Discussiones Mathematicae
Graph Theory 38 (2018) 233–243
doi:10.7151/dmgt.1986

THE GRAPHS WHOSE PERMANENTAL POLYNOMIALS ARE SYMMETRIC

Wei Li

Department of Applied Mathematics, School of Science
Northwestern Polytechnical University
Xi'an, Shaanxi, 710129, P.R. China

e-mail: liw@nwpu.edu.cn

Abstract

The permanental polynomial $\pi(G,x) = \sum_{i=0}^{n} b_i x^{n-i}$ of a graph G is symmetric if $b_i = b_{n-i}$ for each i. In this paper, we characterize the graphs with symmetric permanental polynomials. Firstly, we introduce the rooted product $H(K)$ of a graph H by a graph K, and provide a way to compute the permanental polynomial of the rooted product $H(K)$. Then we give a sufficient and necessary condition for the symmetric polynomial, and we prove that the permanental polynomial of a graph G is symmetric if and only if G is the rooted product of a graph by a path of length one.

Keywords: permanental polynomial, rooted product, matching.

2010 Mathematics Subject Classification: 05C31, 05C75.

1. Introduction

The graphs considered in this paper are simple undirected graphs. The vertex set of a graph G is $V(G) = \{v_1, \ldots, v_n\}$, the edge set of G is $E(G)$ and $|E(G)|$ denotes the number of edges in G. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is a matrix such that $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The *permanental polynomial* of G is [15]

$$\pi(G,x) = \mathrm{per}(xI - A(G)) = \sum_{i=0}^{n} b_i x^{n-i},$$

where I is an identity matrix of order n. For a matrix $A = (a_{ij})_{n\times n}$,

$$\mathrm{per(A)} = \sum_{\sigma\in\Gamma_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where Γ_n denotes the set of all the permutations of $\{1, 2, \ldots, n\}$.

A *linear subgraph* (or *basic figure*) U_i of a graph G is a subgraph on i vertices such that each component is a cycle or a single edge. It was proved that the coefficients of the permanental polynomial of a graph can be expressed in terms of linear subgraphs as follows [5, 11]:

$$b_i = (-1)^i \sum_{U_i \subset G} 2^{c(U_i)} \qquad \text{for} \quad 1 \leq i \leq n, \tag{1}$$

where the summation takes over all linear subgraphs U_i of G, and $c(U_i)$ is the number of cycles of U_i. Particularly, $b_0 = 1$ and $b_n = (-1)^n \mathrm{per}(A(G))$. In a bipartite graph, no linear subgraph with an odd number of vertices exists, so the permanental polynomial of a bipartite graph G can be expressed as $\pi(G, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{2i} x^{n-2i}$.

A *matching* of a graph G is a set of edges that have no common end-vertices. The *size of a matching* is the number of edges contained in it. A *perfect matching* of G is a matching covering all the vertices of G. Let $m(G)$ denote the number of perfect matchings of G. It holds for a bipartite graph G that [10]

$$b_n = m^2(G).$$

The permanental polynomial was first introduced to discriminate cospectral graphs [11, 13], but it does not seem better than the characteristic polynomial when it comes to distinguish trees [2]. Lately, it has been shown that the permanental polynomial really performs better than the characteristic polynomial when we use them to distinguish some non-tree graphs. For example, stars, complete graphs and some of its edge-deleted subgraphs [14, 18].

