, we try to characterize 2-self-centered graphs. An edge-maximal 2-self-centered graph can be easily characterized via a condition on its complement. We do this in Section 2, using a lemma which gives a necessary and sufficient condition for a graph to be 2-self-centered. For edge-minimal 2-self-centered graphs we need to divide the discussion into two cases: triangle-free or not. We do these in Section 3. Then, the characterization is done in sight of the following theorem.

Theorem 1. *A finite graph G is 2-self-centered if and only if there is an edge-minimal 2-self-centered graph G' and an edge-maximal 2-self-centered graph G'' such that G' is a spanning subgraph of G while G is itself a spanning subgraph of G''.*

Proof. The proof is clear. Note that $G' \subseteq G$ implies $\mathrm{rad}(G) \geqslant \mathrm{rad}(G') = 2$ and $G \subseteq G''$ implies $\mathrm{diam}(G) \leqslant \mathrm{diam}(G'') = 2$. ∎

Throughout this paper G is a connected finite simple graph and its complement is denoted by $\overline{G}$. If G is a graph and e is an edge in G, then $G \setminus e$ is the graph obtained from G by omitting e. Moreover, the graph obtained by adding an edge $e \notin E(G)$ to G is denoted by $G + e$. Whenever two vertices u and v are adjacent, we might write $u \sim v$. For concepts and notations of graph theory, the reader is referred to [5].

2. Edge-Maximal 2-Self-Centered Graphs

In this section, we present a characterization for edge-maximal 2-self-centered graphs. The following lemma is not only essential to do so, but it is also going to be used all over this paper.

Lemma 2. *Let $G = (V, E)$ be a graph with n vertices. Then G is 2-self-centered if and only if the following two conditions are true:*

(i) $2 \leqslant \deg(v) \leqslant n - 2$ *for all* $v \in V$*;*

(ii) *for each* $u, v \in V$ *with* $uv \notin E$ *there is a* $w \in V$ *such that* $uw, wv \in E$.

Proof. The proof is obvious. Note that if G has a vertex v with $\deg(v) = n - 1$ then $\mathrm{rad}(G) = 1$ and if there is a vertex u with $\deg(u) = 1$ then its neighbour should be adjacent to any vertex of G, since otherwise $\mathrm{ecc}(u) > 2$. ∎

Remark 3. If we show that for a graph G item (ii) of Lemma 2 holds and no vertex is adjacent to all vertices, then we can deduce that no vertex has degree 1 and therefore G is 2-self-centered.

A 2-self-centered graph G is said to be *edge-maximal* if there are no non-adjacent $u, v \in V(G)$ such that $G + uv$ is 2-self-centered. The following theorem is a characterization for edge-maximal 2-self-centered graphs.

Theorem 4. *Let G be a 2-self-centered graph. Then G is edge-maximal if and only if $\overline{G}$ is disconnected and each connected component of $\overline{G}$ is a star with at least two vertices.*

Proof. Let $H_1, \ldots, H_r$ be the connected components of $\overline{G}$, where r is a positive integer. At first, note that each H_i should be a tree with at least two vertices. To see this, if H_i has only one vertex v, then the degree of v in $\overline{G}$ is zero and thus its degree should be $n - 1$ in G which contradicts to (i) of Lemma 2. Furthermore, if the connected component H_i is not a tree, then there is an edge e with end vertices u_0 and v_0 in H_i which is not a cut edge. Let $H = G + e$. Since G is edge-maximal, H cannot be 2-self-centered. Using Lemma 2, we can deduce that the degree of u_0 or v_0 in G must be $n - 2$. This means that the degree of u_0 or v_0 in $\overline{G}$ is 1 and consequently e is a cut edge, a contradiction.

Now, we show that each connected component H_i is a star. Let u be a vertex with maximum degree k in H_i. If $k = 1$ then H_i is $K_{1,1}$. Let $k \geqslant 2$. If H_i is not $K_{1,k}$ then one of the neighbours of u, say v, has a neighbour $w \neq u$. Let e' be the edge between u and v in $\overline{G}$ and $H' = G + e'$. Since G is edge-maximal, H' cannot be 2-self-centered. Using Lemma 2, we can again deduce that the degree of u or v in G should be $n - 2$. This means that the degree of u or v in $\overline{G}$ is 1; which is a contradiction.

