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TWIN MINUS TOTAL DOMINATION NUMBERS IN DIRECTED GRAPHS

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Abstract

Let $D = (V, A)$ be a finite simple directed graph (shortly, digraph). A function $f : V \longrightarrow \{-1, 0, 1\}$ is called a twin minus total dominating function (TMTDF) if $f(N^-(v)) \geq 1$ and $f(N^+(v)) \geq 1$ for each vertex $v \in V$. The twin minus total domination number of D is $\gamma^*_{mt}(D) = \min\{w(f) \mid f \text{ is a TMTDF of } D\}$. In this paper, we initiate the study of twin minus total domination numbers in digraphs and we present some lower bounds for $\gamma^*_{mt}(D)$ in terms of the order, size and maximum and minimum in-degrees and out-degrees. In addition, we determine the twin minus total domination numbers of some classes of digraphs.

Keywords: twin minus total dominating function, twin minus total domination number, directed graph.

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1. Introduction

In this paper, D is a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$ (briefly, V and A). A digraph without directed cycles of length 2 is an *oriented digraph*. We write $d_D^+(v) = d^+(v)$ for the out-degree of a vertex v and $d_D^-(v) = d^-(v)$ for its in-degree. The *minimum* and *maximum in-degree* and *minimum* and *maximum out-degree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. If (u, v) is an arc of D, then we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v, and we also say that u *dominates* v or v is *dominated* by u. The sets $N^-(v) = N_D^-(v) = \{x \mid (x, v) \in A(D)\}$ and $N^+(v) = N_D^+(v) = \{x \mid (v, x) \in A(D)\}$ are called the *in-neighborhood* and *out-neighborhood* of the vertex v. Likewise, $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$. For $S \subseteq V(D)$, we define $N^-(S) = N_D^-(S) = \bigcup_{v\in S} N^-(v)$, $N^+(S) = N_D^+(S) = \bigcup_{v\in S} N^+(v)$, $N^-[S] = N_D^-[S] = N^-(S) \cup S$ and $N^+[S] = N_D^+[S] = N^+(S) \cup S$. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from X to v. We denote by $A(X, Y)$ the set of arcs from a subset X to a subset Y. The notation D^{-1} is used for the digraph obtained from D by reversing the arcs of D. With any digraph D, we can associate a graph G with the same vertex set simply by replacing each arc by an edge with the same vertices. This graph is the *underlying graph* of D, denoted $G(D)$. The *complete digraph* of order n, K_n^*, is a digraph D such that $(u, v), (v, u) \in A(D)$ for any two distinct vertices $u, v \in V(D)$. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the *weight* of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$. Consult [13] for the notation and terminology which are not defined here.

A *signed total dominating function* (abbreviated STDF) of D is a function $f : V \to \{-1, 1\}$ such that $f(N^-(v)) \geq 1$ for every $v \in V$. The *signed total domination number* of a digraph D is

$$\gamma_{st}(D) = \min\{w(f) \mid f \text{ is a STDF of } D\}.$$

A $\gamma_{st}(D)$*-function* is a STDF of D of weight $\gamma_{st}(D)$. The signed total domination number of a digraph was introduced by Sheikholeslami [12].

Recently, Atapour *et al.* [1] studied the twin signed total domination numbers in digraphs. A signed total dominating function of a digraph D is called a *twin signed total dominating function* (briefly, TSTDF) if it is also a signed total dominating function of D^{-1}, i.e., $f(N^+(v)) \geq 1$ for every $v \in V$. The *twin signed total domination number* of a digraph D is $\gamma^*_{st}(D) = \min\{w(f) \mid f \text{ is a TSTDF of } D\}$.

Let D be digraph with $\min\{\delta^+(D), \delta^-(D)\} \geq 1$. A *minus total dominating function* (abbreviated MTDF) of D is a function $f : V \to \{-1, 0, 1\}$ such that $f(N^-(v)) \geq 1$ for every $v \in V$. The *minus total domination number* for a digraph D is

$$\gamma_{mt}(D) = \min\{w(f) \mid f \text{ is a MTDF of } D\}.$$

A $\gamma_{mt}(D)$*-function* is a MTDF of D of weight $\gamma_{mt}(D)$. The minus total domination number of a digraph was introduced by Li *et al.* [10]. We define a *twin minus total dominating function* of D as a minus total dominating function of both D and D^{-1}, i.e., $f(N^-(v)) \geq 1$ and $f(N^+(v)) \geq 1$ for every $v \in V$. The *twin minus total domination number* for a digraph D is $\gamma^*_{mt}(D) = \min\{w(f) \mid f \text{ is a TMTDF of } D\}$. As the assumption $\delta^-(D), \delta^+(D) \geq 1$ is necessary, we always assume that when we discuss $\gamma^*_{mt}(D)$, all digraphs involved satisfy $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$.

Let G be a graph with vertex V and edge set E. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A *minus total dominating* function of G, introduced by Harris *et al.* [7], is a function $f : V \to \{-1, 0, 1\}$ such that $f(N(v)) \geq 1$ for every $v \in V$. The *minus total domination number* of G, denoted by $\gamma_{mt}(G)$, is the minimum weight of a minus total dominating function on G. The minus total domination number in graphs and its related parameters was studied by several authors, for example [8, 9, 11, 15].