The study on the coefficients of the permanental polynomials also attracted much attention of graph-theoreticians [3, 4, 8, 7, 6, 12, 15, 16, 17]. For bipartite graphs without cycles of length $k = 0 \pmod 4$, the coefficients of the permanental polynomial and characteristic polynomial were proven to have the same magnitude [3], and the structure characterizations of such graphs were shown in [7]. For a bipartite graph without even subdivision of $K_{2,3}$, the permanental polynomial can be expressed by the characteristic polynomial of some orientation graph [15, 16]. Moreover, this result can be generalized to the permanental polynomials of matrices. See [8] for details. Recently, we find that the permanental polynomials of some graphs are symmetric. (A polynomial $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ is said to be *symmetric* if $a_i = a_{n-i}$ for each i.) For example, for the graphs G_1 and G_2 shown in Figure 1, $\pi(G_1, x) = x^6 + 5x^4 + 5x^2 + 1$ and $\pi(G_2, x) = x^{10} + 10x^8 + 30x^6 - 2x^5 + 30x^4 + 10x^2 + 1$. Now, an interesting problem arises naturally: characterize the graphs whose permanental polynomials are symmetric. In this paper, we will solve this problem.

Throughout this paper, P_n means a path of length n and C_n means a cycle of length n. A *null graph* is a graph without edges, and N_n denotes a null graph on n vertices. Let H be a graph on n vertices and K a rooted graph on m vertices. Let $K^1, \ldots, K^n$ be a sequence of n copies of K. The graph obtained by identifying the i-th vertex of H with the root of K^i for each i is called the *rooted product* of H by K, denoted by $H(K)$.

(a) (b)

Figure 1. (a) $G_1 = P_3(P_2)$; (b) $G_2 = C_5(P_2)$.

The rest of this paper is organized as follows. In Section 2, we derive the permanental polynomial of the rooted product of two graphs. As a corollary, we obtain the permanental polynomial of the rooted product of a graph by P_2. In Section 3, we give a criterion for the symmetric polynomial, and then we prove that the permanental polynomial of a graph G is symmetric if and only if G is the rooted product of a graph by P_2.

2. The Permanental Polynomial of the Rooted Product of Two Graphs

Firstly, we deduce the permanental polynomial of the rooted product of a graph H by a graph K. Following this, we show the permanental polynomial of the rooted product of a graph H by P_2.

Let H be a graph with a root u and let K be a graph with a root v. The graph $H - u$ denotes the one obtained from H by deleting the vertex u. The *coalescence* $H \cdot K$ is the graph obtained from H and K by identifying the two roots u and v. It has been proved that [1]

$$\begin{aligned}\pi(H \cdot K, x) = &\ \pi(H, x)\pi(K - v, x) + \pi(H - u, x)\pi(K, x) \\ &- x\pi(H - u, x)\pi(K - v, x).\end{aligned}$$

The permanental polynomial of the coalescence $H \cdot K$ can be derived by the permanental polynomials of H, K and their subgraphs. How about the permanental polynomial of the rooted produdt $H(K)$? To answer this question, we introduce the polynomial $p(G, x, y)$. For a given polynomial $p(G, x) = \sum_{i=0}^{n} a_i x^{n-i}$ associated with a graph G, we define the polynomial $p(G, x, y)$ to be $\sum_{i=0}^{n} a_i x^{n-i} y^i$.

Theorem 1. *Let H be a graph on n vertices and K a rooted graph on m vertices. Let v be the root of K. Then the permanental polynomial of the rooted product $H(K)$ is*

$$\pi(H(K),x)=\pi(H,\pi(K,x),\pi(K-v,x)). \tag{2}$$

Proof. Suppose $\pi(H(K),x)=\sum_{i=0}^{nm} a_i x^{nm-i}$. We show that if there is a linear subgraph contributing m to the coefficient of x^{nm-i} on the right side of equation (2), then there is a corresponding one contributing m to the coefficient a_i of x^{nm-i} on the left side of equation (2).

For a linear subgraph U_i of $H(K)$, each component of U_i either belongs to H or belongs to one of the n copies of K. Thus, we may write U_i as $U_i^0\cup U_i^1\cup\cdots\cup U_i^n$, where U_i^0 is a linear subgraph of H, and each U_i^j is a linear subgraph of the j-th copy of K for $1\le j\le n$ (here the symbol i in U_i^j does not mean the number of vertices of U_i^j). Denote the end-vertices of H by $u_1,\dots,u_n$. If $u_k\in U_i^0$, then $u_k\notin U_i^k$. Thus we view U_i^k as a linear subgraph of $H-v$ when $u_k\in U_i^0$. We can see that U_i and $U_i^0\cup U_i^1\cup\cdots\cup U_i^n$ form a one-to-one correspondence between the linear subgraphs of $H(K)$ and the union of linear subgraphs of H, s copies of K and t copies of $K-v$ with $s,t\ge 0$ and $s+t=n$.

