*Discussiones Mathematicae*
*Graph Theory 38 (2018) 177–188*
*doi:10.7151/dmgt.1985*

# ON THE NUMBER OF $\alpha$-LABELED GRAPHS

Christian Barrientos

and

Sarah Minion

*Department of Mathematics*
*Clayton State University*
*Morrow, Georgia 30260, USA*

**e-mail:** chr_barrientos@yahoo.com
sarah.m.minion@gmail.com

**Abstract**

When a graceful labeling of a bipartite graph places the smaller labels in one of the stable sets of the graph, it becomes an $\alpha$-labeling. This is the most restrictive type of difference-vertex labeling and it is located at the very core of this research area. Here we use an extension of the adjacency matrix to count and classify $\alpha$-labeled graphs according to their size, order, and boundary value.

**Keywords:** $\alpha$-labeling, $\alpha$-graph, graceful triangle.

**2010 Mathematics Subject Classification:** 05C30, 05C78.

## 1. Introduction

A *graceful labeling* of a graph $G$ of order $m$ and size $n$, where $m \leq n+1$, is an injective mapping $f : V(G) \to \{0, 1, \ldots, n\}$ such that for every edge $xy$ of $G$, $f$ induces a *weight* defined by $|f(x) - f(y)|$ and the set of weights is $\{1, 2, \ldots, n\}$. In this case, $G$ is said to be a *graceful graph.* If the graceful labeling $f$ has the property that there exists an integer $\lambda$ such that for each edge $xy$ either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, $f$ is called an *$\alpha$-labeling* and $G$ is an *$\alpha$-graph.* We refer to the number $\lambda$ as the *boundary value* of $f$.

There is a long list of families of $\alpha$-graphs. Some general characteristics of this type of graph are known. For example, an $\alpha$-graph is bipartite; if the graph has size $n$ and degree sequence $d_1, d_2, \ldots, d_m$ then $\gcd(d_1, d_2, \ldots, d_m, n)$

divides $\frac{n(n-1)}{2}$. In the area of graph labeling $\alpha$-graphs are very important, it is well-known that an $\alpha$-graph $G$ of size $n$ can be used to decompose cyclically the complete graph $K_{2nm+1}$ into copies of $G$ for any positive integer $m$, as well as $K_{n,n}$. Furthermore, $\alpha$-graphs have been widely used to produce both new graceful and $\alpha$-graphs [6, 8, 10, 15, 16], however one of the most interesting aspects is that $\alpha$-labelings can be modified to create several other types of labelings such as: sequential [16], $k$-graceful [17], $(k, d)$-graceful [7], odd-graceful [14], $(k, d)$-arithmetic [1], strongly super edge-magic [11], and super edge-magic [21] when $G$ is a tree. Moreover, when $G$ is a tree and the sizes of its stable sets differ by at most one, $\alpha$-labelings can be transformed into $(a, 1)$- and $(3, 2)$-antimagic vertex labelings [3] as well as mean labelings [4]. All $\alpha$-graphs of order $m$ and size $m + 1$, whose stable sets have the previous property, also have super $(a, d)$-edge antimagic total labelings for $d = 0, 1, 2, 3$ [3]. To conclude this list we must mention that any $\alpha$-labeling with boundary value $\lambda$ can be transformed into a $(a^{\lambda+1}, a)$-geometric labeling, for every $a > 1$.

To summarize, $\alpha$-labelings are located at the very core of the graph labeling area. A better understanding of these labelings and graphs is essential. This work is devoted to the enumeration and classification of $\alpha$-labeled graphs according to their size, order, and boundary value. We use an extension of the adjacency matrix of a graph to count, in the first place, the number of $\alpha$-labeled graphs of size $n$ and boundary value $\lambda$; these values are used to determine the number of them that have $m$ vertices.

In Section 2 we present the tools used to count $\alpha$-labeled graphs and the known results. Section 3 contains the main results, there we determine the number of $\alpha$-labeled graphs of size $n$, order $m$, and boundary value $\lambda$ for all feasible values of $m$ and $\lambda$ given $n$. In Section 4 we present some open questions. We end this work with a detailed bibliography.

