# THE GRAPHS WHOSE PERMANENTAL POLYNOMIALS ARE SYMMETRIC 

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#### Abstract

The permanental polynomial $\pi(G, x)=\sum_{i=0}^{n} b_{i} x^{n-i}$ of a graph $G$ is symmetric if $b_{i}=b_{n-i}$ for each $i$. In this paper, we characterize the graphs with symmetric permanental polynomials. Firstly, we introduce the rooted product $H(K)$ of a graph $H$ by a graph $K$, and provide a way to compute the permanental polynomial of the rooted product $H(K)$. Then we give a sufficient and necessary condition for the symmetric polynomial, and we prove that the permanental polynomial of a graph $G$ is symmetric if and only if $G$ is the rooted product of a graph by a path of length one.


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## 1. Introduction

The graphs considered in this paper are simple undirected graphs. The vertex set of a graph $G$ is $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$, the edge set of $G$ is $E(G)$ and $|E(G)|$ denotes the number of edges in $G$. The adjacency matrix $A(G)=\left(a_{i j}\right)_{n \times n}$ of $G$ is a matrix such that $a_{i j}=1$ if $v_{i}$ is adjacent to $v_{j}$, and $a_{i j}=0$ otherwise. The permanental polynomial of $G$ is [15]

$$
\pi(G, x)=\operatorname{per}(x I-A(G))=\sum_{i=0}^{n} b_{i} x^{n-i}
$$

where $I$ is an identity matrix of order $n$. For a matrix $A=\left(a_{i j}\right)_{n \times n}$,

$$
\operatorname{per}(\mathrm{A})=\sum_{\sigma \in \Gamma_{n}} \prod_{i=1}^{n} a_{i \sigma(i)}
$$

where $\Gamma_{n}$ denotes the set of all the permutations of $\{1,2, \ldots, n\}$.

A linear subgraph (or basic figure) $U_{i}$ of a graph $G$ is a subgraph on $i$ vertices such that each component is a cycle or a single edge. It was proved that the coefficients of the permanental polynomial of a graph can be expressed in terms of linear subgraphs as follows [5, 11]:

$$
\begin{equation*}
b_{i}=(-1)^{i} \sum_{U_{i} \subset G} 2^{c\left(U_{i}\right)} \quad \text { for } \quad 1 \leq i \leq n, \tag{1}
\end{equation*}
$$

where the summation takes over all linear subgraphs $U_{i}$ of $G$, and $c\left(U_{i}\right)$ is the number of cycles of $U_{i}$. Particularly, $b_{0}=1$ and $b_{n}=(-1)^{n} \operatorname{per}(A(G))$. In a bipartite graph, no linear subgraph with an odd number of vertices exists, so the permanental polynomial of a bipartite graph $G$ can be expressed as $\pi(G, x)=$ $\sum_{i=0}^{\lfloor n / 2\rfloor} b_{2 i} x^{n-2 i}$.

A matching of a graph $G$ is a set of edges that have no common end-vertices. The size of a matching is the number of edges contained in it. A perfect matching of $G$ is a matching covering all the vertices of $G$. Let $m(G)$ denote the number of perfect matchings of $G$. It holds for a bipartite graph $G$ that [10]

$$
b_{n}=m^{2}(G)
$$

The permanental polynomial was first introduced to discriminate cospectral graphs [11, 13], but it does not seem better than the characteristic polynomial when it comes to distinguish trees [2]. Lately, it has been shown that the permanental polynomial really performs better than the characteristic polynomial when we use them to distinguish some non-tree graphs. For example, stars, complete graphs and some of its edge-deleted subgraphs [14, 18].