We write $\pi(H,x)$ as $\sum_{j=0}^{n} b_j x^{n-j}$. Then

$$\pi(H,\pi(K,x),\pi(K-v,x))=\sum_{j=0}^{n} b_j[\pi(K,x)]^{n-j}[\pi(K-v,x)]^{j}. \tag{3}$$

Now we consider the contributions of linear subgraphs of H, the copy of K and the copy of $K-v$. We know that U_i^0 contributes $(-1)^{|U_i^0|}2^{c(U_i^0)}$ to $b_{|U_i^0|}$. If $u_k\notin U_i^0$, then U_i^k contributes $(-1)^{|U_i^k|}2^{c(U_i^k)}$ to the coefficient of $x^{m-|U_i^k|}$ of $\pi(K,x)$. If $u_k\in U_i^0$, then U_i^k contributes $(-1)^{|U_i^k|}2^{c(U_i^k)}$ to the coefficient of $x^{m-1-|U_i^k|}$ of $\pi(K-v,x)$. By equation (3), the product of all the individual contribution of U_i^k for $k\ge 0$ is exactly the contribution of $U_i^0\cup U_i^1\cup\cdots\cup U_i^n$ to the right side of equation (2). Explicitly, it is

$$\begin{aligned}
&(-1)^{\sum_{k\ge0}|U_i^k|}2^{\sum_{k\ge0}c(U_i^k)}x^{nm-|U_i^0|-\sum_{k\ge1}|U_i^k|}\\
&=(-1)^{\sum_{k\ge0}|U_i^k|}2^{\sum_{k\ge0}c(U_i^k)}x^{nm-\sum_{k\ge0}|U_i^k|}\\
&=(-1)^{|U_i|}2^{c(U_i)}x^{nm-|U_i|}=(-1)^{i}2^{c(U_i)}x^{nm-i},
\end{aligned}$$

which is exactly the contribution of U_i to the left side of equation (2). ■

For a graph H with a root u and a graph K with a root v, $H\cup K\cup(u,v)$ denotes the graph formed from H and K by joining an edge between u and v. Suppose that the graph H has n vertices. Let $K^1,K^2,\dots,K^n$ be a sequence of copies of K. The graph formed by joining an edge between the i-th vertex of H and the root of K^i for each i is called the *rooted join* of H by K, denoted by $H \sim K$. It holds for the permanental polynomial of $H \cup K \cup (u,v)$ that [1]

$$\pi(H \cup K \cup (u,v), x) = \pi(H,x)\pi(K,x) + \pi(H-u,x)\pi(K-v,x). \tag{4}$$

As a corollary of Theorem 1, we obtain the following.

Corollary 2. *For a graph H and a graph K with a root v, the permanental polynomial of the rooted join of H by K is*

$$\pi(H \sim K, x) = \pi(H, x\pi(K,x) + \pi(K-v,x), \pi(K,x)).$$

Proof. Let $K+e$ denote the graph obtained from K by adding an edge e incident to the root vertex v. By equation (4), we have $\pi(K+e,x) = x\pi(K,x)+\pi(K-v,x)$. We can see that the graph $H \sim K$ is the rooted product of H by $K+e$. It follows from Theorem 1 that

$$\begin{aligned}\pi(H \sim K, x) &= \pi(H(K+e),x) = \pi(H, \pi(K+e,x), \pi(K,x)) \\ &= \pi(H, x\pi(K,x) + \pi(K-v,x), \pi(K,x)).\end{aligned}$$