Conversely, suppose that $\overline{G}$ is a disconnected graph whose connected components are all stars, each of which has at least two vertices. Then, $2 \leqslant \deg(v) \leqslant n - 2$ for all $v \in V(G)$ and whenever u and v are two non-adjacent vertices of G, there must be a $w \in V(G)$ such that u and v are both adjacent to w. Therefore, by Lemma 2 G is a 2-self-centered graph. Moreover, since every connected component of $\overline{G}$ is a star with at least two vertices, adding an edge between two non-adjacent vertices in G makes the complement to have a singleton as a connected component, which means that the resulted graph is not 2-self-centered. ■

3. Edge-Minimal 2-Self-Centered Graphs

A 2-self-centered graph G is said to be *edge-minimal* if for each $e \in E(G)$, $G \setminus e$ is not a 2-self-centered graph. In this section, we determine all edge-minimal 2-self-centered graphs. To do so, let at first suppose that $\overline{G}$ is disconnected.

Proposition 5. *Let G be a graph. Then G is an edge-minimal 2-self-centered graph such that $\overline{G}$ is disconnected if and only if it is the complete bipartite graph $K_{k,\ell}$ for some $k, \ell \geqslant 2$.*

Proof. Let $H_1, \ldots, H_r$ be the connected components of $\overline{G}$, where $r \geqslant 2$. At first we prove that each H_i is a clique in $\overline{G}$, or in another word, each H_i is an independent set in G. Let e be an edge in G between two vertices u and v of H_i. If $H = G \setminus e$, then edge minimality of G implies that H cannot be 2-self-centered.

Let u' and v' be two non-adjacent vertices of H. Then u' and v' are belonged to a connected component H_j of $\overline{G}$. Let w' be any vertex of $H_{j'}$, where $j' \neq j$. Thus $u'w', w'v' \in E(H)$. This shows that H satisfies part (ii) of Lemma 2.

Since H is not 2-self-centered, Lemma 2 implies that the degree of u or v in H is 1. Let the degree of u in H be 1. Thus u has a neighbour w in H. This implies that all other vertices of G are in H_i. We know that v is also in H_i. Thus H_i contains all vertices except w and w is itself a component. Hence, the degree of w in G is $n - 1$ which contradicts to Lemma 2.

Now we show that $r = 2$. Let $r \geqslant 3$. Choose x, y and z in three different components. Let $e = xy$ and $H = G \setminus e$. Due to the existence of z, H is clearly 2-self-centered which contradicts the edge-minimality of G.

Conversely, the complete bipartite graph $K_{k,\ell}$ for $k, \ell \geqslant 2$ is an edge-minimal 2-self-centered graph such that its complement is disconnected. ■

For those 2-self-centered graphs that have connected complements, Proposition 5 is not useful. So, we may develop the characterization in some separate propositions for them, or, we can prove a more general statement which covers this case as a special case. In this paper, we do the later one, for which some preliminaries are needed.

Definition 6. Let G be a 2-self-centered graph. A vertex x in G is called *critical for u and v* if $uv \notin E$ and x is the only common neighbour of u and v.

Lemma 7. *Let G be an edge minimal 2-self-centered graph with no critical vertex for any pair of vertices. Then G is triangle-free. Furthermore, every triangle-free 2-self-centered graph is edge-minimal.*

Proof. Suppose in contrary that there are $u, v, w \in V(G)$ such that $uv, vw, wu \in E(G)$. If $\deg(u) = \deg(v) = \deg(w) = 2$, then G is itself a triangle which contradicts to $\mathrm{rad}(G) = 2$. If $\deg(u) = \deg(v) = 2$, then $\mathrm{diam}(G) = 2$ implies that all other vertices of G are neighbours of w. Thus $\deg(w) = n - 1$ which contradicts the fact that $\mathrm{rad}(G) = 2$. Hence, at most one of the vertices u, v and w has degree 2. Suppose that $\deg(u)$ and $\deg(v)$ are both greater than 2.

Let $e = uv$ and $H = G \setminus e$. Then edge-minimality of G implies that H is not a 2-self-centered graph. Since $\deg_G(u)$ and $\deg_G(v)$ are at least 3, this happens only if there are two vertices x and y such that $d_H(x, y) > 2$. Since $d_G(x, y) \leqslant 2$ it can be deduced that $\{x, y\} \cap \{u, v\} \neq \emptyset$. The cases $x = u$ and $y = v$ cannot happen at the same time because we have the path $x \sim w \sim y$ in H. If $x = u$ and y is a vertex other than v, then there is a path xt, ty in G for some vertex t, since v is not critical for u and y. For the case $y = v$ and any other vertex x the argument is similar. Therefore, H is a 2-self-centered graph, a contradiction.