The study on the coefficients of the permanental polynomials also attracted much attention of graph-theoreticians $[3,4,8,7,6,12,15,16,17]$. For bipartite graphs without cycles of length $k=0(\bmod 4)$, the coefficients of the permanental polynomial and characteristic polynomial were proven to have the same magnitude [3], and the structure characterizations of such graphs were shown in [7]. For a bipartite graph without even subdivision of $K_{2,3}$, the permanental polynomial can be expressed by the characteristic polynomial of some orientation graph $[15,16]$. Moreover, this result can be generalized to the permanental polynomials of matrices. See [8] for details. Recently, we find that the permanental polynomials of some graphs are symmetric. (A polynomial $p(x)=\sum_{i=0}^{n} a_{i} x^{n-i}$ is said to be symmetric if $a_{i}=a_{n-i}$ for each $i$.) For example, for the graphs $G_{1}$ and $G_{2}$ shown in Figure 1, $\pi\left(G_{1}, x\right)=x^{6}+5 x^{4}+5 x^{2}+1$ and $\pi\left(G_{2}, x\right)=x^{10}+$ $10 x^{8}+30 x^{6}-2 x^{5}+30 x^{4}+10 x^{2}+1$. Now, an interesting problem arises naturally: characterize the graphs whose permanental polynomials are symmetric. In this paper, we will solve this problem.

Throughout this paper, $P_{n}$ means a path of length $n$ and $C_{n}$ means a cycle of length $n$. A null graph is a graph without edges, and $N_{n}$ denotes a null graph
on $n$ vertices. Let $H$ be a graph on $n$ vertices and $K$ a rooted graph on $m$ vertices. Let $K^{1}, \ldots, K^{n}$ be a sequence of $n$ copies of $K$. The graph obtained by identifying the $i$-th vertex of $H$ with the root of $K^{i}$ for each $i$ is called the rooted product of $H$ by $K$, denoted by $H(K)$.

(a)

(b)

Figure 1. (a) $G_{1}=P_{3}\left(P_{2}\right) ;(\mathrm{b}) G_{2}=C_{5}\left(P_{2}\right)$.
The rest of this paper is organized as follows. In Section 2, we derive the permanental polynomial of the rooted product of two graphs. As a corollary, we obtain the permanental polynomial of the rooted product of a graph by $P_{2}$. In Section 3, we give a criterion for the symmetric polynomial, and then we prove that the permanental polynomial of a graph $G$ is symmetric if and only if $G$ is the rooted product of a graph by $P_{2}$.

## 2. The Permanental Polynomial of the Rooted Product of Two Graphs

Firstly, we deduce the permanental polynomial of the rooted product of a graph $H$ by a graph $K$. Following this, we show the permanental polynomial of the rooted product of a graph $H$ by $P_{2}$.

Let $H$ be a graph with a root $u$ and let $K$ be a graph with a root $v$. The graph $H-u$ denotes the one obtained from $H$ by deleting the vertex $u$. The coalescence $H \cdot K$ is the graph obtained from $H$ and $K$ by identifying the two roots $u$ and $v$. It has been proved that [1]

$$
\begin{aligned}
\pi(H \cdot K, x) & =\pi(H, x) \pi(K-v, x)+\pi(H-u, x) \pi(K, x) \\
& -x \pi(H-u, x) \pi(K-v, x) .
\end{aligned}
$$

The permanental polynomial of the coalescence $H \cdot K$ can be derived by the permanental polynomials of $H, K$ and their subgraphs. How about the permanental polynomial of the rooted produdt $H(K)$ ? To answer this question, we introduce the polynomial $p(G, x, y)$. For a given polynomial $p(G, x)=$ $\sum_{i=0}^{n} a_{i} x^{n-i}$ associated with a graph $G$, we define the polynomial $p(G, x, y)$ to be $\sum_{i=0}^{n} a_{i} x^{n-i} y^{i}$.