■

Corollary 3. *Let H be a graph on n vertices. Then*

$$\pi(H(P_2),x) = x^n \pi\left(H, x + \frac{1}{x}\right).$$

Proof. Suppose $\pi(H,x) = \sum_{i=0}^{n} b_i x^{n-i}$. We know that $\pi(P_2,x) = x^2+1$ and $\pi(P_2 - v, x) = x$, where v is any vertex of P_2. Following Theorem 1, we have

$$\begin{aligned}\pi(H(P_2),x) &= \pi(H, \pi(P_2,x), \pi(P_2-v,x)) = \sum_{i=0}^{n} b_i (x^2+1)^{n-i} x^i \\ &= x^n \sum_{i=0}^{n} b_i (x^2+1)^{n-i} x^{i-n} = x^n \sum_{i=0}^{n} b_i \left(x + \frac{1}{x}\right)^{n-i} = x^n \pi\left(H, x + \frac{1}{x}\right).\end{aligned}$$

■

As applications of Corollary 3, we deduce the permanental polynomials of $P_n(P_2)$ and $C_n(P_2)$. It is known that $\pi(P_n,x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}$ and $\pi(C_n,x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n}{n-i}\binom{n-i}{i} x^{n-2i} + b_n(C_n)$, where $b_n(C_n) = -2$ when n is odd and $b_n(C_n) = 4$ when n is even [9]. Then Corollary 3 implies

$$\pi(P_n(P_2),x) = x^n \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \left(x + \frac{1}{x}\right)^{n-2i}$$

and

$$\pi(C_n(P_2),x) = x^n \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{n}{n-i} \binom{n-i}{i} \left(x + \frac{1}{x}\right)^{n-2i} + b_n(C_n).$$

3. The Graphs Whose Permanental Polynomials are Symmetric

In this section, we give first a sufficient and necessary condition for the symmetric polynomial. Then we provide some helpful lemmas, which will play important roles in the proof of our main result. Based on these, we characterize the graphs with symmetric permanental polynomials.

Theorem 4. *A polynomial $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ of degree n is symmetric if and only if $p\left(\frac{1}{x}\right) = x^{-n}p(x)$.*

Proof. Since $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$, we have $p\left(\frac{1}{x}\right) = \sum_{i=0}^{n} a_i x^{i-n} = x^{-n}\sum_{i=0}^{n} a_i x^{i}$. If $p\left(\frac{1}{x}\right) = x^{-n}p(x) = x^{-n}\sum_{i=0}^{n} a_i x^{n-i} = x^{-n}\sum_{i=0}^{n} a_{n-i} x^{i}$ holds, then $\sum_{i=0}^{n} a_i x^{i}$ $= \sum_{i=0}^{n} a_{n-i} x^{i}$. It is obvious that $a_i = a_{n-i}$ for each i. Thus the polynomial $p(x)$ is symmetric.

If $p(x)$ is symmetric, then $a_i = a_{n-i}$ holds for each i. Thus we have $p\left(\frac{1}{x}\right) = x^{-n}\sum_{i=0}^{n} a_i x^{i} = x^{-n}\sum_{i=0}^{n} a_{n-i} x^{i} = x^{-n}\sum_{i=0}^{n} a_i x^{n-i} = x^{-n}p(x)$. ■

Let M be any matching of a graph G. A path $P = v_1, v_2, \ldots, v_m$ (m is even) in G is said to be an *M-augmenting path* if the edge $(v_i, v_{i+1}) \in M$ for odd i and the edge $(v_i, v_{i+1}) \notin M$ for even i. For two graphs G and H, the *symmetric difference* of G and H contains only the edges that are in exactly one of G or H, and is denoted by $G \triangle H$.

To prove the main result, we need to consider the matchings with one edge less than the perfect matching. The lemma below describes the structure property of such a matching.