Graphs considered here are finite, undirected, with no loops, no isolated vertices, and no multiple edges. For all undefined terminologies see [9] and [13]. For a comprehensive account of graph labelings, the interested reader is referred to [13].

## 2. Counting with Graceful Triangles

A large amount of articles in the area of graph labeling focuses on finding a labeling of some specific family of graphs. Not much is known from an enumerative perspective. Sheppard [19] shows that there are $n!$ gracefully labeled graphs of size $n$; the formula for $\alpha$-labeled graphs is more complex. Let $\alpha(n)$ denote the number of $\alpha$-labeled graphs of size $n$; Sheppard's formula for $\alpha(n)$ is

$$\alpha(n) = \begin{cases} 2\displaystyle\sum_{\lambda=0}^{\frac{n-2}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda} & \text{if } n \text{ is even,} \\ 2\displaystyle\sum_{\lambda=0}^{\frac{n-3}{2}} (\lambda!)^2(\lambda+1)^{n-2\lambda} + \left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)! & \text{if } n \text{ is odd.} \end{cases}$$

Sheppard also counted the number of symmetric $\alpha$-labeled graphs. Using similar arguments, Fukś and Sullivan [12] obtained the same formula when they were counting the amount of number-conserving cellular automata with $n$ inputs. The authors in [5] arrived to the formulas using a different technique, based on an extension of the adjacency matrix. The new approach allows us to determine the exact number of gracefully labeled graphs of order $m$ and size $n$. Furthermore, the same technique can be used to count the number of graphs labeled with other, apparently unrelated, types of labelings.

Let $f$ be a graceful labeling of a graph $G$ of order $m$ and size $n$. The *graceful matrix* of $G$ is the square matrix of order $n+1$, $A(G) = [a_{ij}]$ where for all $0 \leq i, j \leq n$, $a_{ij} = 1$ if there is $uv \in E(G)$ such that $f(u) = i$ and $f(v) = j$, and $a_{ij} = 0$ otherwise; $A(G)$ is an extension of the adjacency matrix; it is symmetric and all the elements in the main diagonal equal zero. Therefore all the characteristics of the labeled graph are contained in the triangular arrangement composed by the cells $a_{ij}$ where $i < j$, $0 \leq i \leq n-1$ and $1 \leq j \leq n$. We refer to this arrangement as the *graceful triangle*. The graceful triangle contains $n$ diagonals $D_1, D_2, \ldots, D_n$, where $D_k$ consists of the cells $a_{ij}$ where $j - i = n + 1 - k$ for every $k \in \{1, 2, \ldots, n\}$. Since $f$ is a graceful labeling, the weights of the $n$ edges of $G$ are the integers $1, 2, \ldots, n$. Therefore, every diagonal contains only one cell equal to 1, all the other must be 0, otherwise the arrangement would not be a graceful triangle. We refer to this nonzero entry as *adjacency* and it is going to be represented by a dot in the graceful triangle. In Figure 1, we present an example for a graph of order 10 and size 10 which $\alpha$-labeling has boundary value 4.

When an $\alpha$-labeled graph with boundary value $\lambda$ is represented in a graceful triangle, all its adjacencies lie inside a rectangle whose corner cells have coordinates $(0, n), (0, \lambda+1), (\lambda, \lambda+1)$, and $(\lambda, n)$. We refer to this rectangle as the *rectangle determined by* $\lambda$. This rectangle has been highlighted in Figure 1. Shiue and Lu use this rectangle in their work on trees that are not $\alpha$-trees [20].

Using this representation we can easily count the number of $\alpha$-labeled graphs of size $n$ and boundary value $\lambda$. We just need to determine the number of cells in each diagonal of the rectangle determined by $\lambda$; once this is done we apply the Product Principle to these numbers to obtain the desired quantity. Taking the sum of these products over all the possible values of $\lambda$ we obtain the number $\alpha(n)$ of $\alpha$-labeled graphs of size $n$.

