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POWER DOMINATION IN KNÖDEL GRAPHS AND HANOI GRAPHS

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Abstract

In this paper, we study the power domination problem in Knödel graphs $W_{\Delta,2\nu}$ and Hanoi graphs H_p^n. We determine the power domination number of $W_{3,2\nu}$ and provide an upper bound for the power domination number of $W_{r+1,2^{r+1}}$ for $r \geq 3$. We also compute the k-power domination number and the k-propagation radius of H_p^2.

Keywords: domination, power domination, Knödel graph, Hanoi graph.

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1. Introduction

Electric power networks must be monitored continuously and this can be accomplished efficiently by placing *phasor measurement units* (PMUs) at selected network locations. The power domination problem, as introduced in [2], is to find the minimum number of PMUs needed to monitor a given electric power system. This problem has been formulated as a graph theoretic problem by Haynes *et al.* [10]. The additional propagational behaviour in power domination is due to the use of Kirchhoff's laws in an electrical network [1].

Let G be a graph and $S \subseteq V(G)$. The *open neighbourhood* of a vertex v of G, denoted by $N_G(v)$, is the set of vertices adjacent to v. The *closed neighbourhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For a subset S of vertices, the *open* (respectively *closed*) *neighbourhood* $N_G(S)$ (respectively $N_G[S]$) of S is the union of the open (respectively closed) neighbourhoods of its elements. A vertex v in a graph G is said to *dominate* its closed neighbourhood $N_G[v]$. A subset $S \subseteq V(G)$ of vertices is a *dominating set* if $N_G[S] = V(G)$. The minimum cardinality among dominating sets of G is called its *domination number*, denoted by $\gamma(G)$. The propagational behaviour of the set of monitored vertices distinguishes power domination from the standard domination in the following way.

The *set monitored by* S, denoted by $M(S)$, is defined algorithmically as follows:

- (domination) $M(S) \leftarrow S \cup N(S)$,
- (propagation) as long as there exists $v \in M(S)$ such that $N(v) \cap (V(G) \setminus M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

Equivalently, the set $M(S)$ of vertices monitored by the set S is obtained from S as follows. The set of vertices *monitored* by a set S, denoted by $M(S)$, initially consists of all vertices in $N_G[S]$. This step is called the domination step. Then this set is iteratively extended by including all vertices $w \in V(G)$ that have a neighbour v in $M(S)$ such that all the other neighbours of v, except w, are already in $M(S)$. This second part is called the *propagation step*. This step is continued until no such vertex w exists, at which stage the set monitored by S has been constructed. The set S is called a *power dominating set* (PDS) of G if $M(S) = V(G)$. The *power domination number* of a graph G, denoted by $\gamma_{\mathrm{P}}(G)$, is the minimum cardinality among power dominating sets of G.

Later, the definition of $M(S)$ was formally described with the following sets definition, where $\mathcal{P}^i_{G,1}$ is the set of vertices monitored after i propagation rounds. This definition was first introduced by Aazami in [1] and then Chang *et al.* generalized this definition in [4] to introduce k-power domination for a nonnegative integer k. The corresponding definition for the monitored set, $M(S)$, is obtained by replacing k by 1 in the following.

Definition 1.1 (Monitored vertices)**.** Let $k \in \mathbb{N}_0 \colon = \{0, 1, \ldots\}$. If G is a graph and $S \subseteq V(G)$, then the sets $\left(\mathcal{P}^i_{G,k}(S)\right)_{i \in \mathbb{N}_0}$ of vertices *monitored* by S at step i are as follows:

$\mathcal{P}^0_{G,k}(S) = N_G[S]$, (domination step) and

$\mathcal{P}^{i+1}_{G,k}(S) = \bigcup \left\{ N_G[v] \colon v \in \mathcal{P}^i_{G,k}(S), \left| N_G[v] \setminus \mathcal{P}^i_{G,k}(S) \right| \leq k \right\}$ (propagation steps).

We remark that for $i \geq 0$ we have $\mathcal{P}^i_{G,k}(S) \subseteq \mathcal{P}^{i+1}_{G,k}(S)$. Furthermore, every time a vertex of the set $\mathcal{P}^i_{G,k}(S)$ has at most k neighbours outside the set, we add its neighbours to the next generation $\mathcal{P}^{i+1}_{G,k}(S)$. If $\mathcal{P}^{i_0+1}_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$ for some i_0, then $\mathcal{P}^j_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$ for every $j \geq i_0$. We thus define $\mathcal{P}^\infty_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$. When the graph G is clear from the context, we will simplify the notations to $\mathcal{P}^i_k(S)$ and $\mathcal{P}^\infty_k(S)$.

Definition 1.2 [4]**.** A *k-power dominating set* of G (k-PDS) is a set $S \subseteq V(G)$ such that $\mathcal{P}^\infty_{G,k}(S) = V(G)$. The *k-power domination number* of G, denoted by $\gamma_{\mathrm{P},k}(G)$, is the minimum cardinality among k-power dominating sets of G.