Lemma 5. *Let G be a graph on $2n$ vertices with exactly one perfect matching M. Then each matching of size $n-1$ in G can be obtained either by deleting an edge of M or by $M \triangle P$, where P is an M-augmenting path in G.*

Proof. Let M_{n-1} be any matching of size $n-1$ in G. We prove that if M_{n-1} is not obtained from M by deleting an edge, then M_{n-1} is the symmetric difference of M and some M-augmenting path P in G.

Suppose that there are k_1 edges $(u_1, v_1), \ldots, (u_k, v_k)$ of M_{n-1}, which are different from the edges in M, and the remaining edges of M_{n-1} are the same as some edges in M. Let $S_1 = \{(u_i, v_i) | 1 \leq i \leq k\}$. Denote by u_{k+1} and v_{k+1} the two vertices in G, which are not incident to any edge of M_{n-1}. Since G admits a perfect matching, there must be a set S_2 of $k+1$ edges in M, which join these vertices $u_1, \ldots, u_k, u_{k+1}$ and $v_1, \ldots, v_k, v_{k+1}$. Denote by H the subgraph induced by the edges in $S_1 \cup S_2$. We can see that exactly two vertices of H are of degree one and the other $2k$ vertices of H are of degree two. Thus H is either a path P of odd length or a union of cycles and a path P^1 of odd length (denoted by $C^1 \cup \cdots \cup C^k \cup P^1$). Clearly, the edges in each cycle C^i are alternate with edges in S_1 and S_2, and so do for the edges in P or P^1. Moreover, the end-vertices of P (respectively, P^1) are u_{k+1} and v_{k+1}, which are incident to edges of M. Thus each cycle C^i is of even length, and P (respectively, P^1) is an M-augmenting path. For the case in which $H = C^1 \cup \cdots \cup C^k \cup P^1$, there exist at least two perfect matchings in G. This contradicts that G has exactly one perfect matching. For the case in which H is an M-augmenting path P, we have $M_{n-1} = M \triangle P$. ■

In the following, we use $m_i(G)$ to denote the number of matchings of size i in G. Based on Lemma 5, we derive the consequence below.

Corollary 6. *Let G be a graph on $2n$ vertices with exactly one perfect matching M. Let l be the number of M-augmenting paths in G. Then*

$$m_{n-1}(G) = n + l.$$

The next lemma provides a lower bound of the number of matchings with one edge less than the perfect matching.

Lemma 7. *Let G be a graph on $2n$ $(n \geq 2)$ vertices with symmetric permanental polynomial. Then*

(i) *there is exactly one perfect matching M in G;*
(ii) *there is no triangle in G, which contains an edge of the perfect matching M;*
(iii) *$m_{n-1}(G) \geq |E(G)|$, and equality holds if and only if $G = H(P_2)$ with H a graph on n vertices.*

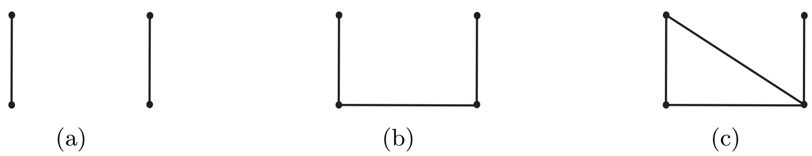

Figure 2. (a) $G_3 = N_2(P_2)$; (b) $G_4 = P_2(P_2)$; (c) G_5.

Proof. Since the permanental polynomial of G is symmetric, we have $b_{2n} = b_0 = 1$. By equation (1), we know that no linear subgraph U_{2n} with at least one cycle exists in G. Otherwise, $b_{2n} > 1$ holds. As $b_{2n} = 1$, there is exactly one linear subgraph U_{2n} whose components are all single edges, and such a linear subgraph is exactly a perfect matching of G, denoted by M. Thus statement (i) is obtained.