Moreover, let G be triangle-free. If G is not edge-minimal, then there is an edge e with ends u and v such that $G \setminus e$ is still a 2-self-centered graph. Thus there is a path of length 2 between u and v in $G \setminus e$. This gives a triangle in G. ■

Nevertheless, there are examples of edge-minimal 2-self-centered graphs possessing some critical vertices *with* or *without* triangles.

Example 8. Let G be the graph with vertex set $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and edge set $E = \{01, 23, 12, 14, 15, 23, 36, 37, 46, 57, 67\}$. Then G is an edge-minimal 2-self-centered graph possessing the critical vertex 6 for the vertices 4 and 7, *with* a triangle on $3, 6, 7$, see Figure 1.

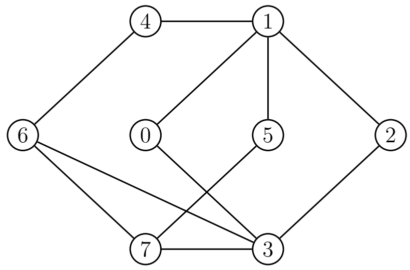

Figure 1. The graph G of Example 8.

Example 9. Let H be a graph constructed in the following way: consider the graph $K_{3,3}$ with two vertices y and z in different parts connected by the edge e. Omit e and add a vertex x with two edges xy and xz to obtain H. Then H is an edge minimal 2-self-centered graph possessing the critical vertex x for the vertices y and z, *without* any triangle.

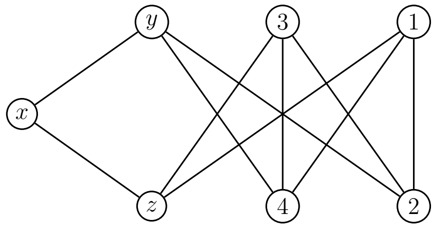

Figure 2. The graph H of Example 9.

Definition 10. A graph G is called to have a *specialized bi-independent covering via* $(\mathbb{A}_r, \mathbb{B}_s)$ if

(i) G is triangle-free,

(ii) there are two families $\mathbb{A}_r = \{A_1, \ldots, A_r\}$ and $\mathbb{B}_s = \{B_1, \ldots B_s\}$ of not necessarily distinct independent subsets of G such that we have $V(G) = \bigcup_{i=1}^{r} A_i = \bigcup_{j=1}^{s} B_j$,

(iii) for all $u, v \in V(G)$ if $d(u,v) \geq 3$, then there is an $1 \leq i \leq r$ such that $u, v \in A_i$ or there is $1 \leq j \leq s$ such that $u, v \in B_j$,

(iv) for all $u \in V(G)$ and $i \in \{1, \ldots, r\}$ if $d(u, A_i) \geq 2$, then there is a $j \in \{1, \ldots, s\}$ such that $A_i \cap B_j = \emptyset$ and $u \in B_j$, and

(v) for all $u \in V(G)$ and $j \in \{1, \ldots, s\}$ if $d(u, B_j) \geq 2$, then there is an $i \in \{1, \ldots, r\}$ such that $A_i \cap B_j = \emptyset$ and $u \in A_i$.

To make it easy, we shorten the name "specialized bi-independent covering" to *SBIC*. It is straightforward to check that every triangle-free graph G has two families of independent sets $\mathbb{A}_r, \mathbb{B}_s$ such that G has a SBIC via $(\mathbb{A}_r, \mathbb{B}_s)$. To see this, fix two independent coverings of G, and by adding enough independent sets to them, we can always satisfy items (iii) to (v) of Definition 10.

We need the following definition to complete our characterization of triangle-free 2-self-centered graphs.

Definition 11. A graph G is called an *X-generalized complete bipartite*, denoted by $\mathrm{GCB}_X(k, \ell, \mathbb{A}_r, \mathbb{B}_s)$, if X has an SBIC via $(\mathbb{A}_r, \mathbb{B}_s)$ and G is constructed in the following way:

(1) $V(G) = K \cup L \cup Y \cup Z \cup V(X)$ where $|K| = k$, $|L| = \ell$, $Y = \{y_1, \ldots, y_r\}$ and $Z = \{z_1, \ldots, z_s\}$.