Theorem 1. Let $H$ be a graph on $n$ vertices and $K$ a rooted graph on $m$ vertices. Let $v$ be the root of $K$. Then the permanental polynomial of the rooted product $H(K)$ is

$$
\begin{equation*}
\pi(H(K), x)=\pi(H, \pi(K, x), \pi(K-v, x)) . \tag{2}
\end{equation*}
$$

Proof. Suppose $\pi(H(K), x)=\sum_{i=0}^{n m} a_{i} x^{n m-i}$. We show that if there is a linear subgraph contributing $m$ to the coefficient of $x^{n m-i}$ on the right side of equation (2), then there is a corresponding one contributing $m$ to the coefficient $a_{i}$ of $x^{n m-i}$ on the left side of equation (2).

For a linear subgraph $U_{i}$ of $H(K)$, each component of $U_{i}$ either belongs to $H$ or belongs to one of the $n$ copies of $K$. Thus, we may write $U_{i}$ as $U_{i}^{0} \cup U_{i}^{1} \cup$ $\cdots \cup U_{i}^{n}$, where $U_{i}^{0}$ is a linear subgraph of $H$, and each $U_{i}^{j}$ is a linear subgraph of the $j$-th copy of $K$ for $1 \leq j \leq n$ (here the symbol $i$ in $U_{i}^{j}$ does not mean the number of vertices of $U_{i}^{j}$ ). Denote the end-vertices of $H$ by $u_{1}, \ldots, u_{n}$. If $u_{k} \in U_{i}^{0}$, then $u_{k} \notin U_{i}^{k}$. Thus we view $U_{i}^{k}$ as a linear subgraph of $H-v$ when $u_{k} \in U_{i}^{0}$. We can see that $U_{i}$ and $U_{i}^{0} \cup U_{i}^{1} \cup \cdots \cup U_{i}^{n}$ form a one-to-one correspondence between the linear subgraphs of $H(K)$ and the union of linear subgraphs of $H, s$ copies of $K$ and $t$ copies of $K-v$ with $s, t \geq 0$ and $s+t=n$.

We write $\pi(H, x)$ as $\sum_{j=0}^{n} b_{j} x^{n-j}$. Then

$$
\begin{equation*}
\pi(H, \pi(K, x), \pi(K-v, x))=\sum_{j=0}^{n} b_{j}[\pi(K, x)]^{n-j}[\pi(K-v, x)]^{j} . \tag{3}
\end{equation*}
$$

Now we consider the contributions of linear subgraphs of $H$, the copy of $K$ and the copy of $K-v$. We know that $U_{i}^{0}$ contributes $(-1)^{\left|U_{i}^{0}\right|} 2^{c\left(U_{i}^{0}\right)}$ to $b_{\left|U_{i}^{0}\right|}$. If $u_{k} \notin U_{i}^{0}$, then $U_{i}^{k}$ contributes $(-1)^{\left|U_{i}^{k}\right|} 2^{c\left(U_{i}^{k}\right)}$ to the coefficient of $x^{m-\left|U_{i}^{k}\right|}$ of $\pi(K, x)$. If $u_{k} \in U_{i}^{0}$, then $U_{i}^{k}$ contributes $(-1)^{\left|U_{i}^{k}\right|} 2^{c\left(U_{i}^{k}\right)}$ to the coefficient of $x^{m-1-\left|U_{i}^{k}\right|}$ of $\pi(K-v, x)$. By equation (3), the product of all the individual contribution of $U_{i}^{k}$ for $k \geq 0$ is exactly the contribution of $U_{i}^{0} \cup U_{i}^{1} \cup \cdots \cup U_{i}^{n}$ to the right side of equation (2). Explicitly, it is

$$
\begin{aligned}
& (-1)^{\sum_{k \geq 0}\left|U_{i}^{k}\right|} 2^{\sum_{k \geq 0} c\left(U_{i}^{k}\right)} x^{n m-\left|U_{i}^{0}\right|-\sum_{k \geq 1}\left|U_{i}^{k}\right|} \\
& =(-1)^{\sum_{k \geq 0}\left|U_{i}^{k}\right|} 2^{\sum_{k \geq 0} c\left(U_{i}^{k}\right)} x^{n m-\sum_{k \geq 0}\left|U_{i}^{k}\right|} \\
& =(-1)^{\left|U_{i}\right|} 2^{c\left(U_{i}\right)} x^{n m-\left|U_{i}\right|}=(-1)^{i} 2^{c\left(U_{i}\right)} x^{n m-i},
\end{aligned}
$$