For the edges in G, denote the n edges of the perfect matching M by $(u_1, v_1), \ldots, (u_n, v_n)$. If G is bipartite, then there is no triangle in G. Thus we only need to consider the case in which G is non-bipartite. Suppose to the contrary that there is a triangle C_3 in G containing the edge (u_s, v_s). Denote the other end-vertex of C_3 by u_k (respectively v_k), where $k \in \{1, \ldots, n\}$ and $k \neq s$. Then the union of the triangle C_3 and edges (u_i, v_i), for $1 \leq i \leq n$ and $i \neq s, k$, is a linear subgraph of G on $2n-1$ vertices. Thus $b_{2n-1} \neq 0$ by equation (1). However, $b_1 = 0$. This contradicts that the permanental polynomial of G is symmetric. Therefore, statement (ii) is proved.

Now, we show that statement (iii) holds for G. For the case $n = 2$, there are three simple graphs with exactly one perfect matching (see Figure 2), and only two graphs $G_3 = N_2(P_2)$ and $G_4 = P_2(P_2)$ have symmetric permanental polynomials. Clearly, it holds that $m_{n-1}(G_3) = |E(G_3)|$ and $m_{n-1}(G_4) = |E(G_4)|$. Thus we assume $n \geq 3$. Denote by E_1 the set of edges in G that are not in M. It is clear that $|E(G)| = n + |E_1|$. We can see that the number of M-augmenting paths of length three in G is equal to the number of edges in E_1.

If at least one end-vertex of each matching edge in M is of degree one in G, then it is obvious that $G = H(P_2)$, where H is the graph obtained from G by deleting one end-vertex of degree one of (u_i, v_i) for each i. Moreover, in this case there is no M-augmenting path of length greater than three. By Corollary 6, it holds that $m_{n-1}(G) = n + l = n + |E_1| = |E(G)|$, where l is the number of M-augmenting paths in G.

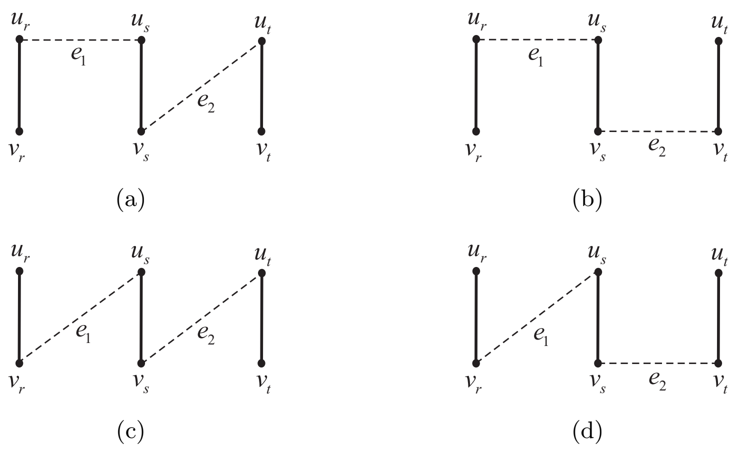

Figure 3. Cases of $e_1 \cup e_2 \cup (u_r, v_r) \cup (u_s, v_s) \cup (u_t, v_t)$ with an M-augmenting path of length 5.

Now we consider the case that the end-vertices u_s and v_s of some matching edge (u_s, v_s) in M are of degrees at least two. Then in this case G is not the rooted product of a graph by P_2. Suppose that u_s is adjacent to some vertex a ($a \neq v_s$) and v_s is adjacent to some vertex b ($b \neq u_s$). By statement (ii), we have $a \neq b$. As G has exactly one perfect matching, $\{a, b\} \neq \{u_i, v_i\}$ for any $i \in \{1, \ldots, n\}$. Thus we assume $a \in \{u_r, v_r\}$ and $b \in \{u_t, v_t\}$ for some $r, t \in \{1, \ldots, n\}$ and $r \neq t$. Then this leads to a graph in any of the forms shown in Figure 3. As each graph shown in Figure 3 admits an M-augmenting path of length 5, there is an M-augmenting path of length at least five in G. By Corollary 6, $m_{n-1}(G) = n + l$. Since l is larger than the number of M-augmenting paths of length three in G, we have $m_{n-1}(G) > n + |E_1| = |E(G)|$. ∎

The following is an upper bound of the number of matchings with one edge less than the perfect matching.