For any function $f : V(D) \to \{-1, 0, 1\}$, on a digraph D, we define $P = P_f = \{v \in V \mid f(v) = 1\}$, $Z = Z_f = \{v \in V \mid f(v) = 0\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every TMTDF of D is a MTDF on both D and D^{-1} and since the constant function 1 is a TMTDF of D, we have

$$\max\{\gamma_{mt}(D), \gamma_{mt}(D^{-1})\} \leq \gamma^*_{mt}(D) \leq |V(D)|. \tag{1}$$

Since every TSTDF of a digraph D is a TMTDF, we have

$$\gamma^*_{mt}(D) \leq \gamma^*_{st}(D). \tag{2}$$

In this paper, we initiate the study of the twin minus total domination number in digraphs and we present some lower bounds on this parameter.

2. Basic Properties

In this section, we present basic properties of the twin minus total domination number. By (1), $\gamma^*_{mt}(D) \leq |V(D)|$ for any digraph D. The next proposition provides conditions to establish the equality.

Proposition 1. *Let D be a digraph of order n. Then $\gamma^*_{mt}(D) = n$ if and only if every vertex has either an out-neighbor with in-degree at most* 1 *or an in-neighbor with out-degree at most* 1.

Proof. The sufficiency is clear. Thus, we verify the necessity of the condition. Assume to the contrary that there exists a vertex $v \in V(D)$ such that $d^-(u) \geq 2$

for each $u \in N^+(v)$ and $d^+(w) \geq 2$ for each $w \in N^-(v)$. Define $f : V(D) \to \{-1, 0, 1\}$ by $f(v) = 0$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a twin minus total dominating function of D of weight less than n, a contradiction. This completes the proof. ∎

The next result is an immediate consequence of Proposition 1.

Corollary 2. *If $\overrightarrow{C}_n$ is the directed cycle on n vertices, then $\gamma^*_{mt}(\overrightarrow{C}_n) = n$.*

Now we show that the twin minus total domination and also the twin signed total domination number of digraphs can be arbitrarily small.

Theorem 3. *For any positive integer $k \geq 2$, there exists a digraph D such that*

$$\gamma^*_{mt}(D) \leq 6k - 4k^2.$$

Proof. Let $k \geq 2$ be an integer and D be a digraph obtained from a complete digraph of order $2k$ with vertex set $V(K^*_{2k}) = \{u_{i_1}, u_{i_2} \mid 1 \leq i \leq k\}$ by adding the set $\{v_{i_j}, w_{i_j} \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq 2k-2\}$ of new vertices and the set

$$\{(v_{i_j}, u_{i_1}), (u_{i_1}, w_{i_j}), (w_{i_j}, u_{i_2}), (u_{i_2}, v_{i_j}) \mid 1 \leq i \leq k,\ 1 \leq j \leq 2k-2\}$$

of new arcs. It is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(x) = 1$ for $x \in \{u_{i_1}, u_{i_2} \mid 1 \leq i \leq k\}$ and $f(x) = -1$ otherwise, is a TMTDF of D and so $\gamma^*_{mt}(D) \leq 6k - 4k^2$. ∎

The function defined in the proof of Theorem 3 is also a TSTDF of D and so $\gamma^*_{st}(D) \leq 6k - 4k^2$. Then the twin signed total domination number of digraphs can be arbitrarily small.

As we observed in (1), $\gamma^*_{mt}(D) \geq \max\{\gamma_{mt}(D), \gamma_{mt}(D^{-1})\}$. Now we show that the difference $\gamma^*_{mt}(D) - \max\{\gamma_{mt}(D), \gamma_{mt}(D^{-1})\}$ can be arbitrarily large.

Theorem 4. *For every positive integer $k \geq 3$, there exists a digraph D such that*

$$\gamma^*_{mt}(D) - \max\{\gamma_{mt}(D), \gamma_{mt}(D^{-1})\} \geq k.$$

Proof. Let $k \geq 3$ be an integer and D be a digraph obtained from the directed cycle $\overrightarrow{C}_k = (v_1, \ldots, v_k)$ by adding new vertices u_i, $1 \leq i \leq 2k$, and arcs $\{(v_i, u_i), (u_i, u_{k+i}), (u_{k+i}, v_i) \mid 1 \leq i \leq k\}$. Then the order of D is $n = 3k$. Obviously, $D \cong D^{-1}$ and so, $\gamma_{mt}(D) = \gamma_{mt}(D^{-1})$. By Proposition 1, $\gamma^*_{mt}(D) = n$. On the other hand, it is easy to verify that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(x) = 1$ for $x \in \{u_i, v_i \mid 1 \leq i \leq k\}$ and $f(x) = 0$ otherwise, is a MTDF of D and so $\gamma_{mt}(D) \leq 2k$. This implies that $\gamma^*_{mt}(D) - \max\{\gamma_{mt}(D), \gamma_{mt}(D^{-1})\} \geq k$ and the proof is complete. ∎

A *tournament* is a digraph in which for every pair u and v of different vertices, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but not both. Next we determine the exact value of the twin minus total domination number for a particular type of tournaments.

Let $n = 2r + 1$ for some positive integer r. We define the *circulant tournament* $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$ and for each i, the arcs go from u_i to the vertices $u_{i+1}, \ldots, u_{i+r}$ where the sum being taken modulo n.