Clearly, $\gamma_{\mathrm{P},0}(G) = \gamma(G)$ and $\gamma_{\mathrm{P},1}(G) = \gamma_{\mathrm{P}}(G)$. Upper bounds for the power domination number are studied in [10, 19]. The power domination problem for various products of graphs is studied in [7, 8, 18] and exact values are determined for some product graphs. The generalized power domination is further studied in [5, 6]. In [6], the authors introduced the *k-propagation radius* of a graph G, motivated from the studies in [1], as a way to measure the efficiency of a minimum k-PDS. It gives the minimum number of propagation steps needed to monitor the entire graph G over all minimum k-PDS. The k-power domination number and propagation radius of Sierpiński graphs (cf. [14]) are determined in [6].

Definition 1.3 [6]**.** The radius of a k-PDS is defined by

$$\mathrm{rad}_{\mathrm{P},k}(G, S) = 1 + \min \left\{ i : \mathcal{P}^i_{G,k}(S) = V(G) \right\}.$$

The *k-propagation radius* of the graph can be expressed as

$$\mathrm{rad}_{\mathrm{P},k}(G) = \min \left\{ \mathrm{rad}_{\mathrm{P},k}(G, S) : \ S \text{ is a } k\text{-PDS of } G, \ |S| = \gamma_{\mathrm{P},k}(G) \right\}.$$

Knödel graphs $W_{\Delta,2\nu}$ $(0 \leq \Delta - 1 \leq \lfloor \log_2(\nu) \rfloor)$ have been introduced by Knödel in [15] as the network topology underlying an optimal-time algorithm for gossiping among n nodes. They have been widely studied as interconnection networks mainly because of their favourable properties in terms of broadcasting and gossiping [3]. $W_{r,2^r}$ is one of the three nonisomorphic infinite graph families known to be minimum broadcast and gossip graphs. The other two families are the hypercube of dimension r, H_r [16] and the recursive circulant graph

$G(2^r, 4)$ [17]. Vertex transitivity as a Cayley graph [11], high vertex and edge connectivity, dimensionality and embedding properties [9] make the Knödel graph a suitable candidate for a network topology and an appropriate architecture for parallel computing. For a survey about the Knödel graphs, see [9].

We will use the notations $\mathbb{N}_t := \{t, t+1, \ldots\} \subseteq \mathbb{N}_0$, $[t] := \{1, \ldots, t\} \subseteq \mathbb{N}_1$, and $[t]_0 := \{0, \ldots, t-1\} \subseteq \mathbb{N}_0$, $t \in \mathbb{N}_0$, in the sequel; note that $|[t]_0| = t = |[t]|$.

Definition 1.4 [9]**.** The *Knödel graph* on 2ν vertices, where $\nu \in \mathbb{N}_1$, and of maximum degree $\Delta \in [1 + \lfloor \log_2(\nu) \rfloor]$ is denoted by $W_{\Delta,2\nu}$. The vertices of $W_{\Delta,2\nu}$ are the pairs (i, j) with $i \in [2]$ and $j \in [\nu]_0$. For every such j, there is an edge between vertex $(1, j)$ and any vertex $(2, j + 2^\ell - 1 \bmod \nu)$ with $\ell \in [\Delta]_0$.

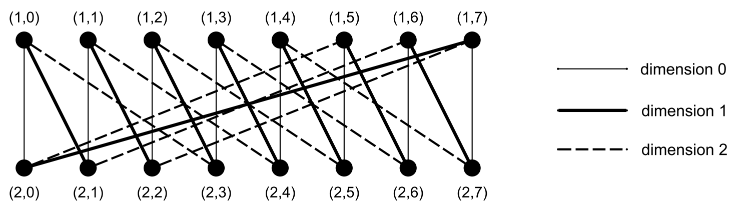

Figure 1. The graph $W_{3,16}$.

An edge of $W_{\Delta,2\nu}$ which connects a vertex $(1, j)$ with the vertex $(2, j + 2^\ell - 1 \bmod \nu)$ is called an edge *in dimension* ℓ; cf. Figure 1.

The Tower of Hanoi (TH) problem, invented by the French number theorist É. Lucas in 1883, has presented a challenge in mathematics as well as in computer science and psychology. The classical problem consists of three pegs and is thoroughly studied in [12]. On the other hand, as soon as there are at least four pegs, the problem turned into a notorious open question. The general TH problem has $p \in \mathbb{N}_3$ pegs and $n \in \mathbb{N}_0$ discs of mutually different size. A *legal move* is a transfer of the topmost disc from one peg to another peg, no disc being placed onto a smaller one. Initially, all discs lie on one peg in small-on-large ordering, that is, in a *perfect state*. The objective is to transfer all the discs from one perfect state to another in the minimum number of legal moves. A state (= distribution of discs on pegs) is called *regular* if on every peg the discs lie in the small-on-large ordering. The Hanoi graphs H_p^n form a natural mathematical model for the TH problem. Each graph is constructed with all regular states as vertices, and two states are adjacent whenever one is obtained from the other by a legal move. For any $n \in \mathbb{N}_0$, H_1^n is the graph K_1. For two pegs, only the smallest disc can be moved in any regular state. Hence, for $n \in \mathbb{N}_1$, H_2^n is the disjoint union of 2^{n-1} copies of K_2, i.e., $H_2^n \cong W_{1,2^n}$. Many properties of Hanoi graphs have been studied in [13] and literature therein.