which is exactly the contribution of $U_{i}$ to the left side of equation (2).
For a graph $H$ with a root $u$ and a graph $K$ with a root $v, H \cup K \cup(u, v)$ denotes the graph formed from $H$ and $K$ by joining an edge between $u$ and $v$. Suppose that the graph $H$ has $n$ vertices. Let $K^{1}, K^{2}, \ldots, K^{n}$ be a sequence of
copies of $K$. The graph formed by joining an edge between the $i$-th vertex of $H$ and the root of $K^{i}$ for each $i$ is called the rooted join of $H$ by $K$, denoted by $H \sim K$. It holds for the permanental polynomial of $H \cup K \cup(u, v)$ that [1]

$$
\begin{equation*}
\pi(H \cup K \cup(u, v), x)=\pi(H, x) \pi(K, x)+\pi(H-u, x) \pi(K-v, x) . \tag{4}
\end{equation*}
$$

As a corollary of Theorem 1, we obtain the following.
Corollary 2. For a graph $H$ and a graph $K$ with a root $v$, the permanental polynomial of the rooted join of $H$ by $K$ is

$$
\pi(H \sim K, x)=\pi(H, x \pi(K, x)+\pi(K-v, x), \pi(K, x)) .
$$

Proof. Let $K+e$ denote the graph obtained from $K$ by adding an edge $e$ incident to the root vertex $v$. By equation (4), we have $\pi(K+e, x)=x \pi(K, x)+\pi(K-v, x)$. We can see that the graph $H \sim K$ is the rooted product of $H$ by $K+e$. It follows from Theorem 1 that

$$
\begin{aligned}
\pi(H \sim K, x) & =\pi(H(K+e), x)=\pi(H, \pi(K+e, x), \pi(K, x)) \\
& =\pi(H, x \pi(K, x)+\pi(K-v, x), \pi(K, x)) .
\end{aligned}
$$

Corollary 3. Let $H$ be a graph on $n$ vertices. Then

$$
\pi\left(H\left(P_{2}\right), x\right)=x^{n} \pi\left(H, x+\frac{1}{x}\right) .
$$

Proof. Suppose $\pi(H, x)=\sum_{i=0}^{n} b_{i} x^{n-i}$. We know that $\pi\left(P_{2}, x\right)=x^{2}+1$ and $\pi\left(P_{2}-v, x\right)=x$, where $v$ is any vertex of $P_{2}$. Following Theorem 1, we have

$$
\begin{aligned}
\pi\left(H\left(P_{2}\right), x\right) & =\pi\left(H, \pi\left(P_{2}, x\right), \pi\left(P_{2}-v, x\right)\right)=\sum_{i=0}^{n} b_{i}\left(x^{2}+1\right)^{n-i} x^{i} \\
& =x^{n} \sum_{i=0}^{n} b_{i}\left(x^{2}+1\right)^{n-i} x^{i-n}=x^{n} \sum_{i=0}^{n} b_{i}\left(x+\frac{1}{x}\right)^{n-i}=x^{n} \pi\left(H, x+\frac{1}{x}\right) .
\end{aligned}
$$