The proof of the next result can be found in [10].

Proposition 5. *For $n \geq 3$, $\gamma_{mt}(\mathrm{CT}(n)) = 3$.*

The next proposition shows that $\gamma^*_{mt}(\mathrm{CT}(n)) = \gamma_{mt}(\mathrm{CT}(n))$

Proposition 6. *Let $n \geq 3$ and $n = 2r + 1$ where r is a positive integer. Then $\gamma^*_{mt}(\mathrm{CT}(n)) = \gamma_{mt}(\mathrm{CT}(n))$.*

Proof. By (1) and Proposition 5, we have $\gamma^*_{mt}(\mathrm{CT}(n)) \geq 3$. On the other hand, the function $f : V(\mathrm{CT}(n)) \to \{-1, 0, 1\}$ defined by $f(u_0) = f(u_r) = f(u_{2r}) = 1$ and $f(x) = 0$ otherwise, is TMTDF of $\mathrm{CT}(n)$ of weight 3. This completes the proof. ■

As we observed in (2), $\gamma^*_{st}(D) \geq \gamma^*_{mt}(D)$. Next we show that $\gamma^*_{st}(D) - \gamma^*_{mt}(D)$ can be arbitrarily large.

The proof of the following proposition can be found in [1].

Proposition 7. *Let D be a digraph of order n with $\delta^+(D), \delta^-(D) \geq 1$. Then $\gamma^*_{st}(D) = n$ if and only if every vertex has either an out-neighbor with indegree at most 2 or an in-neighbor with outdegree at most 2.*

Theorem 8. *For every positive integer k, there exists a digraph D such that*

$$\gamma^*_{st}(D) - \gamma^*_{mt}(D) \geq 2k.$$

Proof. Let $k \geq 1$ be an integer and for $1 \leq j \leq k$, let D_j be a circulant tournament $\mathrm{CT}(5)$ with vertex set $\{u_{i_j} \mid 1 \leq i \leq 5\}$. Let D be a digraph obtained from the union of D_j's by adding the set $\{(u_{3_1}, u_{3_2}), (u_{3_2}, u_{3_3}), \ldots, (u_{3_{k-1}}, u_{3_k}), (u_{3_k}, u_{3_1})\}$ of new arcs. Then the order of D is $5k$. By Proposition 7, $\gamma^*_{st}(D) = 5k$. On the other hand, it is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(u_{1_j}) = f(u_{3_j}) = f(u_{5_j}) = 1$ and $f(x) = 0$ otherwise, is a TMTDF of D and so $\gamma^*_{mt}(D) \leq 3k$. It follows that $\gamma^*_{st}(D) - \gamma^*_{mt}(D) \geq 2k$ and the proof is complete. ■

3. Lower Bounds on $\gamma^*_{mt}(D)$

In this section we present some lower bounds for $\gamma^*_{mt}(D)$ in terms of the order, size, the maximum and minimum in-degrees and out-degrees of D. We begin with some results on the minus total domination number of a digraph.

Observation 9. *Let f be any $\gamma_{mt}(D)$-function of a digraph D of order n. Then*
1. $n = |M| + |P| + |Z|$.
2. $w(f) = |P| - |M|$.

Theorem 10. *Let f be an MTDF on a digraph D of order n. If $\Delta^+ = \Delta^+(D)$ and $\delta^+ = \delta^+(D)$, then*
(a) $(\Delta^+ - 1)|P| \geq (\delta^+ + 1)|M| + |Z|$.
(b) $(\Delta^+ + \delta^+)|P| + \delta^+|Z| \geq (\delta^+ + 1)n$.
(c) $\delta^+ w(f) \geq (\delta^+ - \Delta^+)|P| + n$.
(d) $w(f) \geq \frac{2\delta^+ - \Delta^+ + 1}{\Delta^+ - \delta^+} n + |P|$.

Proof. (a) It follows from Observation 9 (part 1) that

$$\begin{aligned}|P| + |M| + |Z| &= n \leq \sum_{v \in V} \sum_{x \in N^-(v)} f(x) = \sum_{v \in V} d^+(v) f(v) \\ &= \sum_{v \in P} d^+(v) - \sum_{v \in M} d^+(v) \leq \Delta^+|P| - \delta^+|M|.\end{aligned}$$

This inequality chain yields to the desired bound in (a).

(b) Observation 9 (part 1) implies that $|M| = n - |P| - |Z|$. Using this identity and part (a), we arrive at (b).

(c) According to Observation 9 and part (b), we obtain part (c) as follows:

$$w(f) = 2|P| - n + |Z|$$

and

$$\begin{aligned}\delta^+ w(f) &= \delta^+(2|P| - n + |Z|) = (\Delta^+ + \delta^+)|P| + (\delta^+ - \Delta^+)|P| - \delta^+ n + \delta^+|Z| \\ &\geq (\delta^+ - \Delta^+)|P| - \delta^+ n + (\delta^+ + 1)n = (\delta^+ - \Delta^+)|P| + n.\end{aligned}$$