Definition 1.5 [13]**.** The *Hanoi graphs* H_p^n for *base* $p \in \mathbb{N}_3$ and *exponent* $n \in \mathbb{N}_0$ are defined as follows.

$$V(H_p^n) = \{s_n \cdots s_1 \colon s_d \in [p]_0 \text{ for } d \in [n]\},$$
$$E(H_p^n) = \big\{\{\underline{s}i\overline{s}, \underline{s}j\overline{s}\} \colon i, j \in [p]_0,\ i \neq j,\ \underline{s} \in [p]_0^{n-d},\ \overline{s} \in ([p]_0 \backslash \{i,j\})^{d-1},\ d \in [n]\big\}.$$

The edge sets of Hanoi graphs can also be expressed in a recursive definition (cf. Figure 2).

$$E(H_p^0) = \emptyset,$$
$$\forall\, n \in \mathbb{N}_0 \colon E(H_p^{1+n}) = \big\{\{ir, is\} \colon i \in [p]_0,\ \{r,s\} \in E(H_p^n)\big\}$$
$$\cup \big\{\{ir, jr\} \colon i, j \in [p]_0,\ i \neq j,\ r \in ([p]_0 \backslash \{i,j\})^n\big\}.$$

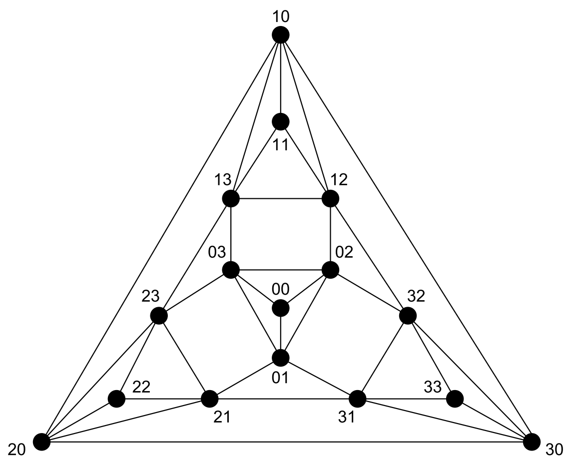

Figure 2. The graph H_4^2.

The vertices of the form $i^n := \overbrace{i \ldots i}^{n \text{ times}}$ are called *extreme vertices* of H_p^n.

In this paper, we study the power domination problem in Knödel graphs and Hanoi graphs. We determine the power domination number of $W_{3,2\nu}$ and provide an upper bound for the power domination number of $W_{r+1,2^{r+1}}$ for $r \in \mathbb{N}_3$. We also compute the k-power domination number and k-propagation radius of H_p^2.

2. Power Domination in Knödel Graphs

In this section, we study the power domination number of Knödel graphs. For $\Delta = 1, W_{1,2\nu}$ consists of ν disjoint copies of K_2 and therefore $\gamma_{\mathrm{P}}(W_{1,2\nu}) = \nu$. For

$\nu \in \mathbb{N}_2$ and $\Delta = 2, W_{2,2\nu}$ is a cycle on 2ν vertices and clearly $\gamma_{\mathrm{P}}(W_{2,2\nu}) = 1$. We have the following theorem for the case $\Delta = 3$, if $\nu \in \mathbb{N}_4$.

Theorem 2.1. *For $\nu \in \mathbb{N}_4$, $\gamma_{\mathrm{P}}(W_{3,2\nu}) = 2$.*

Proof. We prove that the set $S = \{(1,0),(2,2)\}$ is a PDS of $W_{3,2\nu}$. Clearly, $\mathcal{P}_1^0(S) = \{(i,j)\colon i \in [2], j \in [3]_0\} \cup \{(1,\nu-1),(2,3)\}$. For $\nu = 4$, S is a dominating set of $W_{3,8}$ and for $\nu = 5,6$, we can easily observe that all vertices of $W_{3,2\nu}$ get monitored after stage 1 and therefore S is a PDS. Let $\nu \in \mathbb{N}_7$. Depending on whether ν is odd or even, we write $\nu = 2m-1$ or $\nu = 2m$, $m \in \mathbb{N}_4$, respectively. Then for $i \in [m-3]$,

$$\begin{aligned}\mathcal{P}_1^i(S) = &\big(\{(1,j)\colon j \in [i+3]_0\} \cup \{(1,\nu-j)\colon j \in [i+2]\}\big)\\ &\cup \big(\{(2,j)\colon j \in [i+5]_0\} \cup \{(2,\nu-j)\colon j \in [i]\}\big).\end{aligned}$$

We get that $\mathcal{P}_1^{m-3}(S) = V(W_{3,2\nu})$, if ν is odd, and $\mathcal{P}_1^{m-2}(S) = \mathcal{P}_1^{m-3}(S) \cup \{(1,m), (2,m+2)\} = V(W_{3,2\nu})$, if ν is even. Hence, in both cases we see that every vertex of $W_{3,2\nu}$ gets monitored after stage $\lfloor \frac{\nu}{2} \rfloor - 2$ and therefore S is a PDS of $W_{3,2\nu}$.