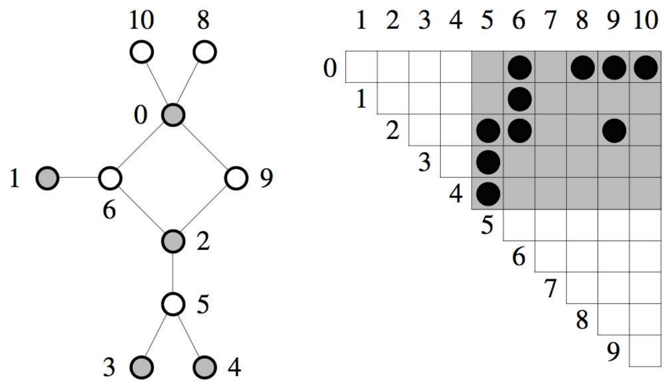


Figure 1. Graceful triangle of an $\alpha$-graph.

Let $R_\lambda = \{a_{ij} \in A(G) : 0 \leq i \leq \lambda \text{ and } \lambda + 1 \leq j \leq n\}$ be the rectangle determined by $\lambda$, the diagonal $D_k$ of $R_\lambda$ consists of all $a_{ij} \in R_\lambda$ such that $j = i + (n + 1 - k)$. We want to determine $d_\lambda(k)$, that is, the number of cells in $D_k$. By observing $R_\lambda$ we see that $d_\lambda(k)$ starts as an increasing sequence, becomes stable, and finally decreases. To show that, we consider the linear programming problem of optimizing the function $z = i - j + (n+1-k)$ over the feasible region $R_\lambda$. Thus we may conclude that the minimum value of $z = d_\lambda(k)$ is 1 when $k = 1$ or $k = n$, and the maximum value of $z$ is $\lambda + 1$ when $k = \lambda + 1$ or $k = n - \lambda$. Let $\mu = \min\{\lambda + 1, n - \lambda\}$ and $M = \max\{\lambda + 1, n - \lambda\}$. Thus,

$$d_\lambda(k) = \begin{cases} k & \text{if } 1 \leq k < \mu, \\ \lambda + 1 & \text{if } \mu \leq k \leq M, \\ n + 1 - k & \text{if } M < k \leq n. \end{cases}$$

Note that when $\lambda = 0$ or $\lambda = n - 1$, $d_\lambda(k) = 1$ for every $1 \leq k \leq n$. Hence the number $\alpha_\lambda(n)$ of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ is

$$\alpha_\lambda(n) = \prod_{k=1}^{n} d_\lambda(k),$$

and the number $\alpha(n)$ of $\alpha$-labeled graphs of size $n$ is given by

$$\alpha(n) = \sum_{\lambda=0}^{n-1} \alpha_\lambda(n) = \sum_{\lambda=0}^{n-1} \prod_{k=1}^{n} d_\lambda(k).$$

In Table 1 we show all the values of $\alpha_\lambda(n)$ for $n$ up to 10.

| $n\backslash\lambda$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | |
| 3 | 1 | 2 | 1 | | | | | | | |
| 4 | 1 | 4 | 4 | 1 | | | | | | |
| 5 | 1 | 8 | 12 | 8 | 1 | | | | | |
| 6 | 1 | 16 | 36 | 36 | 16 | 1 | | | | |
| 7 | 1 | 32 | 108 | 144 | 108 | 32 | 1 | | | |
| 8 | 1 | 64 | 324 | 576 | 576 | 324 | 64 | 1 | | |
| 9 | 1 | 128 | 972 | 2304 | 2880 | 2304 | 972 | 128 | 1 | |
| 10 | 1 | 256 | 2916 | 9216 | 14400 | 14400 | 9216 | 2916 | 256 | 1 |

Table 1. $\alpha$-labeled graphs of size $n$ and boundary value $\lambda$.

Since our goal is to determine the number of $\alpha$-labeled graphs of size $n$ and order $m$, we must observe first that $m \le n+1$ and any $\alpha$-labeling $f$ with boundary value $\lambda$ assigns, at least, the integers $0, \lambda, \lambda+1$, and $n$ as labels. Note that eventually $0 = \lambda$ or $\lambda + 1 = n$, except when $n = 1$, $\lambda = 0$, and $\lambda + 1 = n$; any other integer $x$ in $L = \{1, 2, \ldots, \lambda - 1, \lambda + 2, \lambda + 3, \ldots, n - 1\}$ could or could not be assigned by $f$.