(d) The inequality chain in the proof of part (a) and Observation 9 (part 1) show that

$$n \leq \Delta^+|P \cup Z| - \delta^+(n - |P \cup Z|) = (\Delta^+ - \delta^+)|P \cup Z| - \delta^+ n$$

and so

$$|P \cup Z| \geq \frac{\delta^+ + 1}{\Delta^+ - \delta^+} n.$$

Using this inequality and Observation 9, we obtain

$$w(f) = |P| - n + |P \cup Z| \geq \frac{\delta^+ + 1}{\Delta^+ - \delta^+}n - n + |P| = \frac{2\delta^+ - \Delta^+ + 1}{\Delta^+ - \delta^+}n + |P|.$$

This is the bound in part (d), and the proof is complete. ∎

Corollary 11. *Let D be a digraph of order n, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then*

$$\gamma_{mt}(D) \geq \frac{2\delta^+ - \Delta^+ + 2}{\Delta^+}n.$$

Proof. Multiplying both sides of the inequality in Theorem 10 (d) by $(\Delta^+ - \delta^+)$ and adding the resulting inequality to the inequality in Theorem 10 (c), we obtain the desired lower bound. ∎

Since $\delta^+(D^{-1}) = \delta^-(D)$ and $\Delta^+(D^{-1}) = \Delta^-(D)$ for any digraph D, Corollary 11 implies the following corollary.

Corollary 12. *Let D be a digraph of order n, minimum in-degree δ^- and maximum in-degree Δ^-. If $\delta^- < \Delta^-$, then*

$$\gamma_{mt}(D^{-1}) \geq \frac{2\delta^- - \Delta^- + 2}{\Delta^-}n.$$

The next corollary is a consequence of (1) and Corollaries 11 and 12.

Corollary 13. *Let D be a digraph of order n, minimum in-degree δ^-, maximum in-degree Δ^-, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^- < \Delta^-$ and $\delta^+ < \Delta^+$, then*

$$\gamma^*_{mt}(D) \geq \max\left\{\frac{2\delta^+ - \Delta^+ + 2}{\Delta^+}n, \frac{2\delta^- - \Delta^- + 2}{\Delta^-}n\right\}.$$

Proposition 14. *Let D be a digraph of order $n \geq 3$. Then $\gamma^*_{mt}(D) \geq 6 - n$ with equality if and only if $D = \overrightarrow{C}_3$.*

Proof. Let f be a $\gamma^*_{mt}(D)$-function. Since D is a simple digraph, we have $|P \cap (N^+(v) \cup N^-(v))| \geq 2$ for every $v \in P$. This implies that $|P| \geq 3$ and so $|M| \leq n - 3$. Hence $\gamma^*_{mt}(D) \geq 6 - n$.

Let now D be a digraph such that $\gamma^*_{mt}(D) = 6 - n$ and f be a $\gamma^*_{mt}(D)$-function. Then $|P| = 3$, $|M| = n - 3$ and $|Z| = 0$. Also since D is a simple digraph, for every $v \in P$, $|N^-(v) \cap P| = |N^+(v) \cap P| = 1$ and so $|N^-(v) \cap M| = |N^+(v) \cap M| = 0$. It follows that $M = \emptyset$ and so $D = \overrightarrow{C}_3$. On the other hand, if $D = \overrightarrow{C}_3$, then $\gamma^*_{mt}(D) = 3 = 6 - n$. ∎

Proposition 15. *If D is a digraph with maximum out-degree $\Delta^+(D) \leq 4$, then $\gamma^*_{mt}(D) \geq 0$.*

Proof. Let f be a twin minus total dominating function for which $\omega(f) = \gamma^*_{mt}(D)$. If $M = \emptyset$, then the result follows. Assume that $M \neq \emptyset$. Since $f(N^-(v)) \geq 1$, for each $v \in M$, we have $|A(P, v)| \geq 1$. It follows that $|A(P, M)| \geq |M|$. On the other hand, since $f(N^+(v)) \geq 1$ for each $v \in P$, we have $|A(v, P)| \geq |A(v, M)| + 1$. Since $|A(v, P)| + |A(v, M)| = d^+(v) \leq \Delta^+(D) \leq 4$, it follows that $|A(v, M)| \leq 1$. Hence, we have $|A(P, M)| = \sum_{v \in P} |A(v, M)| \leq |P|$. Combining these, we have $|M| \leq |P|$, and so $\gamma^*_{mt}(D) = |P| - |M| \geq 0$. ∎

The condition $\Delta^-(D) \leq 4$, in the Proposition 15, implies the following proposition.

Proposition 16. *If D is a digraph with maximum in-degree $\Delta^-(D) \leq 4$, then $\gamma^*_{mt}(D) \geq 0$.*

A twin minus total dominating function f of D is called *minimal* if there exists no twin minus total dominating function f' of D such that $f' \neq f$ and $f'(v) \leq f(v)$ for every $v \in V(D)$.