(2) $a \sim t$ for all $a \in K$ and $t \in L \cup Y$.

(3) $b \sim t$ for all $b \in L$ and $t \in K \cup Z$.

(4) $y_i \sim t$ for all $t \in A_i$ and $1 \leq i \leq r$.

(5) $z_j \sim t$ for all $t \in B_j$ and $1 \leq j \leq s$.

(6) $y_i \sim z_j$ if and only if $A_i \cap B_j = \emptyset$.

Moreover, there are some special cases that must be treated separately:

(7) If $k = 0$ then every member of Y has a neighbour in Z and for all $i, j \in \{1, \ldots, r\}$ we have $A_i \cap A_j \neq \emptyset$ or there is a $p \in \{1, \ldots, s\}$ such that $A_i \cap B_p = A_j \cap B_p = \emptyset$.
(8) If $\ell = 0$ then every member of Z has a neighbour in Y and for all $i, j \in \{1, \ldots, r\}$ we have $A_i \cap A_j \neq \emptyset$ or there is a $p \in \{1, \ldots, s\}$ such that $A_i \cap B_p = A_j \cap B_p = \emptyset$.
(9) If $r = 0$ then $k \neq 0$ and if $s = 0$ then $\ell \neq 0$.
(10) $r = s = 0$ if and only if $X = \emptyset$ and $k, \ell \geq 2$.
(11) If $|X| = 1$ then at least one of k or ℓ is non-zero.

Proposition 12. *Any generalized complete bipartite graph is a triangle-free* 2-*self-centered graph.*

Proof. Let $G = \mathrm{GCB}_X(k, \ell, \mathbb{A}_r, \mathbb{B}_s)$ and $t = |X|$. Then $n := |V(G)| = k + \ell + r + s + t$. We show that G has no vertex of degree $n - 1$ and then we show that item (ii) of Lemma 2 holds for G. Then by Remark 3 we deduce that G is 2-self-centered. By proving that G has no triangle and using Lemma 7, we actually show that G is also edge-minimal.

For $a \in K$, $\deg(a) = \ell + r = n - s - k - t$. Thus if $r = s = 0$ then by item (10) of Definition 11 we have $k \geq 2$ and hence $\deg(a) \leqslant n - 2$. If r or s is non-zero, then by item (10) we have $t \neq 0$ and therefore we have $\deg(a) \leqslant n - 2$ (by items (2) and (9) of Definition 11, and because no element of K is adjacent to a vertex of X).

For $b \in L$, by a similar proof to the case $a \in K$ we can deduce that $\deg(b) \leqslant n - 2$.

For $y_i \in Y$, if $\ell \neq 0$ then $\deg(y_i) \leqslant n - 2$ because no element of L is adjacent to y_i. If $\ell = 0$ then either y_i is not adjacent to all vertices of X or if y_i is adjacent to all vertices of x then it is not adjacent to z_j for some $j \in \{1, \ldots, s\}$ (which its existence is supported by item (8) of Definition 11), each of which cases yields to $\deg(y_i) \leqslant n - 2$.

For $z_j \in Z$ we have the same argument to $y_i \in Y$.

Finally, for each $x \in X$, item (11) of Definition 11 guarantees that $\deg(x) \leqslant n - 2$ whenever X has only one vertex. So, assume that $t \geqslant 2$. Therefore, there are two possibilities: either there is $\hat{x} \in X$ such that x is not adjacent to $\hat{x}$, or, x is adjacent to all other vertices of X. If the former case is true then $\deg(x) \leqslant n - 2$. For the later case, since x is not in any independent set with other vertices of X, we have there is some $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$ such that $\{x\} \cap A_i = \{x\} \cap B_j = \emptyset$. Thus, by items (4) and (5) of Definition 11, we have x is not adjacent to y_i and z_j and hence $\deg(x) \leqslant n - 2$.

To show that (ii) of Lemma 2 is also satisfied, we should choose two vertices u and v in G and show that whenever they are not adjacent, they have at least one common neighbour. There are 15 different ways for choosing u and v from $G = K \cup L \cup Y \cup Z \cup X$.

If $(u, v) \in (K \times L) \cup (K \times Y) \cup (L \times Z)$ then u and v are adjacent to each other.

If $(u, v) \in (K \times K) \cup (L \times L)$ then there is a path of length 2 between u and v via one of the sets $L \cup Y$ or $K \cup Z$.