Suppose $x \in L$ is not assigned by $f$ as a label. We want to determine the number of $\alpha$-labeled graphs of size $n$ and boundary value $\lambda$ that do not have $x$ as a label. To achieve this goal, our first step is to find the number $\delta_\lambda(k, x)$ of forbidden cells in $D_k$ when $x$ is not used as a label.

When $x < \lambda$, for every $\lambda+1 \le j \le n$, the cells $a_{xj} \in R_\lambda$ are forbidden. Thus, for $x < \lambda$,

$$\delta_\lambda(k,x) = \begin{cases} 0 & \text{if } 1 \le k \le x, \\ 1 & \text{if } x < k \le x + n - \lambda, \\ 0 & \text{if } x + n - \lambda < k \le n. \end{cases}$$

Similarly, for $x > \lambda + 1$,

$$\delta_\lambda(k,x) = \begin{cases} 0 & \text{if } 1 \le k \le n - x, \\ 1 & \text{if } n - x < k < n - x + \lambda + 2, \\ 0 & \text{if } n - x + \lambda + 2 \le k \le n. \end{cases}$$

Hence, the value of $d_\lambda(k) - \delta_\lambda(k, x)$ corresponds to the number of cells in $D_k$ where a dot can be placed to create an $\alpha$-labeled graph satisfying the requested conditions. Applying the Product Rule to these numbers, we obtain the quantity of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ that do not have the integer $x \in L$ as a label; in other terms, we have proved the following theorem.

**Theorem 1.** *The number of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ that do not have the integer $x \in L$ as a label is given by*

$$a(n,\lambda,x) = \prod_{k=1}^{n} (d_\lambda(k) - \delta_\lambda(k,x)).$$

A consequence of this result is that by taking the sum of these products over all the possible values of $x$, we obtain the number of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ in which at least one element of $L$ is not used as a label. These numbers can be found in the Online Encyclopedia of Integer Sequences under the sequence A245518 [18].

**Corollary 2.** *The number $a_1(n,\lambda)$ of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ in which at least one element of $L$ is not used is*

$$a_1(n,\lambda) = \sum_{x\in L} \left( \prod_{k=1}^{n} (d_\lambda(k) - \delta_\lambda(k,x)) \right).$$

This last number is the corner stone of the forthcoming calculations, where we find a formula to calculate the number of $\alpha$-labeled graphs of size $n$ and order $m$.

## 3. Enumerating $\alpha$-Graphs

Let $f$ be an $\alpha$-labeling of a graph $G$ of size $n$ with boundary value $\lambda$. Recall that if $x \in \{0,1,\ldots,n\}$ is not assigned by $f$ as a label of $G$, then $x \in L = \{1,2,\ldots,\lambda-1,\lambda+2,\lambda+3,\ldots,n-1\}$. Suppose that $t$ elements of $L$ are not used by $f$, thus the order of $G$ is $m = n+1-t$. Let $U = \{x_1,x_2,\ldots,x_t\}$ be the subset of $L$ formed by these numbers, and assume that for all $1 \le i \le t-1$, $x_i < x_{i+1}$. Consider the following partition of $U$:

$$U_\lambda = \{x \in U : x < \lambda\} \text{ and } U^\lambda = \{x \in U : x > \lambda + 1\}.$$

Let $x_i, x_j \in U$. If $x_i, x_j \in U_\lambda$ or $x_i, x_j \in U^\lambda$, then there is no cell in $R_\lambda$ with coordinates $(x_i, x_j)$. Therefore, the number of cells in $D_k$ where a dot can be placed is given by

$$d_\lambda(k) - \delta_\lambda(k,x_i) - \delta_\lambda(k,x_j).$$

On the other side, if $x_i \in U_\lambda$ and $x_j \in U^\lambda$, the cell with coordinates $(x_i, x_j)$ belongs to $R_\lambda$, which implies that the number of cells in $D_k$, that is, where a dot can be placed, is given by

$$d_\lambda(k) - \delta_\lambda(k,x_i) - \delta_\lambda(k,x_j) + 1,$$

because the cell $a_{x_i x_j}$ should not be eliminated twice.