Proposition 17. *A twin minus total dominating function f on a digraph D is minimal if and only if for every vertex $v \in V$ with $f(v) \geq 0$, there exists a vertex $u \in N^+(v)$ with $f(N^-(u)) = 1$ or there exists a vertex $w \in N^-(v)$ with $f(N^+(w)) = 1$.*

Proof. Let f be a minimal twin minus total dominating function and assume that there is a vertex $v \in V$ with $f(v) \geq 0$, $f(N^+(u)) > 1$ for every $u \in N^-(v)$ and $f(N^-(w)) > 1$ for every $w \in N^+(v)$. Define a new function $g : V \to \{-1, 0, 1\}$ by $g(v) = f(v) - 1$ and $g(x) = f(x)$ for all $x \neq v$. Then for all $u \in N^+(v)$, $g(N^+(u)) = f(N^+(u)) - 1 \geq 1$, $g(N^-(u)) = f(N^-(u)) \geq 1$, for all $w \in N^-(v)$, $g(N^+(w)) = f(N^+(w)) \geq 1$, $g(N^-(w)) = f(N^-(w)) - 1 \geq 1$ and for $z \notin N^+(v) \cup N^-(v)$, $g(N^+(z)) = f(N^+(z)) \geq 1$ and $g(N^-(z)) = f(N^-(z)) \geq 1$. This implies that g is a twin minus total dominating function of D, contradiction to the minimality of f.

Conversely, let f be a twin minus total dominating function such that for all $v \in V$ with $f(v) \geq 0$, there exists a vertex $u \in N^+(v)$ with $f(N^-(u)) = 1$ or there exists a vertex $w \in N^-(v)$ with $f(N^+(w)) = 1$. Assume that f is not minimal, i.e., there is a twin minus total dominating function g such that $g \neq f$ and $g(x) \leq f(x)$ for all $x \in V$. Then there is at least one $v \in V$ with $g(v) < f(v)$. It follows that $f(v) \geq 0$, and by assumption, there exists a vertex $u \in N^+(v)$ with $f(N^-(u)) = 1$ or there exists a vertex $w \in N^-(v)$ with $f(N^+(w)) = 1$. Since $g(x) \leq f(x)$ for all $x \in V$ and $g(v) < f(v)$, we have $g(N^-(u)) < f(N^-(u)) = 1$ or $g(N^+(w)) < f(N^+(w)) = 1$. This contradicts the fact that g is a twin minus total dominating function. Hence f is a minimal twin minus total dominating function and this completes the proof. ■

Theorem 18. *Let D be a digraph of order n and size m. Then*

$$\gamma^*_{mt}(D) \geq 2n - m,$$

and this bound is sharp.

Proof. Let f be a $\gamma^*_{mt}(D)$-function. For any $v \in M$, we have $|A(v, P)| \geq 1$ and $|A(P, v)| \geq 1$ which implies that $|A(M, P)| \geq |M|$ and $|A(P, M)| \geq |M|$. Also for any $v \in Z$, we have $|A(v, P)| \geq 1$ and $|A(P, v)| \geq 1$ which implies that $|A(Z, P)| \geq |Z|$ and $|A(P, Z)| \geq |Z|$. On the other hand, if $x \in P$, then it follows from $f(N^+(x)) \geq 1$ that $|A(x, P)| \geq |A(x, M)| + 1$ implying that $|A(P, P)| \geq |A(P, M)| + |P| \geq |M| + |P|$. Therefore,

$$\begin{aligned} m &\geq |A(M, P)| + |A(P, M)| + |A(Z, P)| + |A(P, Z)| + |A(P, P)| \\ &\geq 2|M| + 2|Z| + |M| + |P| = 2|M| + 2|Z| + n - |Z| = n + 2|M| + |Z|. \end{aligned} \tag{3}$$

Hence, we have

$$\gamma^*_{mt}(D) = w(f) = |P| - |M| = n - 2|M| - |Z| \geq 2n - m.$$

To prove the sharpness, suppose that $\overrightarrow{C}_t$ is a directed cycle of order $t \geq 3$. Let D be a digraph obtained from $\overrightarrow{C}_t$ by adding the set $\{u_1, \ldots, u_k \mid \ k \geq 1\}$ of new vertices and arcs from vertices of $\overrightarrow{C}_t$ to k new vertices and from k new vertices to vertices of $\overrightarrow{C}_t$ such that $d^+_D(v) = d^-_D(v) = 1$ for every new vertex v. Then the order of D is $n = t + k$ and the size of D is $m = t + 2k$. So $2n - m = t$. Now define $f : V(D) \to \{-1, 0, 1\}$ which assigns $f(x) = 1$ for $x \in V(\overrightarrow{C}_t)$ and $f(x) = 0$ otherwise. Obviously, f is a TMTDF of D and $\omega(f) = t$. This completes the proof. ■

A set $S \subseteq V(G)$ is a 2-*packing* if for each pair of vertices $x, y \in S$, $N[x] \cap N[y] = \emptyset$. The 2-*packing number* $\rho(G)$ is the cardinality of a maximum 2-packing.

Proposition 19. *Let G be a graph of order n with minimum degree $\delta \geq 2$ and let D be an orientation of G such that $\delta^+(D) \geq 1$, $\delta^-(D) \geq 1$. Then*

$$\gamma^*_{mt}(D) \geq \rho(G)(\delta + 1) - n.$$

Proof. Let S be a maximum 2-packing of G and f be a $\gamma^*_{mt}(D)$-function. Since $f(N^+(v)) \geq 1$ and $f(N^-(v)) \geq 1$, we have $f(N_G(v)) = f(N^+(v)) + f(N^-(v)) \geq 2$ for each $v \in S$. This implies that

$$\begin{aligned}\gamma^*_{mt}(D) &= \sum_{v\in S} f(N_G(v)) + \sum_{v\in V(G)-N_G(S)} f(v)\\ &\geq |S| + \sum_{v\in V(G)-N_G(S)} (-1) \geq |S| - (n - |S|\delta) = \rho(G)(\delta+1) - n,\end{aligned}$$

and the proof is complete. ■

The next theorem presents a lower bound on twin signed total domination numbers in a digraph in terms of its order.