To prove that $\gamma_{\mathrm{P}}(W_{3,2\nu}) \geq 2$, let us assume that $\{v\}$ is a PDS of $W_{3,2\nu}$. Then, since $W_{3,2\nu}$ is bipartite, after the domination step, each of the neighbours of v has exactly two unmonitored neighbours which prevents the further propagation. Hence $\gamma_{\mathrm{P}}(W_{3,2\nu}) = 2$. ∎

We now focus on the family of Knödel graphs $W_{r+1,2^{r+1}}$. In the next theorem, we prove that the power domination number of $W_{r+1,2^{r+1}}$ is at most 2^{r-2}. For that, we construct a PDS of cardinality 2^{r-2} in $W_{r+1,2^{r+1}}$. One can easily check that $S' = \{(1,1),(2,6)\}$ is a PDS of $W_{4,16}$. It is proved in [9] that $W_{r+1,2^{r+1}}$ can be constructed by taking two copies of $W_{r,2^r}$ and linking the vertices of each copy by a certain perfect matching. Therefore, in order to construct a PDS for $W_{5,32}$, we take two copies of the set S', each from a copy of $W_{4,16}$ that lies in $W_{5,32}$ and then prove that the new set is a PDS of $W_{5,32}$. We now extend the same idea to construct a PDS of $W_{r+1,2^{r+1}}$ for larger values of r. In the proof of the following theorem, we first produce a set S and then give the set of vertices that are dominated by $\mathcal{P}_1^0(S)$. After that we give the elements in $\mathcal{P}_1^1(S)$ and $\mathcal{P}_1^2(S)$, the sets of vertices that get monitored at the first and second propagation step, respectively. We obtain that the entire graph gets monitored in two propagation steps and thus S is a PDS of $W_{r+1,2^{r+1}}$.

Theorem 2.2. *For $r \in \mathbb{N}_3, \gamma_{\mathrm{P}}(W_{r+1,2^{r+1}}) \leq 2^{r-2}$.*

Proof. Let $\nu = 2^r$ and $S = \{(1, 2^{r-3}+j), (2, 7\cdot 2^{r-3}-1+j)\colon j \in [2^{r-3}]_0\}$. Then

$$\begin{aligned}\mathcal{P}_1^0(S) = S \cup \{&(1, 7\cdot 2^{r-3}+j-2^{\ell} \bmod \nu), (2, 2^{r-3}+j+2^{\ell}-1 \bmod \nu)\ :\\ &j \in [2^{r-3}]_0,\ \ell = r-3, r-2, r-1, r\}.\end{aligned}$$

For $r = 3$, the vertex $(1, 2j + 1)$ monitors $(2, 2j + 1)$ for every $j \in [3]$ and the vertex $(2, 2j)$ monitors $(1, 2j)$ for every $j \in [3]_0$. Thus we get $\mathcal{P}_1^1(S) = V(W_{4,16})$. Assume now that $r \in \mathbb{N}_4$. Then, for each j and ℓ, where $j \in [2^{r-4}]_0$, $\ell = r-2, r-1, r$, the vertices in the set $\{(1, 7 \cdot 2^{r-3} + j - 2^\ell \bmod \nu)\}$ monitor the vertices in the set $\{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \bmod \nu)\}$ by propagation. Also, for each j and ℓ, where $2^{r-4} \le j \le 2^{r-3} - 1$, $\ell = r-2, r-1, r$, the vertices in the set $\{(2, 2^{r-3} + j + 2^\ell - 1 \bmod \nu)\}$ monitor the vertices in the set $\{(1, j + 2^\ell \bmod \nu)\}$ by propagation. Hence the set of vertices monitored at stage 1 is given by

$$\begin{aligned}\mathcal{P}_1^1(S) = {} & \{(1, j + 2^\ell \bmod \nu) \colon 2^{r-4} \le j \le 2^{r-3} - 1,\ \ell = r-2, r-1, r\} \\ & \cup \{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \bmod \nu) \colon j \in [2^{r-4}]_0,\ \ell = r-2, r-1, r\} \cup \mathcal{P}_1^0(S).\end{aligned}$$