As applications of Corollary 3, we deduce the permanental polynomials of $P_{n}\left(P_{2}\right)$ and $C_{n}\left(P_{2}\right)$. It is known that $\pi\left(P_{n}, x\right)=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-i}{i} x^{n-2 i}$ and $\pi\left(C_{n}, x\right)=$ $\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1} \frac{n}{n-i}\binom{n-i}{i} x^{n-2 i}+b_{n}\left(C_{n}\right)$, where $b_{n}\left(C_{n}\right)=-2$ when $n$ is odd and $b_{n}\left(C_{n}\right)$ $=4$ when $n$ is even [9]. Then Corollary 3 implies

$$
\pi\left(P_{n}\left(P_{2}\right), x\right)=x^{n} \sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-i}{i}\left(x+\frac{1}{x}\right)^{n-2 i}
$$

and

$$
\pi\left(C_{n}\left(P_{2}\right), x\right)=x^{n} \sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1} \frac{n}{n-i}\binom{n-i}{i}\left(x+\frac{1}{x}\right)^{n-2 i}+b_{n}\left(C_{n}\right) .
$$

## 3. The Graphs Whose Permanental Polynomials are Symmetric

In this section, we give first a sufficient and necessary condition for the symmetric polynomial. Then we provide some helpful lemmas, which will play important roles in the proof of our main result. Based on these, we characterize the graphs with symmetric permanental polynomials.

Theorem 4. A polynomial $p(x)=\sum_{i=0}^{n} a_{i} x^{n-i}$ of degree $n$ is symmetric if and only if $p\left(\frac{1}{x}\right)=x^{-n} p(x)$.

Proof. Since $p(x)=\sum_{i=0}^{n} a_{i} x^{n-i}$, we have $p\left(\frac{1}{x}\right)=\sum_{i=0}^{n} a_{i} x^{i-n}=x^{-n} \sum_{i=0}^{n} a_{i} x^{i}$. If $p\left(\frac{1}{x}\right)=x^{-n} p(x)=x^{-n} \sum_{i=0}^{n} a_{i} x^{n-i}=x^{-n} \sum_{i=0}^{n} a_{n-i} x^{i}$ holds, then $\sum_{i=0}^{n} a_{i} x^{i}$ $=\sum_{i=0}^{n} a_{n-i} x^{i}$. It is obvious that $a_{i}=a_{n-i}$ for each $i$. Thus the polynomial $p(x)$ is symmetric.

If $p(x)$ is symmetric, then $a_{i}=a_{n-i}$ holds for each $i$. Thus we have $p\left(\frac{1}{x}\right)=$ $x^{-n} \sum_{i=0}^{n} a_{i} x^{i}=x^{-n} \sum_{i=0}^{n} a_{n-i} x^{i}=x^{-n} \sum_{i=0}^{n} a_{i} x^{n-i}=x^{-n} p(x)$.

Let $M$ be any matching of a graph $G$. A path $P=v_{1}, v_{2}, \ldots, v_{m}$ ( $m$ is even) in $G$ is said to be an $M$-augmenting path if the edge $\left(v_{i}, v_{i+1}\right) \in M$ for odd $i$ and the edge $\left(v_{i}, v_{i+1}\right) \notin M$ for even $i$. For two graphs $G$ and $H$, the symmetric difference of $G$ and $H$ contains only the edges that are in exactly one of $G$ or $H$, and is denoted by $G \triangle H$.

To prove the main result, we need to consider the matchings with one edge less than the perfect matching. The lemma below describes the structure property of such a matching.

Lemma 5. Let $G$ be a graph on $2 n$ vertices with exactly one perfect matching $M$. Then each matching of size $n-1$ in $G$ can be obtained either by deleting an edge of $M$ or by $M \triangle P$, where $P$ is an $M$-augmenting path in $G$.

Proof. Let $M_{n-1}$ be any matching of size $n-1$ in $G$. We prove that if $M_{n-1}$ is not obtained from $M$ by deleting an edge, then $M_{n-1}$ is the symmetric difference of $M$ and some $M$-augmenting path $P$ in $G$.