Theorem 20. *Let D be a digraph of order n. Then*

$$\gamma^*_{st}(D) \geq 1 + \sqrt{1+4n} - n.$$

Proof. Let f be a $\gamma^*_{st}(D)$-function. For every $v \in M$, since $f(N^+(v)) \geq 1$, we have $|A(v,P)| \geq 1$ and thus $|A(M,P)| \geq |M|$. If $x \in P$, then it follows from $f(N^+(x)) \geq 1$ that $|A(x,P)| \geq |A(x,M)| + 1$. This implies that

$$|A(P,P)| \geq |A(P,M)| + |P| \geq |M| + (n - |M|) = n.$$

On the other hand, $|A(P,P)| \leq |P|(|P|-1)$. It follows that $|P|(|P|-1) \geq n$ and so $|P|^2 - |P| - n \geq 0$. This implies that

$$|P| \geq \frac{1+\sqrt{4n+1}}{2},$$

and thus we obtain

$$\gamma^*_{st}(D) = 2|P| - n \geq 1 + \sqrt{4n+1} - n.$$

■

The next theorem presents a lower bound on twin signed total domination numbers in a bipartite digraph in terms of its order.

Theorem 21. *Let D be a bipartite digraph of order n. Then*

$$\gamma^*_{st}(D) \geq 2\sqrt{2n} - n.$$

Proof. Let f be a $\gamma^*_{st}(D)$-function. In view of the proof of Theorem 20, $|A(P,P)| \geq n$. Since the subdigraph induced by P is bipartite, we have $|A(P,P)| \leq |P|^2/2$. It follows that $|P|^2/2 \geq n$ and so $|P| \geq \sqrt{2n}$. This implies that

$$\gamma^*_{st}(D) = 2|P| - n \geq 2\sqrt{2n} - n.$$

■

Next, we present lower bounds on twin minus total domination numbers in digraphs in terms of their orders.

Theorem 22. *Let D be a digraph of order n. Then*

$$\gamma^*_{mt}(D) \geq 1 + \sqrt{1+4n} - n.$$

Proof. Let f be a $\gamma^*_{mt}(D)$-function. If $Z = \emptyset$, then f is a TSTDF on D and by Theorem 20, $\gamma^*_{mt}(D) = w(f) \geq \gamma^*_{st}(D) \geq 1 + \sqrt{1+4n} - n$. Suppose $Z \neq \emptyset$. Let $n_1 = n - |Z|$ and D_1 be a subdigraph of D induced by the set $V(D) - Z$. Then $f|_{V(D_1)}$ is a TSTDF on D_1 and by Theorem 20, $\gamma^*_{mt}(D) = w(f) \geq \gamma^*_{st}(D_1) \geq 1+\sqrt{1+4n_1}-n_1$. Now we can easily see that the function $g(x) = 1+\sqrt{1+4x}-x$ is a non increasing function for any integer $x \geq 2$ and so $g(n_1) \geq g(n)$. This implies that $\gamma^*_{mt}(D) \geq 1 + \sqrt{1+4n_1} - n_1 \geq 1 + \sqrt{1+4n} - n$. ■

Theorem 23. *Let D be a bipartite digraph of order n. Then*

$$\gamma^*_{mt}(D) \geq 2\sqrt{2n} - n.$$

Proof. Let f be a $\gamma^*_{mt}(D)$-function. If $Z = \emptyset$, then f is a TSTDF on D and by Theorem 21, $\gamma^*_{mt}(D) = w(f) \geq \gamma^*_{st}(D) \geq 2\sqrt{2n} - n$. Suppose $Z \neq \emptyset$. Let $n_1 = n - |Z|$ and D_1 be a subdigraph of D induced by the set $V(D) - Z$. Then $f|_{V(D_1)}$ is a TSTDF on D_1 and by Theorem 21, $\gamma^*_{mt}(D) = w(f) \geq \gamma^*_{st}(D_1) \geq 2\sqrt{2n_1} - n_1$. Now we can easily see that the function $g(x) = 2\sqrt{2x} - x$ is a non increasing function for any integer $x \geq 2$ and so $g(n_1) \geq g(n)$. This implies that $\gamma^*_{mt}(D) \geq 2\sqrt{2n_1} - n_1 \geq 2\sqrt{2n} - n$. ■

The *associated digraph* $D(G)$ of a graph G is the digraph obtained in such a way that each edge e of G is replaced by two oppositely oriented arcs with the same end vertices as e. Since $N^-_{D(G)}(v) = N^+_{D(G)}(v) = N_G(v)$ for each $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 24. *If $D(G)$ is the associated digraph of a graph G, then $\gamma^*_{mt}(D(G)) = \gamma_{mt}(G)$.*

Theorems 22, 23 and Observation 24 lead to the next well-known result.

Corollary 25 [14]**.** *If G is a graph of order n, then $\gamma_{mt}(G) \geq \sqrt{1+4n} + 1 - n$. If G is a bipartite graph of order n, then $\gamma_{mt}(G) \geq 2\sqrt{2n} - n$.*

Xing *et al.* [14] have presented examples with equality in the two inequalities of Corollary 25. The associated digraphs of these examples show that Theorems 22 and 23 are both sharp.