If $(u, v) \in (Y \times Y)$ then if $k \neq 0$ there is a path of length 2 between u and v via any member of K. If $k = 0$ then, by item (7) of Definition 11, we have either there is a $z_p \in Z$ which is a common neighbour of u and v, or, u and v are both adjacent to a vertex $x \in X$. The case $(u, v) \in (Z \times Z)$ is also similar.

If $(u, v) \in (L \times Y)$ then if $k \neq 0$ there is a path of length 2 between u and v via any member of K. If $k = 0$ then, by item (7) of Definition 11, we have every member of Y has a neighbour in Z, so Z is non-empty and v has a neighbour in Z, namely $\hat{z}$. Since u is also adjacent to $\hat{z}$ by item (3) of Definition 11, there is a path of length 2 between u and v. The case $(u, v) \in (K \times Z)$ is also similar.

If $(u, v) \in Y \times Z$ then $u = y_i$ and $v = z_j$ for some i and j. If $A_i \cap B_j \neq \emptyset$ then we can choose a c in $A_i \cap B_j$ such that there is a path of length 2 between u and v via c. If $A_i \cap B_j = \emptyset$ then u is adjacent to v, by item (6) of Definition 11.

If $(u, v) \in X \times X$ then either $d_X(u, v) \leqslant 2$ or by item (iii) of Definition 10 there is an $i \in \{1, \ldots, r\}$ or a $j \in \{1, \ldots, s\}$ such that both u and v are adjacent to y_i or z_j.

If $(u, v) \in (K \times X) \cup (L \times X)$ then there is an $i \in \{1, \ldots, r\}$ or a $j \in \{1, \ldots, s\}$ such that v is adjacent to y_i and z_j. Then, since u is adjacent to y_i or z_j, we have $d_G(u, v) = 2$.

If $(u, v) \in (Y \times X)$ then then there is an $i \in \{1, \ldots, r\}$ such that $u = y_i$. Then, either $d(v, A_i) \leqslant 1$ which means that $d(u, v) \leqslant 2$, or, if $d(v, A_i) \geqslant 2$ then by item (iv) of Definition 10 there is a $j \in \{1, \ldots, s\}$ such that $A_i \cap B_j = \emptyset$ and $v \in B_j$. Hence by items (5) and (6) of Definition 11 we have z_j is adjacent to both u and v. The case $(u, v) \in (Z \times X)$ is also similar.

So, for each of 15 ways of choosing u and v from vertices of G we have $d(u, v) \leqslant 2$.

We finally show that G is triangle-free. On contrary, suppose that u, v, w are vertices of a triangle in G. The case that none of u, v and w is a vertex of X cannot happen because $K \cup Z$ and $L \cup Y$ are independent sets. Since X is triangle-free, u, v and w are not all together vertices of X. Meanwhile, if only two vertices of $\{u, v, w\}$ are in X, then the third is not adjacent to the other two because they cannot be in the same independent set in X. So, at most one of $\{u, v, w\}$ is a vertex of X. Let for instance w be a vertex of X. Then u and v are not members of Y or Z at the same time, because otherwise they are not adjacent together. The case that one of u and v is in Y and the other in Z is also impossible because it is contrary to item (6) of Definition 11. ∎

Theorem 13. *A graph G is a triangle-free 2-self-centered graph if and only if there are positive integers k, ℓ, r, s and a graph X which has a SBIC via $(\mathbb{A}_r, \mathbb{B}_s)$ such that $G = \mathrm{GCB}_X(k, \ell, \mathbb{A}_r, \mathbb{B}_s)$.*

Proof. Let Y' be a maximal independent subset of G, let Z' be a maximal independent subset of $G \setminus Y'$ and let $X = G \setminus (Y' \cup Z')$. Suppose that K (respectively L) is the set of all vertices in Z' (respectively Y') which are not adjacent to any member of X and put $Y = Y' \setminus L, Z = Z' \setminus K$.

Let $a \in K$ and $y' \in Y'$. We claim that $ay' \in E$. Suppose on the contrary that $ay' \notin E$. Since $\mathrm{diam}(G) = 2$ there is a u in G such that $au, uy' \in E$. The vertex u cannot be in Y' or Z' since Y' and Z' are independent sets. Hence $u \in X$. This contradicts to the definition of K.

A similar argument shows that each member of L is adjacent to each member of Z'.