Again following the propagation rule, for each j and ℓ, where $2^{r-4} \le j \le 2^{r-3} - 1$, $\ell = r-2, r-1, r$, the vertices in the set $\{(1, 7 \cdot 2^{r-3} + j - 2^\ell \bmod \nu)\}$ monitor the vertices in the set $\{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \bmod \nu)\}$ and for each j and ℓ, where $j \in [2^{r-4}]_0$, $\ell = r-2, r-1, r$, the vertices in the set $\{(2, 2^{r-3} + j + 2^\ell - 1 \bmod \nu)\}$ monitor the vertices in the set $\{(1, j + 2^\ell \bmod \nu)\}$. Hence the set of vertices monitored at stage 2 is given by

$$\begin{aligned}\mathcal{P}_1^2(S) = {} & \{(1, j + 2^\ell \bmod \nu) \colon j \in [2^{r-4}]_0,\ \ell = r-2, r-1, r\} \\ & \cup \{(2, 8 \cdot 2^{r-3} + j - 2^\ell - 1 \bmod \nu) \colon 2^{r-4} \le j \le 2^{r-3} - 1,\ \ell = r-2, r-1, r\} \\ & \cup \mathcal{P}_1^1(S) = V(W_{r+1, 2^{r+1}}).\end{aligned}$$

Therefore every vertex of $W_{r+1,2^{r+1}}$ gets monitored after stage 2 and hence S is a PDS of $W_{r+1,2^{r+1}}$ and $\gamma_{\mathrm{P}}(W_{r+1,2^{r+1}}) \le |S| = 2^{r-2}$. ■

For $r = 3$, any singleton set $\{v\}$, $v \in W_{4,16}$, cannot itself power dominate the entire graph, as each of the neighbours of v will have exactly three unmonitored neighbours after the domination step. Hence the bound in Theorem 2.2 is sharp for $r = 3$. We further illustrate Theorem 2.2 for the graph $W_{5,32}$. The vertices of the set S as defined in the theorem are coloured black in Figure 3.

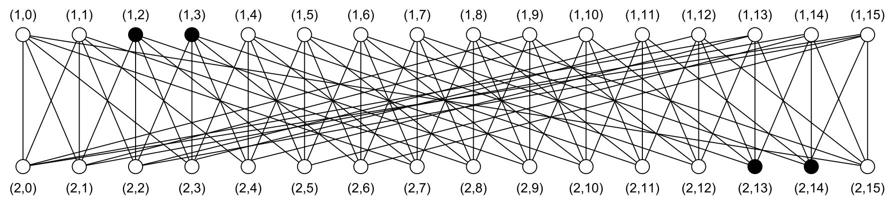

Figure 3. A power dominating set in the graph $W_{5,32}$.

In Figure 4, the set of dominated vertices, $\mathcal{P}_1^0(S)$, is coloured black and the remaining vertices are white.

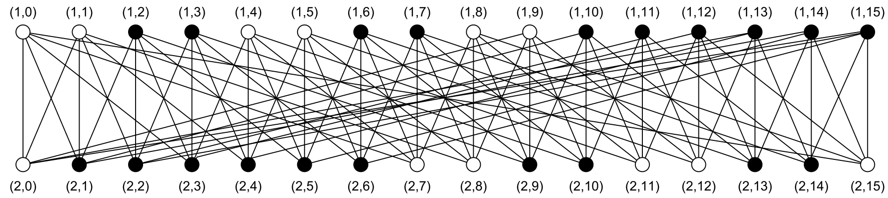

Figure 4. Neighbourhood is monitored.

The black vertices in Figure 5 and Figure 6 represent the sets $\mathcal{P}_1^1(S)$ and $\mathcal{P}_1^2(S)$, respectively.

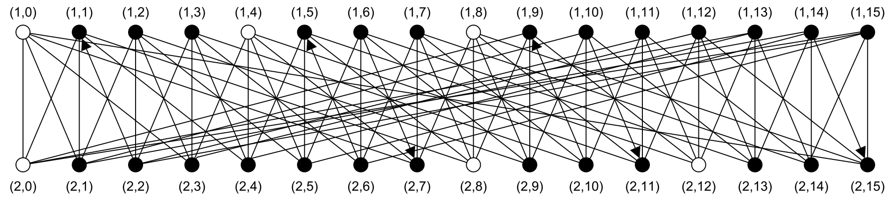

Figure 5. Propagation occurs.

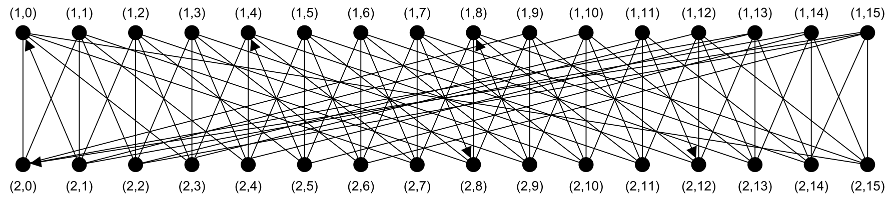

Figure 6. End of propagation.

The directed edges in the figures indicate the direction in which the propagation occurs at each stage. For instance, the directed edge $[(2,2),(1,1)]$ in Figure 5 indicates that $(2,2)$ monitors $(1,1)$ in the first propagation step. We observe that all the vertices get monitored by stage 2 and therefore S is a PDS of $W_{5,32}$.