4. Twin Minus Total Domination in Oriented Graphs

Let G be the complete bipartite graph $K_{4,4}$ with bipartite sets $V_1 = \{v_1, v_2, v_3, v_4\}$ and $V_2 = \{u_1, u_2, u_3, u_4\}$. Let D_1 be a 2-regular oriented graph of G and D_2 be an orientation of G such that $A(D_2) = \{(v_i, u_i), (u_j, v_r) \mid 1 \leq i, j, r \leq 4 \text{ and } j \neq r\}$. It is easy to see that $\gamma^*_{mt}(D_1) = 4$ and $\gamma^*_{mt}(D_2) = 8$. Thus two distinct orientations of a graph can have distinct twin minus total domination numbers. Motivated by this observation, we define *lower orientable twin minus total domination number* $\mathrm{dom}^*_{mt}(G)$ and *upper orientable twin minus total domination number* $\mathrm{Dom}^*_{mt}(G)$ of a graph G with $\delta(G) \geq 2$ as follows:

$$\mathrm{dom}^*_{mt}(G) = \min\{\gamma^*_{mt}(D) \mid \text{D is an orientation of } G\},$$

and

$$\mathrm{Dom}^*_{mt}(G) = \max\{\gamma^*_{mt}(D) \mid \text{D is an orientation of } G\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) [6], twin domination number [5], twin signed domination number [3], twin signed total domination number [1], twin minus domination number [2] and twin signed Roman domination number [4].

Note that the definitions are well-defined because every graph G with $\delta(G) \geq 2$, has an orientation D such that $\delta^+(D), \delta^-(D) \geq 1$. Since for any orientation D of a graph G, $\gamma^*_{mt}(D) \leq \gamma^*_{st}(D)$, we have

$$\mathrm{dom}^*_{mt}(G) \leq \mathrm{dom}^*_{st}(G) \tag{4}$$

Proposition 26. *For any graph G of order n with $\delta(G) \geq 2$, $\gamma_{mt}(G) \leq \mathrm{dom}^*_{mt}(G)$.*

Proof. Let D be an orientation of G and let f be a $\gamma^*_{mt}(D)$-function. Then $f(N_G(v)) = f(N^+_D(v)) + f(N^-_D(v))$ for each $v \in V$. Since $f(N^+_D(v)) \geq 1$ and $f(N^-_D(v)) \geq 1$, we have $f(N_G(v)) \geq 2$ for each $v \in V$, and so f is a MTDF of G. Therefore $\gamma_{mt}(G) \leq w(f) = \mathrm{dom}^*_{mt}(G)$ as desired. ■

The proof of the next result is straightforward and therefore omitted.

Proposition 27. *Let G be a graph of order n and $v \in V(G)$. If $\deg(u) \leq 3$, for some $u \in N_G(v)$, then for any orientation D of G and any $\gamma^*_{mt}(D)$-function f, we have $f(v) \geq 0$.*

Proposition 28. *For $n \geq 3$, $\mathrm{dom}^*_{mt}(C_n) = n$.*

Proof. If D is an orientation of C_n with $\delta^+(D) \geq 1$ and $\delta^-(D) \geq 1$, then obviously D is a directed cycle and the result follows from Corollary 2. ■

We now proceed to determine the lower orientable twin minus total domination numbers of several classes of graphs including complete graphs, complete bipartite graphs and wheels.

Lemma 29. *For* $n \geq 3$, $\mathrm{dom}^*_{mt}(K_n) \geq 3$.

Proof. The result is immediate for $n = 3$. Let $n \geq 4$, D be an orientation of K_n and let f be a $\gamma^*_{mt}(D)$-function. Assume that $v \in P$. Since $f(N^+_D(v)) \geq 1$ and $f(N^-_D(v)) \geq 1$, we have

$$\mathrm{dom}^*_{mt}(K_n) = w(f) = f(N^+_D(v)) + f(N^-_D(v)) + f(v) \geq 3,$$

as desired. ■

Theorem 30. *For* $n \geq 3$, $\mathrm{dom}^*_{mt}(K_n) = 3$.

Proof. The result is immediate for $n = 3$, so assume $n \geq 4$. Let D_1 be an orientation of K_{n-3} with vertex set $V(K_{n-3}) = \{u_i \mid 1 \leq i \leq n-3\}$. Suppose that D is obtained from D_1 by adding the set $\{v_1, v_2, v_3\}$ of new vertices and the set

$$\{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_1, u_i), (v_2, u_i), (u_i, v_3) \mid\ 1 \leq i \leq n-3\}$$

of new arcs. Then D is an orientation of K_n. It is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(v_1) = f(v_2) = f(v_3) = 1$ and $f(x) = 0$ otherwise, is a TMTDF of D of weight 3. This implies that $\mathrm{dom}^*_{mt}(K_n) \leq w(f) = 3$. Now the result follows from Lemma 29. ■

Lemma 31. *For* $m, n \geq 2$, $\mathrm{dom}^*_{mt}(K_{m,n}) \geq 4$.