Consequently, in our counting process, we need to introduce a new expression, that represents the number of cells in $D_k$ with coordinates $(x_i, x_j)$ when $(x_i, x_j) \in U_\lambda \times U^\lambda$. Let $\delta_\lambda(k, x_i, x_j)$ be this number, where

$$\delta_\lambda(k, x_i, x_j) = \begin{cases} 1 & \text{if } k = n + 1 - (x_j - x_i), (x_i, x_j) \in U_\lambda \times U^\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that at least one of $U_\lambda$ or $U^\lambda$ is empty, that is $U = U_\lambda$ or $U = U^\lambda$. Then the expression

$$d_\lambda(k) - \sum_{x \in U} d_\lambda(k, x)$$

gives us the total number of cells in $D_k$ where a dot can be placed when the elements of $U$ are not used as labels. Assume now that both, $U_\lambda$ and $U^\lambda$, are nonempty; then there are cells in $R_\lambda$ that could be eliminated twice by using the previous expression, eventually, some of these cells are in the same diagonal. To fix this problem, we introduce the function $\Delta_\lambda(k)$ defined as

$$\Delta_\lambda(k) = \sum_{x_i, x_j \in U} \delta_\lambda(k, x_i, x_j).$$

Hence, the number of available cells in $D_k$, i.e., where a dot can be placed, when the elements of $U$ are not used as labels, is given by

$$d_\lambda(k) + \Delta_\lambda(k) - \sum_{x \in U} d_\lambda(k, x).$$

Therefore, the number of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ that do not have the integers $x_1, x_2, \ldots, x_t$ as labels, is given by the product of the numbers above. This proves the next theorem.

**Theorem 3.** *The number $a(n, \lambda, U)$ of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ that do not have the integers in $U = \{x_1, x_2, \ldots, x_t\}$ as labels, is given by*

$$a(n, \lambda, U) = \prod_{k=1}^{n} \left( d_\lambda(k) + \Delta_\lambda(k) - \sum_{x \in U} d_\lambda(k, x) \right).$$

Recall that $L = \{1, 2, \ldots, \lambda - 1, \lambda + 2, \lambda + 3, \ldots, n - 1\}$. Taking the sum of these products, over all the $t$-element subsets $U$ of $L$, we obtain the following:

**Theorem 4.** *The number $a_t(n, \lambda)$, of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ in which at least one element of $L$ is not used as a label is*

$$a_t(n, \lambda) = \sum_{\substack{U \subseteq L \\ |U| = t}} \left( \prod_{k=1}^{n} \left( d_\lambda(k) + \Delta_\lambda(k) - \sum_{x \in U} d_\lambda(k, x) \right) \right).$$

Theorems 3 and 4 use $\lambda$ as a constant; in the following theorem, we use it as a variable to determine a formula for the number of $\alpha$-labeled graphs of size $n$ and order at most $n+1-t$, when $t \geq 1$.

**Theorem 5.** *The number $a_t(n)$ of $\alpha$-labeled graphs of size $n$ and order at most $n+1-t$, $t \geq 1$, is given by*

$$a_t(n) = \sum_{\lambda=1}^{n-2} a_t(n, \lambda).$$

***Proof.*** Since $a_t(n, \lambda)$ is the number of $\alpha$-labeled graphs of size $n$ with boundary value $\lambda$ in which $t$ elements from $\{1, 2, \ldots, n-1\}$ are not used, the order of these graphs is at most $n+1-t$. When $\lambda = 0$ or $\lambda = n-1$, there is only one $\alpha$-graph of size $n$ with this boundary value, that graph is the star $S_n \cong K_{1,n}$, which has order $n+1$. Then, when $t \geq 1$ the number $a_t(n)$ is obtained by taking the sum, over all the possible values of $\lambda$, i.e., $\lambda \in \{1, 2, \ldots, n-2\}$, of the numbers $a_t(n, \lambda)$. ■

Let $\hat{a}_t(n)$ denote the number of $\alpha$-labeled graphs of size $n$ and order $m = n+1-t$. If $s$ is the cardinality of the largest subset of $\{1, 2, \ldots, n-1\}$ such that there exists an $\alpha$-labeled graph of size $n$ and order $n+1-s$, then $\hat{a}_s(n) = a_s(n)$. For $0 \leq t < s$, we want to determine $\hat{a}_t(n)$.