Suppose that there are $k_{1}$ edges $\left(u_{1}, v_{1}\right), \ldots,\left(u_{k}, v_{k}\right)$ of $M_{n-1}$, which are different from the edges in $M$, and the remaining edges of $M_{n-1}$ are the same as some edges in $M$. Let $S_{1}=\left\{\left(u_{i}, v_{i}\right) \mid 1 \leq i \leq k\right\}$. Denote by $u_{k+1}$ and $v_{k+1}$ the two vertices in $G$, which are not incident to any edge of $M_{n-1}$. Since $G$ admits a perfect matching, there must be a set $S_{2}$ of $k+1$ edges in $M$, which join these vertices $u_{1}, \ldots, u_{k}, u_{k+1}$ and $v_{1}, \ldots, v_{k}, v_{k+1}$. Denote by $H$ the subgraph induced by the edges in $S_{1} \cup S_{2}$. We can see that exactly two vertices of $H$ are of degree one and the other $2 k$ vertices of $H$ are of degree two. Thus $H$ is either a path $P$ of odd length or a union of cycles and a path $P^{1}$ of odd length (denoted by $C^{1} \cup \cdots \cup C^{k} \cup P^{1}$ ). Clearly, the edges in each cycle $C^{i}$ are alternate with edges
in $S_{1}$ and $S_{2}$, and so do for the edges in $P$ or $P^{1}$. Moreover, the end-vertices of $P$ (respectively, $P^{1}$ ) are $u_{k+1}$ and $v_{k+1}$, which are incident to edges of $M$. Thus each cycle $C^{i}$ is of even length, and $P$ (respectively, $P^{1}$ ) is an $M$-augmenting path. For the case in which $H=C^{1} \cup \cdots \cup C^{k} \cup P^{1}$, there exist at least two perfect matchings in $G$. This contradicts that $G$ has exactly one perfect matching. For the case in which $H$ is an $M$-augmenting path $P$, we have $M_{n-1}=M \triangle P$.

In the following, we use $m_{i}(G)$ to denote the number of matchings of size $i$ in $G$. Based on Lemma 5, we derive the consequence below.

Corollary 6. Let $G$ be a graph on $2 n$ vertices with exactly one perfect matching $M$. Let $l$ be the number of $M$-augmenting paths in $G$. Then

$$
m_{n-1}(G)=n+l .
$$

The next lemma provides a lower bound of the number of matchings with one edge less than the perfect matching.

Lemma 7. Let $G$ be a graph on $2 n(n \geq 2)$ vertices with symmetric permanental polynomial. Then
(i) there is exactly one perfect matching $M$ in $G$;
(ii) there is no triangle in $G$, which contains an edge of the perfect matching $M$;
(iii) $m_{n-1}(G) \geq|E(G)|$, and equality holds if and only if $G=H\left(P_{2}\right)$ with $H$ a graph on $n$ vertices.


Figure 2. (a) $G_{3}=N_{2}\left(P_{2}\right) ; ~(b) ~ G_{4}=P_{2}\left(P_{2}\right) ; ~(c) ~ G_{5}$.

Proof. Since the permanental polynomial of $G$ is symmetric, we have $b_{2 n}=b_{0}$ $=1$. By equation (1), we know that no linear subgraph $U_{2 n}$ with at least one cycle exists in $G$. Otherwise, $b_{2 n}>1$ holds. As $b_{2 n}=1$, there is exactly one linear subgraph $U_{2 n}$ whose components are all single edges, and such a linear subgraph is exactly a perfect matching of $G$, denoted by $M$. Thus statement (i) is obtained.

For the edges in $G$, denote the $n$ edges of the perfect matching $M$ by $\left(u_{1}, v_{1}\right), \ldots,\left(u_{n}, v_{n}\right)$. If $G$ is bipartite, then there is no triangle in $G$. Thus we only need to consider the case in which $G$ is non-bipartite. Suppose to the contrary that there is a triangle $C_{3}$ in $G$ containing the edge $\left(u_{s}, v_{s}\right)$. Denote
the other end-vertex of $C_{3}$ by $u_{k}$ (respectively $v_{k}$ ), where $k \in\{1, \ldots, n\}$ and $k \neq s$. Then the union of the triangle $C_{3}$ and edges $\left(u_{i}, v_{i}\right)$, for $1 \leq i \leq n$ and $i \neq s, k$, is a linear subgraph of $G$ on $2 n-1$ vertices. Thus $b_{2 n-1} \neq 0$ by equation (1). However, $b_{1}=0$. This contradicts that the permanental polynomial of $G$ is symmetric. Therefore, statement (ii) is proved.