However, we found that for $r = 5$, $W_{6,64}$ has a power dominating set of cardinality 6, namely $S = \{(1,1),(1,2),(1,11),(2,22),(2,27),(2,31)\}$. Therefore $\gamma_{\mathrm{P}}(W_{6,64}) < 2^3$. Hence the bound in Theorem 2.2 is not sharp for $r = 5$. This has to be compared with a conjecture stated in [5]. This conjecture says that, for $k \in \mathbb{N}_1$ and $r \in \mathbb{N}_2$, if $G \neq K_{r+1,r+1}$ is a connected $r+1$-regular graph of order n, then $\gamma_{\mathrm{P},k}(G) \leq \frac{n}{r+2}$. In the present example this means, for $k = 1$, $\gamma_{\mathrm{P}}(W_{6,64}) \leq 9$.

3. Generalized Power Domination in H_p^2

In this section, we study the behaviour of power domination in H_p^2. The cases $p \in [2]$ are trivial with $\gamma_{\mathrm{P},k}(H_1^2) = \gamma_{\mathrm{P},k}(K_1) = 1$ and $\gamma_{\mathrm{P},k}(H_2^2) = 2 = \gamma_{\mathrm{P},k}(W_{1,4})$, respectively, for all k.

Recall that for $p \in \mathbb{N}_3$ and $n = 2$, $V(H_p^2) = \{s_2 s_1 \colon s_1, s_2 \in [p]_0\}$ and $E(H_p^2) = \big\{\{ri, rj\}, \{i\ell, j\ell\} \colon r, i, j \in [p]_0,\ i \neq j,\ \ell \in [p]_0 \backslash \{i, j\}\big\}$.

Note that the extreme vertices are of degree $p-1$ and all the other vertices are of degree $2p-3$ in H_p^2. It is easy to observe that $\gamma(H_p^2) = p$. Indeed, any set containing a vertex from each of the p cliques in H_p^2 forms a dominating set of H_p^2. Since each of the p cliques contains an extreme vertex, any dominating set of H_p^2 must contain at least p vertices and hence $\gamma(H_p^2) = p$.

For $p = 3$, H_3^n is isomorphic to the Sierpiński graph S_3^n; see [13, p. 143 ff]. It is proved in [6] that

$$\gamma_{\mathrm{P},k}(S_3^n) = \begin{cases} 1, & n = 1 \text{ or } k \in \mathbb{N}_2; \\ 2, & n = 2 \text{ and } k = 1; \\ 3^{n-2}, & n \in \mathbb{N}_3 \text{ and } k = 1. \end{cases}$$

Therefore $\gamma_{\mathrm{P},1}(H_3^2) = 2$ and $\gamma_{\mathrm{P},k}(H_3^2) = 1$ for $k \in \mathbb{N}_2$.

For $p \in \mathbb{N}_4$, the Hanoi graphs do not contain perfect codes for $n \in \mathbb{N}_3$ [13, p. 195]. The domination number of these graphs is not known. Therefore we concentrate on $n = 2$. (For $n = 1$, $H_p^1 \cong K_p \cong S_p^1$.)

Theorem 3.1. *Let $k \in \mathbb{N}_1$ and $p \in \mathbb{N}_4$. Then*

$$\gamma_{\mathrm{P},k}(H_p^2) = \begin{cases} 1, & k \in \mathbb{N}_{p-2}; \\ p-k-1, & k \in [p-3]. \end{cases}$$

Proof. *Case* 1. $k \in \mathbb{N}_{p-2}$. Let v be an arbitrary vertex of H_p^2. Let K_p^i denote the subgraph induced by the vertices $\{ij \colon j \in [p]_0\}$. Assume that $v \in K_p^i$ for some i. Let $S = \{v\}$. Then $V(K_p^i) \subseteq \mathcal{P}_k^0(S)$. Since each vertex in K_p^i other than the vertex ii has $p-2$ neighbours outside K_p^i, for any $j \neq i$, $V(K_p^j) \backslash \{jj, ji\} \subseteq \mathcal{P}_k^1(S)$. Hence any vertex $j\ell$ in K_p^j, $\ell \neq i, j$, will have two unmonitored neighbours, namely jj and ji. Since $k \geq p-2 \geq 2$, these vertices will get monitored by propagation, i.e., $V(K_p^j) \subseteq \mathcal{P}_k^2(S)$. Since this is true for any $j \neq i$, S is a k-PDS of H_p^2.

Case 2. $k \in [p-3]$. We first prove that $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-1$. Let S be the set of vertices $\{i(i-1) \colon i \in [p-k-2]\} \cup \{0(p-k-2)\}$. Then $\mathcal{P}_k^0(S) = \bigcup\{V(K_p^i) \colon i \in [p-k-1]_0\} \cup \{ij \colon p-k-1 \leq i \leq p-1,\ j \in [p-k-2]_0\} \cup \{i(p-k-2) \colon p-k-1 \leq i \leq p-1\}$. Let Y be the set of vertices $\{ij \colon i \in [p-k-1]_0,\ p-k-1 \leq j \leq p-1\}$. Then any vertex $v = i'j'$ in Y has exactly k unmonitored neighbours given by

$\{\ell j'\colon p-k-1 \leq \ell \leq p-1,\, \ell \neq j'\}$ which will get monitored by propagation. Therefore, the remaining set of unmonitored vertices is given by $\{jj\colon V(K_p^j) \cap S = \emptyset\}$, which will then get monitored by propagation by its neighbours in K_p^j. Thus S is a k-PDS of H_p^2, which implies $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-1$.