Lemma 8. *Let G be a graph on $2n$ vertices and $\pi(G,x) = \sum_{i=0}^{2n} b_i x^{2n-i}$. Then*

$$b_{2n-2} \geq m_{n-1}(G).$$

Moreover, if $G = H(P_2)$, then equality holds, where H is a graph on n vertices.

Proof. By equation (1), $b_{2n-2} = \sum_{U_{2n-2} \subset G} 2^{c(U_{2n-2})}$. A matching M of size $n-1$ in G is a linear subgraph on $2n-2$ vertices. Since $c(M) = 0$, M contributes one to b_{2n-2}. Thus $b_{2n-2} \geq m_{n-1}(G)$.

If $G = H(P_2)$, we show that no linear subgraph U_{2n-2} containing at least one cycle exists in G. If not, suppose that there is a linear subgraph U_{2n-2} containing a cycle C on k ($k \geq 3$) vertices. Since $H(P_2)$ has at least n vertices of degree one, the vertices of C must belong to H and those vertices of degree one adjacent to $V(C)$ do not lie in U_{2n-2}. Thus such a linear subgraph U_{2n-2} contains at most $2n - k \leq 2n - 3$ vertices. This contradicts that U_{2n-2} has $2n - 2$ vertices. Thus all the linear subgraphs U_{2n-2} in $H(P_2)$ contain only single edges. Therefore, $b_{2n-2} = m_{n-1}(G)$ holds for $G = H(P_2)$. ∎

Now we characterize the graphs with symmetric permanental polynomials.

Theorem 9. *Let G be a graph on $2n$ vertices. Then the permanental polynomial of G is symmetric if and only if $G = H(P_2)$, where H is a graph on n vertices.*

Proof. If $n = 1$, there are exactly two simple graphs P_2 and N_2 on two vertices. We know that $P_2 = N_1(P_2)$ and $\pi(P_2, x) = x^2 + 1$; while $\pi(N_2, x) = x^2$. Thus for the case $n = 1$, $\pi(G, x)$ is symmetric if and only if $G = N_1(P_2)$. Thus we only need to consider the case $n \geq 2$.

Sufficiency. By Corollary 3, $\pi(H(P_2), x) = x^n \pi\left(H, x + \frac{1}{x}\right)$, and so $\pi\left(H(P_2), \frac{1}{x}\right) = \left(\frac{1}{x}\right)^n \pi\left(H, x + \frac{1}{x}\right) = x^{-2n} x^n \pi\left(H, x + \frac{1}{x}\right) = x^{-2n} \pi\left(H(P_2), x\right)$. Since the polynomial $\pi(H(P_2), x)$ is of degree $2n$, Theorem 4 implies that $\pi(H(P_2), x)$ is symmetric.

Necessity. Suppose $\pi(G, x) = \sum_{i=0}^{2n} b_i x^{2n-i}$. Since $\pi(G, x)$ is symmetric, we have $b_{2n} = b_0 = 1$ and $b_2 = b_{2n-2} = |E(G)|$. By Lemma 7(i), G admits exactly one perfect matching. Let $m_{n-1}(G)$ be the number of matchings of size $n - 1$ in G. By Lemma 7(iii), we know $m_{n-1}(G) \geq |E(G)| = b_2$, and equality holds if and only if $G = H(P_2)$, where H is a graph on n vertices. By Lemma 8, we have $b_{2n-2} \geq m_{n-1}(G)$. Moreover, if G is the rooted product of a graph by P_2, then equality holds. Thus, we get $b_{2n-2} \geq b_2$, and equality holds if and only if G is the rooted product of a graph H by P_2. Therefore, $G = H(P_2)$ is obtained. ■

Acknowledgments

We are grateful to the anonymous referees for providing many helpful suggestions on this paper. This work is supported by NSFC (Grant No. 11501448), the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2015JQ1025) and the Fundamental Research Funds for the Central Universities (Grant No. 3102015ZY071).