Now, we show that statement (iii) holds for $G$. For the case $n=2$, there are three simple graphs with exactly one perfect matching (see Figure 2), and only two graphs $G_{3}=N_{2}\left(P_{2}\right)$ and $G_{4}=P_{2}\left(P_{2}\right)$ have symmetric permanental polynomials. Clearly, it holds that $m_{n-1}\left(G_{3}\right)=\left|E\left(G_{3}\right)\right|$ and $m_{n-1}\left(G_{4}\right)=\left|E\left(G_{4}\right)\right|$. Thus we assume $n \geq 3$. Denote by $E_{1}$ the set of edges in $G$ that are not in $M$. It is clear that $|E(G)|=n+\left|E_{1}\right|$. We can see that the number of $M$-augmenting paths of length three in $G$ is equal to the number of edges in $E_{1}$.

If at least one end-vertex of each matching edge in $M$ is of degree one in $G$, then it is obvious that $G=H\left(P_{2}\right)$, where $H$ is the graph obtained from $G$ by deleting one end-vertex of degree one of $\left(u_{i}, v_{i}\right)$ for each $i$. Moreover, in this case there is no $M$-augmenting path of length greater than three. By Corollary 6 , it holds that $m_{n-1}(G)=n+l=n+\left|E_{1}\right|=|E(G)|$, where $l$ is the number of $M$-augmenting paths in $G$.


Figure 3. Cases of $e_{1} \cup e_{2} \cup\left(u_{r}, v_{r}\right) \cup\left(u_{s}, v_{s}\right) \cup\left(u_{t}, v_{t}\right)$ with an $M$-augmenting path of length 5.

Now we consider the case that the end-vertices $u_{s}$ and $v_{s}$ of some matching edge $\left(u_{s}, v_{s}\right)$ in $M$ are of degrees at least two. Then in this case $G$ is not the rooted product of a graph by $P_{2}$. Suppose that $u_{s}$ is adjacent to some vertex $a\left(a \neq v_{s}\right)$ and $v_{s}$ is adjacent to some vertex $b\left(b \neq u_{s}\right)$. By statement (ii), we have $a \neq b$. As $G$ has exactly one perfect matching, $\{a, b\} \neq\left\{u_{i}, v_{i}\right\}$ for any $i \in\{1, \ldots, n\}$. Thus we assume $a \in\left\{u_{r}, v_{r}\right\}$ and $b \in\left\{u_{t}, v_{t}\right\}$ for some $r, t \in\{1, \ldots, n\}$ and $r \neq t$. Then
this leads to a graph in any of the forms shown in Figure 3. As each graph shown in Figure 3 admits an $M$-augmenting path of length 5 , there is an $M$-augmenting path of length at least five in $G$. By Corollary $6, m_{n-1}(G)=n+l$. Since $l$ is larger than the number of $M$-augmenting paths of length three in $G$, we have $m_{n-1}(G)>n+\left|E_{1}\right|=|E(G)|$.

The following is an upper bound of the number of matchings with one edge less than the perfect matching.
Lemma 8. Let $G$ be a graph on $2 n$ vertices and $\pi(G, x)=\sum_{i=0}^{2 n} b_{i} x^{2 n-i}$. Then

$$
b_{2 n-2} \geq m_{n-1}(G) .
$$

Moreover, if $G=H\left(P_{2}\right)$, then equality holds, where $H$ is a graph on $n$ vertices.
Proof. By equation (1), $b_{2 n-2}=\sum_{U_{2 n-2} \subset G} 2^{c\left(U_{2 n-2}\right)}$. A matching $M$ of size $n-1$ in $G$ is a linear subgraph on $2 n-2$ vertices. Since $c(M)=0, M$ contributes one to $b_{2 n-2}$. Thus $b_{2 n-2} \geq m_{n-1}(G)$.