We next prove that $\gamma_{\mathrm{P},k}(H_p^2) \geq p-k-1$. Let S be a k-PDS of H_p^2. Suppose on the contrary that $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-2$. Assume first that S has exactly one vertex in p-cliques K_p^i for $i \in \{i_1, \ldots, i_{p-k-2}\}$. Let $\{i_1j_1, \ldots, i_{p-k-2}j_{p-k-2}\}$ be the set of $p-k-2$ vertices in S. Then $S \cap V(K_p^{i'}) = \emptyset$ for any $i' \in I' = [p]_0 \backslash \{i_1, \ldots, i_{p-k-2}\}$. Let $X = \{i'j_1, \ldots, i'j_{p-k-2}\}$. Then $\mathcal{P}_k^0(S) \cap V(K_p^{i'}) \subseteq X$. This holds for any $i' \in I'$. Let $J' = [p]_0 \backslash \{j_1, \ldots, j_{p-k-2}\}$. Then the set of vertices $\{i'j'\colon i' \in I',\, j' \in J'\}$ has an empty intersection with $\mathcal{P}_k^0(S)$. Since every vertex in H_p^2 has either no or more than k neighbours in this set, no vertex from this set can get monitored later on, a contradiction. Assume next that $|S| < p-k-2$ or that S intersects some K_p^i in more than one vertex. Then we can conclude analogously that not all vertices of $K_p^{i'}$ will be monitored and hence $\gamma_{\mathrm{P},k}(H_p^2) \geq p-k-1$. ∎

It is obtained in [6] that $\gamma_{\mathrm{P},k}(S_p^2) = \begin{cases} 1, & k \in \mathbb{N}_{p-1}; \\ p-k, & k \in [p-2]. \end{cases}$

We can observe that for $p \in \mathbb{N}_4$, $\gamma_{\mathrm{P},k}(S_p^2) - \gamma_{\mathrm{P},k}(H_p^2) = 1$ if and only if $k \in [p-2]$ and for $k \in \mathbb{N}_{p-1}$, the two values coincide.

We now compute the propagation radius of H_p^2. For $p = 3$, it is proved in [6] that $\mathrm{rad}_{\mathrm{P},1}(H_3^2) = 2$ and $\mathrm{rad}_{\mathrm{P},k}(H_3^n) = 3$ for $k \in \mathbb{N}_2$.

Theorem 3.2. *Let $k \in \mathbb{N}_1$ and $p \in \mathbb{N}_4$. Then* $\mathrm{rad}_{\mathrm{P},k}(H_p^2) = 3$.

Proof. For $k \in \mathbb{N}_{p-2}$, $\gamma_{\mathrm{P},k}(H_p^2) = 1$ and let $S = \{ij\}$ be a k-PDS of H_p^2. If $i \neq j$, we prove that the vertices ji and jj do not belong to $\mathcal{P}_k^1(S)$. Clearly, $ji, jj \notin \mathcal{P}_k^0(S)$. Also none of the neighbours of ji and jj belongs to $\mathcal{P}_k^0(S)$. Therefore, ji and jj cannot be monitored in stage 1. For $i = j$, we can similarly prove that the vertices ℓi and $\ell\ell$, for $\ell \neq i$, do not belong to $\mathcal{P}_k^1(S)$ and hence $\mathrm{rad}_{\mathrm{P},k}(H_p^2) \geq 3$. To prove the upper bound, consider the set $S = \{ii\}$. Then,

$$\begin{aligned} \mathcal{P}_k^0(S) &= V(K_p^i), \\ \mathcal{P}_k^1(S) &= \mathcal{P}_k^0(S) \cup \bigcup \left\{ V(K_p^\ell) \backslash \{\ell i, \ell\ell\} \colon \ell \in [p]_0 \setminus \{i\} \right\}, \\ \mathcal{P}_k^2(S) &= \mathcal{P}_k^1(S) \cup \{\ell i, \ell\ell \colon \ell \in [p]_0 \setminus \{i\}\} = V(H_p^2). \end{aligned}$$

Hence $\mathrm{rad}_{\mathrm{P},k}(H_p^2) \leq \mathrm{rad}_{\mathrm{P},k}(G,S) = 3$.