Since $a_{s-1}(n)$ includes the $\alpha$-labeled graphs of size $n$ and order $n+1-s$, we have that

$$\hat{a}_{s-1}(n) = a_{s-1}(n) - s\hat{a}_s(n),$$

where the factor $s$ counts the number of $(s-1)$-element subsets of $\{x_1, x_2, \ldots, x_s\}$, which are considered when calculating $a_{s-1}(n)$. We can use this observation recursively together with the Principle of Inclusion and Exclusion to prove the following theorem.

**Theorem 6.** *The number of $\alpha$-labeled graphs of size $n$ and order $n+1$ is given by*

$$\hat{a}_0(n) = \alpha(n) + \sum_{t=1}^{n-1} (-1)^t a_t(n),$$

*where $\alpha(n)$ is the number of $\alpha$-labeled graphs of size $n$.*

***Proof.*** Let $\hat{a}_0(n)$ denote the number of $\alpha$-labeled graphs of size $n$ and order $n+1$. For every $1 \leq t \leq n-2$, $a_t(n)$ includes the number $a_{t+1}(n)$, that is, the labeled graphs counted by $a_{t+1}(n)$ are also counted by $a_t(n)$. Since there are $\alpha(n)$ $\alpha$-labeled graphs of size $n$ and the numbers $a_t(n)$ satisfy the conditions of the Principle of Inclusion and Exclusion, $\hat{a}_0(n)$ is given by the expression above. ■

Note that $\hat{a}_0(n)$ can be used as an upper bound for the number of $\alpha$-labeled trees of size $n$.

**Theorem 7.** *For $t \geq 1$, the number $\hat{a}_t(n)$ of $\alpha$-labeled graphs of size $n$ and order $m = n + 1 - t$ is*

$$\hat{a}_t(n) = a_t(n) - \sum_{i=1}^{s-t} \binom{t+i}{t} \hat{a}_{t+i}(n),$$

*where $s$ is the largest integer such that $\hat{a}_s(n) \neq 0$.*

***Proof.*** Since for every $t$, $a_t(n)$ counts the number of $\alpha$-labeled graphs of size $n$ and order $n - t$, and every $(t + i)$-element subset includes $\binom{t+i}{t}$ subsets with $t$ elements, the number $\binom{t+i}{t}\hat{a}_{t+i}(n)$ corresponds to the number of times $\hat{a}_{t+i}(n)$ has been counted within $a_t(n)$. Thus,

$$a_t(n) - \sum_{i=1}^{s} \binom{t+i}{t} \hat{a}_{t+i}(n)$$

gives the number $\hat{a}_t(n)$. ∎

In the following tables we summarize the numbers obtained. Recall that the graphs considered in this work do not have isolated vertices. Table 2 shows the total amount of $\alpha$-labeled graphs of size $n$ and order $m$ with $n$ up to 10. Table 3 presents the number of $\alpha$-labeled graphs of size $n$ and order $m$ for every possible value of $\lambda$. Two related results are the numbers of $\alpha$-labeled graphs of size $n$ and order at most $n$ and the number of $\alpha$-labeled graphs of size $n$ and order $m$ classified according to the possible boundary values. These are, respectively, the sequences A245519 and A245517, in the Online Encyclopedia of Integer Sequences [18].