Let $k = |K|, \ell = |L|, r = |Y|, s = |Z|, Y = \{y_1, \ldots, y_r\}$ and $Z = \{z_1, \ldots, z_s\}$. Now put $A_i = N_X(y_i)$ and $B_j = N_X(z_j)$. We show that A_i's and B_j's are independent subsets of X and X has a SBIC via $(\mathbb{A}_r, \mathbb{B}_s)$.

Let x be an arbitrary member of X. Since Y' and Z' are maximal independent, there should be neighbours for x in Y' and Z'. We know that these neighbours are in Y and Z. Let y_i and z_j be adjacent to x. Thus $x \in A_i$ and $x \in B_j$. This shows that $X = \bigcup_{i=1}^{r} A_i = \bigcup_{j=1}^{s} B_j$.

Each A_i and each B_j is independent, since G is triangle-free. Moreover, if y_i and z_j are not adjacent to each other, then since $\mathrm{diam}(G) = 2$, there should be an $x \in X$ with $y_i x, x z_j \in E(X)$. Thus $x \in A_i \cap B_j$. If y_i is adjacent to z_j then there must not be such an x, so we have $y_i \sim z_j$ if and only if $A_i \cap B_j = \emptyset$.

Furthermore, X is triangle-free since X is a subgraph of the triangle-free graph G.

Items (iii), (iv) and (v) of Definition 10 must hold because G is a triangle-free 2-self-centered graph. Therefore, X has an SBIC via $(\mathbb{A}_r, \mathbb{B}_s)$.

Items (1) to (6) of Definition 11 have already hold. Moreover, items (7) to (11) of Definition 11 must also hold because G is a triangle-free 2-self-centered graph. Hence $G = \mathrm{GCB}_X(k, \ell, \mathbb{A}_r, \mathbb{B}_s)$.

Since the converse is evident by Proposition 12, we are done with the proof. ∎

The reader should note that every complete bipartite graph $K_{k,\ell}$ with $k, \ell \geqslant 2$ is a generalized complete bipartite graph $\mathrm{GCB}_{\emptyset}(k, \ell, \emptyset, \emptyset)$.

Now, we can consider edge-minimal 2-self-centerd graphs with some triangles. We need the following procedure to proceed.

Procedure 14. *Let G be a graph, u, v, w form a triangle in G and suppose that v is a critical vertex for u and $v_1, \ldots, v_p$ and/or u is a critical vertex for v and $u_1, \ldots, u_q$. Remove the edge uv and add edges $uv_1, \ldots, uv_p$ and $vu_1, \ldots, vu_q$.*

The following theorem characterizes edge-minimal 2-self-centered graphs with triangles, which completes the characterization of all 2-self-centered graphs.

Theorem 15. *Let G be a graph. Then G is an edge-minimal 2-self-centered graph with some triangle if and only if the following two conditions are true:*

(i) *for each edge of every triangle in G, at least one end-vertex is a critical vertex (for the other end-vertex of that edge and some other vertices of G), and*

(ii) *iteration of Procedure 14 on G (at most to the number of triangles of G) transforms G to a triangle-free 2-self-centered graph.*

Proof. Assume that u, v, w form a triangle in G. Since G is edge-minimal, if we omit the edge uv, then the resulting graph is not 2-self-centered. This shows that u or v is a critical vertex. Let u be a critical vertex. Thus there are vertices $u_1, \ldots, u_q$ such that u is the common neighbour of v and each of the u_i's. Moreover, if v is also a critical vertex for u and some other vertices, then we suppose that $v_1, \ldots, v_p$ are the vertices such that v is a common neighbour of u and each of the v_j's.

If we omit uv and add edges $u_1v, \ldots, u_qv, uv_1, \ldots, uv_p$, then the resulting graph G' is clearly 2-self-centered and the number of triangles of G' is less than the number of triangles of G. To see this, note that edges of a triangle on u, v and w are omitted and no new triangle is added. In contrary, suppose that we have a new triangle. Then it should be of the form u_i, v, t (or v_j, u, s) which contradicts to the fact that u (or v) is a critical vertex for u_i and v (for v_j and u).

If G has still some triangle then we can proceed this process. Therefore, we finally transform G into a triangle-free 2-self-centered graph.

Conversely, if the two conditions are true for a graph G with some triangles, then G is an edge-minimal 2-self-centered graph because condition (ii) guarantees that G is 2-self-centered while condition (i) obligates G to be edge-minimal. ■

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