Suppose that $k \in [p-3]$ and let S be a minimum k-PDS of H_p^2. Then $\gamma_{\mathrm{P},k}(H_p^2) = p-k-1$ and thus there exist at least $k+1$ p-cliques K_p^i not containing any vertex of S. Let $K_p^{i'}$ be an arbitrary such clique. We prove that the vertex $i'i'$ is not in $\mathcal{P}_k^1(S)$. Clearly, the vertex $i'i'$ does not belong to $\mathcal{P}_k^0(S)$. Moreover, $|V(K_p^{i'}) \cap \mathcal{P}_k^0(S)| \leq p-k-1$ and therefore $|V(K_p^{i'}) \backslash \mathcal{P}_k^0(S)| \geq k+1$. Hence any neighbour of $i'i'$ has more than k unmonitored vertices preventing any propagation to this vertex on that step. Thus $i'i'$ is not in $\mathcal{P}_k^1(S)$. To prove the upper bound, consider the set $S = \{i(i-1)\colon i \in [p-k-2]\} \cup \{0(p-k-2)\}$. Then,

$$\begin{aligned}
\mathcal{P}_k^0(S) &= \{V(K_p^i)\colon i \in [p-k-1]_0\} \cup \{ij\colon p-k-1 \leq i \leq p-1,\ j \in [p-k-1]_0\},\\
\mathcal{P}_k^1(S) &= \mathcal{P}_k^0(S) \cup \{ij\colon p-k-1 \leq i,j \leq p-1,\ i \neq j\},\\
\mathcal{P}_k^2(S) &= \mathcal{P}_k^1(S) \cup \{ii\colon p-k-1 \leq i \leq p-1\} = V(H_p^2).
\end{aligned}$$

This completes the proof. ∎

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References

[1] A. Aazami, *Domination in graphs with bounded propagation: algorithms, formulations and hardness results*, J. Comb. Optim. **19** (2010) 429–456. doi:10.1007/s10878-008-9176-7

[2] T.L. Baldwin, L. Mili, M.B. Boisen Jr. and R. Adapa, *Power system observability with minimal phasor measurement placement*, IEEE Trans. Power Systems **8** (1993) 707–715. doi:10.1109/59.260810

[3] J.-C. Bermond, H.A. Harutyunyan, A.L. Liestman and S. Perennes, *A note on the dimensionality of modified Knödel graphs*, Internat. J. Found. Comput. Sci. **8** (1997) 109–116. doi:10.1142/S0129054197000094

[4] G.J. Chang, P. Dorbec, M. Montassier and A. Raspaud, *Generalized power domination of graphs*, Discrete Appl. Math. **160** (2012) 1691–1698. doi:10.1016/j.dam.2012.03.007

[5] P. Dorbec, M.A. Henning, C. Löwenstein, M. Montassier and A. Raspaud, *Generalized power domination in regular graphs*, SIAM J. Discrete Math. **27** (2013) 1559–1574. doi:10.1137/120891356

[6] P. Dorbec and S. Klavžar, *Generalized power domination: propagation radius and Sierpiński graphs*, Acta Appl. Math. **134** (2014) 75–86. doi:10.1007/s10440-014-9870-7

[7] P. Dorbec, M. Mollard, S. Klavžar and S. Špacapan, *Power domination in product graphs*, SIAM J. Discrete Math. **22** (2008) 554–567. doi:10.1137/060661879

[8] M. Dorfling and M.A. Henning, *A note on power domination in grid graphs*, Discrete Appl. Math. **154** (2006) 1023–1027. doi:10.1016/j.dam.2005.08.006

[9] G. Fertin and A. Raspaud, *A survey on Knödel graphs*, Discrete Appl. Math. **137** (2004) 173–195. doi:10.1016/S0166-218X(03)00260-9

[10] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi and M.A. Henning, *Domination in graphs applied to electric power networks*, SIAM J. Discrete Math. **15** (2002) 519–529. doi:10.1137/S0895480100375831

[11] M.C. Heydemann, N. Marlin and S. Pérennes, Cayley graphs with complete rotations, Research Report INRIA **3624** (1999).

[12] A.M. Hinz, *The Tower of Hanoi*, Enseign. Math. (2) **35** (1989) 289–321.

[13] A.M. Hinz, S. Klavžar, U. Milutinović and C. Petr, The Tower of Hanoi—Myths and Maths (Springer, Basel, 2013).

[14] A.M. Hinz, S. Klavžar and S.S. Zemljič, *A survey and classification of Sierpiński-type graphs*, Discrete Appl. Math. **217** (2017) 565–600. doi:10.1016/j.dam.2016.09.024

[15] W. Knödel, *New gossips and telephones*, Discrete Math. **13** (1975) 95. doi:10.1016/0012-365X(75)90090-4

[16] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes (Morgan Kaufman, San Francisco CA, 1992).

[17] J.-H. Park and K.-Y. Chwa, *Recursive circulant: a new topology for multicomputers networks*, in: Proc. Int. Symp. Parallel Architectures, Algorithms and Networks (1994) 73–80. doi:10.1109/ISPAN.1994.367162

[18] S. Varghese and A. Vijayakumar, *On the power domination number of graph products*, Lecture Notes in Comput. Sci. **9602** (2016) 357–367. doi:10.1007/978-3-319-29221-2_31

[19] M. Zhao, L. Kang and G.J. Chang, *Power domination in graphs*, Discrete Math. **306** (2006) 1812–1816. doi:10.1016/j.disc.2006.03.037

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