| $n\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\alpha(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | | | | | | | | | | 1 |
| 2 | | | 2 | | | | | | | | | 2 |
| 3 | | | | 4 | | | | | | | | 4 |
| 4 | | | | 2 | 8 | | | | | | | 10 |
| 5 | | | | | 12 | 18 | | | | | | 30 |
| 6 | | | | | 4 | 56 | 46 | | | | | 106 |
| 7 | | | | | | 50 | 236 | 140 | | | | 426 |
| 8 | | | | | | 14 | 398 | 1034 | 484 | | | 1930 |
| 9 | | | | | | 2 | 292 | 2712 | 4796 | 1888 | | 9690 |
| 10 | | | | | | | 100 | 3552 | 18072 | 23880 | 7974 | 53578 |

Table 2. $\alpha$-labeled graphs of size $n$ and order $m$.

| $n$ | $\lambda\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | | 1 | | | | | | | | | |
| 2 | 0 | | | 1 | | | | | | | | |
| | 1 | | | 1 | | | | | | | | |
| 3 | 0 | | | | 1 | | | | | | | |
| | 1 | | | | 2 | | | | | | | |
| | 2 | | | | 1 | | | | | | | |
| 4 | 0 | | | | | 1 | | | | | | |
| | 1 | | | | 1 | 3 | | | | | | |
| | 2 | | | | 1 | 3 | | | | | | |
| | 3 | | | | | 1 | | | | | | |
| 5 | 0 | | | | | | 1 | | | | | |
| | 1 | | | | | 4 | 4 | | | | | |
| | 2 | | | | | 4 | 8 | | | | | |
| | 3 | | | | | 4 | 4 | | | | | |
| | 4 | | | | | | 1 | | | | | |
| 6 | 0 | | | | | | | 1 | | | | |
| | 1 | | | | | 1 | 10 | 5 | | | | |
| | 2 | | | | | 1 | 18 | 17 | | | | |
| | 3 | | | | | 1 | 18 | 17 | | | | |
| | 4 | | | | | 1 | 10 | 5 | | | | |
| | 5 | | | | | | | 1 | | | | |
| 7 | 0 | | | | | | | | 1 | | | |
| | 1 | | | | | | 6 | 20 | 6 | | | |
| | 2 | | | | | | 14 | 60 | 34 | | | |
| | 3 | | | | | | 10 | 76 | 58 | | | |
| | 4 | | | | | | 14 | 60 | 34 | | | |
| | 5 | | | | | | 6 | 20 | 6 | | | |
| | 6 | | | | | | | | 1 | | | |
| 8 | 0 | | | | | | | | | 1 | | |
| | 1 | | | | | | 1 | 21 | 35 | 7 | | |
| | 2 | | | | | | 5 | 81 | 175 | 63 | | |
| | 3 | | | | | | 1 | 97 | 307 | 171 | | |
| | 4 | | | | | | 1 | 97 | 307 | 171 | | |
| | 5 | | | | | | 5 | 81 | 175 | 63 | | |
| | 6 | | | | | | 1 | 21 | 35 | 7 | | |
| | 7 | | | | | | | | | 1 | | |
| 9 | 0 | | | | | | | | | | 1 | |
| | 1 | | | | | | | 8 | 56 | 56 | 8 | |
| | 2 | | | | | | 1 | 56 | 354 | 448 | 113 | |
| | 3 | | | | | | | 60 | 628 | 1156 | 460 | |
| | 4 | | | | | | | 44 | 636 | 1476 | 724 | |
| | 5 | | | | | | | 60 | 628 | 1156 | 460 | |
| | 6 | | | | | | 1 | 56 | 354 | 448 | 113 | |
| | 7 | | | | | | | 8 | 56 | 56 | 8 | |
| | 8 | | | | | | | | | | 1 | |
| 10 | 0 | | | | | | | | | | | 1 |
| | 1 | | | | | | | 1 | 36 | 126 | 84 | 9 |
| | 2 | | | | | | | 24 | 384 | 1256 | 1056 | 196 |
| | 3 | | | | | | | 20 | 704 | 3316 | 4028 | 1148 |
| | 4 | | | | | | | 5 | 652 | 4338 | 6772 | 2633 |
| | 5 | | | | | | | 5 | 652 | 4338 | 6772 | 2633 |
| | 6 | | | | | | | 20 | 704 | 3316 | 4028 | 1148 |
| | 7 | | | | | | | 24 | 384 | 1256 | 1056 | 196 |
| | 8 | | | | | | | 1 | 36 | 126 | 84 | 9 |
| | 9 | | | | | | | | | | | 1 |

Figure 3. $\alpha$-labeled graphs of size $n$ and order $m$ for every possible value of $\lambda$.