If $G=H\left(P_{2}\right)$, we show that no linear subgraph $U_{2 n-2}$ containing at least one cycle exists in $G$. If not, suppose that there is a linear subgraph $U_{2 n-2}$ containing a cycle $C$ on $k(k \geq 3)$ vertices. Since $H\left(P_{2}\right)$ has at least $n$ vertices of degree one, the vertices of $C$ must belong to $H$ and those vertices of degree one adjacent to $V(C)$ do not lie in $U_{2 n-2}$. Thus such a linear subgraph $U_{2 n-2}$ contains at most $2 n-k \leq 2 n-3$ vertices. This contradicts that $U_{2 n-2}$ has $2 n-2$ vertices. Thus all the linear subgraphs $U_{2 n-2}$ in $H\left(P_{2}\right)$ contain only single edges. Therefore, $b_{2 n-2}=m_{n-1}(G)$ holds for $G=H\left(P_{2}\right)$.

Now we characterize the graphs with symmetric permanental polynomials.
Theorem 9. Let $G$ be a graph on $2 n$ vertices. Then the permanental polynomial of $G$ is symmetric if and only if $G=H\left(P_{2}\right)$, where $H$ is a graph on $n$ vertices.

Proof. If $n=1$, there are exactly two simple graphs $P_{2}$ and $N_{2}$ on two vertices. We know that $P_{2}=N_{1}\left(P_{2}\right)$ and $\pi\left(P_{2}, x\right)=x^{2}+1$; while $\pi\left(N_{2}, x\right)=x^{2}$. Thus for the case $n=1, \pi(G, x)$ is symmetric if and only if $G=N_{1}\left(P_{2}\right)$. Thus we only need to consider the case $n \geq 2$.

Sufficiency. By Corollary 3, $\pi\left(H\left(P_{2}\right), x\right)=x^{n} \pi\left(H, x+\frac{1}{x}\right)$, and so $\pi\left(H\left(P_{2}\right)\right.$, $\left.\frac{1}{x}\right)=\left(\frac{1}{x}\right)^{n} \pi\left(H, x+\frac{1}{x}\right)=x^{-2 n} x^{n} \pi\left(H, x+\frac{1}{x}\right)=x^{-2 n} \pi\left(H\left(P_{2}\right), x\right)$. Since the polynomial $\pi\left(H\left(P_{2}\right), x\right)$ is of degree $2 n$, Theorem 4 implies that $\pi\left(H\left(P_{2}\right), x\right)$ is symmetric.

Necessity. Suppose $\pi(G, x)=\sum_{i=0}^{2 n} b_{i} x^{2 n-i}$. Since $\pi(G, x)$ is symmetric, we have $b_{2 n}=b_{0}=1$ and $b_{2}=b_{2 n-2}=|E(G)|$. By Lemma $7(\mathrm{i}), G$ admits exactly one perfect matching. Let $m_{n-1}(G)$ be the number of matchings of size $n-1$ in $G$. By Lemma 7 (iii), we know $m_{n-1}(G) \geq|E(G)|=b_{2}$, and equality holds if
and only if $G=H\left(P_{2}\right)$, where $H$ is a graph on $n$ vertices. By Lemma 8, we have $b_{2 n-2} \geq m_{n-1}(G)$. Moreover, if $G$ is the rooted product of a graph by $P_{2}$, then equality holds. Thus, we get $b_{2 n-2} \geq b_{2}$, and equality holds if and only if $G$ is the rooted product of a graph $H$ by $P_{2}$. Therefore, $G=H\left(P_{2}\right)$ is obtained.

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