References

[1] M. Borowiecki and T. Jóźwiak, *Computing the permanental polynomial of a multigraph*, Discuss. Math. **5** (1982) 9–16.

[2] M. Borowiecki and T. Jóźwiak, *A note on characteristic and permanental polynomials of multigraphs*, in: Graph Theory, M. Borowiecki, J.W. Kenendy and M.M. Syslo (Ed(s)), (Springer-Verlag, Berlin, 1983) 75–78. doi:10.1007/bfb0071615

[3] M. Borowiecki, *On spectrum and per-spectrum of graphs*, Publ. Inst. Math. (Beograd) (N.S.) **38** (1985) 31–33.

[4] Q. Chou, H. Liang and F. Bai, *Computing the permanental polynomial of the high level fullerene C_{70} with high precision*, MATCH Commun. Math. Comput. Chem. **73** (2015) 327–336.

[5] D.M. Cvetković, M. Doob and H. Sachs, Spectra of graphs (Academic Press, New York, 1980).

[6] I. Gutman and G.G. Cash, *Relations between the permanental and characteristic polynomials of fullerenes and benzenoid hydrocarbons*, MATCH Commun. Math. Comput. Chem. **45** (2002) 55–70.

[7] W. Li, *Graphs whose characteristic and permanental polynomials have coefficients of the same magnitude*, Discrete Math. **339** (2016) 2127–2135. doi:10.1016/j.disc.2016.02.019

[8] W. Li and H. Zhang, *On the permanental polynomials of matrices*, Bull. Malays. Math. Sci. Soc. **38** (2015) 1361–1374. doi:10.1007/s40840-014-0093-x

[9] W. Li and H. Zhang, *The permanental polynomials of certain graphs*, MATCH Commum. Math. Comput. Chem. **68** (2012) 871–888.

[10] L. Lovász and M.D. Plummer, Matching Theory (Elsevier, Amsterdam, 1986).

[11] R. Merris, K.R. Rebman and W. Watkins, *Permanental polynomials of graphs*, Linear Algebra Appl. **38** (1981) 273–288.
doi:10.1016/0024-3795(81)90026-4

[12] Y. Shi, M. Dehmer, X. Li and I. Gutman, Graph Polynomials (CRC Press, Boca Raton, 2016).

[13] J. Turner, *Generalized marix functions and the graph isomorphism problem*, SIAM J. Appl. Math. **16** (1968) 520–526.
doi:10.1137/0116041

[14] T. Wu and H. Zhang, *Per-spectral and adjacency spectral characterizations of a complete graph removing six edges*, Discrete Appl. Math. **203** (2016) 158–170.
doi:10.1016/j.dam.2015.09.014

[15] W. Yan and F. Zhang, *On the permanental polynomials of some graphs*, J. Math. Chem. **35** (2004) 175–188.
doi:10.1023/B:JOMC.0000033254.54822.f8

[16] H. Zhang and W. Li, *Computing the permanental polynomials of bipartite graphs by Pfaffian orientation*, Discrete Appl. Math. **160** (2012) 2069–2074.
doi:10.1016/j.dam.2012.04.007

[17] H. Zhang, S. Liu and W. Li, *A note on the permanental roots of bipartite graphs*, Discuss. Math. Graph Theory **34** (2014) 49–56.
doi:10.7151/dmgt.1704

[18] H. Zhang, T. Wu and H. Lai, *Per-spectral characterizations of some edge-deleted subgraphs of a complete graph*, Linear Multilinear Algebra **63** (2015) 397–410.
doi:10.1080/03081087.2013.869592

Received 16 May 2016
Accepted 7 November 2016