## 4. Conclusion

Graceful triangles have been useful in the enumeration of gracefully labeled graphs as well as in the enumeration and classification of $\alpha$-labeled graphs. They can also be used to visualize labeling patters, not only for difference-vertex labelings, they can also be used to study sum-vertex labelings. We hope that both, the counting technique and the graceful triangle can be used in the future to count other kinds of labeled graphs and to produce new families of labeled graphs. For instance it would be interesting to know the number of $\alpha$-labeled trees of size $n$. Another problem is to find closed formulas for the numbers defined in this paper.

## References


[1] B.D. Acharya and S.M. Hegde, *On certain vertex valuations of a graph*, Indian J. Pure Appl. Math. **22** (1991) 553–560.

[2] V. Ajtha, S. Arumugam and K.A. Germina, *On square sum graphs*, AKCE Int. J. Graphs Comb. **6** (2009) 1–10.

[3] M. Bača and C. Barrientos, *Graceful and edge-antimagic labelings*, Ars Combin. **96** (2010) 505–513.

[4] C. Barrientos and E. Krop, *Mean graphs*, AKCE Int. J. Graphs Comb. **11** (2014) 13–26.

[5] C. Barrientos and S. Minion, *Enumerating families of labeled graphs*, J. Integer Seq. **18** (2015) 15.1.7.

[6] C. Barrientos and S. Minion, *Three graceful operations*, J. Algorithms Comput. **45** (2014) 13–24.

[7] C. Bu and J. Zhang, *The properties of $(k, d)$-graceful graphs*, preprint (2014).

[8] G.J. Chang, D.F. Hsu and D.G. Rogers, *Additive variations on a graceful theme: some results on harmonious and other related graphs*, Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. I (Baton Rouge, LA., 1981), Congr. Numer. **32** (1981) 181–197.

[9] G. Chartrand and L. Lesniak, Graphs and Digraphs (Wadsworth and Brooks/Cole, 1986).

[10] W.C. Chen, H.I. Lü and Y.N. Yeh, *Operations of interlaced trees and graceful trees*, Southeast Asian Bull. Math. **21** (1997) 337–348.

[11] R. Figueroa-Centeno, R. Ichishima and F. Muntaner-Batle, *The place of super edgemagic labelings among other classes of labelings*, Discrete Math. **231** (2001) 153–168.
doi:10.1016/S0012-365X(00)00314-9

[12] H. Fukś and L. Sullivan, *Numeration of number-conserving cellular automata rules with two inputs*, J. Cell. Autom. **2** (2007) 141–148.

[13] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. (2015) #DS6.

[14] R.B. Gnanajothi, Topics in Graph Theory (Ph.D. Thesis, Madurai Kamaraj University, 1991).

[15] T. Grace, *On sequential labelings of graphs*, J. Graph Theory **7** (1983) 195–201. doi:10.1002/jgt.3190070208

[16] D. Jungreis and M. Reid, *Labeling grids*, Ars Combin. **34** (1992) 167–182.

[17] M. Maheo and H. Thuillier, *On d-graceful graphs*, Ars Combin. **13** (1982) 181–192.

[18] OEIS Foundation Inc., The Online Encyclopedia of Integer Sequences (2015). http://oeis.org

[19] D.A. Sheppard, *The factorial representation of major balanced graphs*, Discrete Math. **15** (1976) 379–388. doi:10.1016/0012-365X(76)90051-0

[20] C.-L. Shiue and H.-C. Lu, *Trees which admit no $\alpha$-labelings*, Ars Combin. **103** (2012) 453–463.

[21] Y. Yuansheng, X. Yue, X. Xirong and M. Xinhong, *Super edge magic labelings of book graphs*, Ars Combin. **93** (2009) 431–438.

Received 13 April 2015
Revised 28 October 2016
Accepted 28 October 2016