Proof. Let $V(K_{m,n}) = X \cup Y$. Let D be an orientation of $K_{m,n}$ and let f be a $\gamma^*_{mt}(D)$-function. Assume that $v \in X$ and $u \in Y$. Since $f(N^+_D(v)) \geq 1$ and $f(N^-_D(v)) \geq 1$, and since $Y = f(N^+_D(v)) \cup f(N^-_D(v))$, $f(Y) \geq 2$. Similarly, $f(X) \geq 2$. It follows that

$$\mathrm{dom}^*_{mt}(K_{m,n}) = w(f) = f(X) + f(Y) \geq 4,$$

as desired. ■

Theorem 32. *For* $m, n \geq 2$, $\mathrm{dom}^*_{mt}(K_{m,n}) = 4$.

Proof. Let D_1 be an orientation of $K_{m-2,n-2}$ with vertex set $V(K_{m-2,n-2}) = \{u_i, v_j \mid 1 \leq i \leq m-2, 1 \leq j \leq n-2\}$ and suppose that D is obtained from D_1 by adding the set $\{w_1, w_2, w_3, w_4\}$ of new vertices and the set $\{(w_1, w_2), (w_2, w_3), (w_3, w_4), (w_4, w_1), (w_1, u_i), (u_i, w_3), (w_2, v_j), (v_j, w_4) \mid 1 \leq i \leq m-2, 1 \leq j \leq n-2\}$ of new arcs. Then D is an orientation of $K_{m,n}$. It is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(w_k) = 1$ for $1 \leq k \leq 4$ and $f(x) = 0$ otherwise, is a TMTDF of D of weight 4. This implies that $\mathrm{dom}^*_{mt}(K_{m,n}) \leq w(f) = 4$. Now the result follows from Lemma 31. ■

The *wheel*, W_n, is a graph with vertex set $\{w, v_0, \ldots, v_{n-1}\}$ and edge set $\{wv_i, v_iv_{i+1} \mid 0 \le i \le n-1\}$ where the indices are taken modulo n. Next we determine the lower orientable twin minus total domination number of wheels.

Lemma 33. *For $n \ge 3$,*

$$\mathrm{dom}^*_{mt}(W_n) \ge \begin{cases} \lceil \frac{n+3}{2} \rceil & n \equiv 2 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & n \equiv 0 \pmod 4, \\ \lceil \frac{n+2}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let D be an orientation of W_n with $\gamma^*_{mt}(D) = \mathrm{dom}^*_{mt}(W_n)$ and f be a $\gamma^*_{mt}(D)$-function. It follows from Proposition 27 that $f(v_i) \ge 0$ for each $0 \le i \le n-1$. Also since for each $0 \le i \le n-1$, $d^+(v_i) = 1$ or $d^-(v_i) = 1$, then $f(w) \ge 0$. If $f(w) = 0$, then $f(v_i) = 1$ for each $0 \le i \le n-1$ and so

$$\mathrm{dom}^*_{mt}(W_n) = w(f) = n \ge \begin{cases} \lceil \frac{n+3}{2} \rceil & n \equiv 2 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & n \equiv 0 \pmod 4, \\ \lceil \frac{n+2}{2} \rceil & \text{otherwise.} \end{cases}$$

Let $f(w) = 1$. Suppose that $v_i \in P$ for some $0 \le i \le n-1$. Since $d^+(v_i) = 1$ or $d^-(v_i) = 1$, then $f(v_{i-1}) = 1$ or $f(v_{i+1}) = 1$. Also it is easy to see that $f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(v_{i+3}) \ge 2$, for each $0 \le i \le n-1$. It follows that $|\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap P| \ge 2$, for each $0 \le i \le n-1$. Summing the above inequalities for each $0 \le i \le n-1$, we have $|P| \ge |Z| + 3$ when $n \equiv 2 \pmod 4$, $|P| \ge |Z| + 1$ when $n \equiv 0 \pmod 4$ and $|P| \ge |Z| + 2$ otherwise. It follows that

$$\mathrm{dom}^*_{mt}(W_n) = w(f) = |P| \ge \begin{cases} \lceil \frac{n+3}{2} \rceil & n \equiv 2 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & n \equiv 0 \pmod 4, \\ \lceil \frac{n+2}{2} \rceil & \text{otherwise.} \end{cases}$$ ■

Theorem 34. *For $n \ge 3$,*

$$\mathrm{dom}^*_{mt}(W_n) = \begin{cases} \lceil \frac{n+3}{2} \rceil & n \equiv 2 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & n \equiv 0 \pmod 4, \\ \lceil \frac{n+2}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let D be an orientation of W_n such that

$$A(D) = \big\{(v_i, v_{i+1}), (w, v_{4j}), (v_{4j+1}, w), (v_{4j+2}, w), (w, v_{4j+3}) \mid 0 \le i \le n-1, \\ 0 \le j \le \lfloor \tfrac{n}{4} \rfloor \big\}.$$

It is easy to verify that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(x) = 1$ for $x \in \big\{w, v_{4j}, v_{4j+1} \mid 0 \le j \le \lfloor \frac{n}{4} \rfloor \big\}$ and $f(x) = 0$ otherwise, is a TMTDF of

D of weight $\lceil \frac{n+3}{2} \rceil$ when $n \equiv 2 \pmod 4$, $\lceil \frac{n+1}{2} \rceil$ when $n \equiv 0 \pmod 4$ and $\lceil \frac{n+2}{2} \rceil$ otherwise. This implies that

$$\mathrm{dom}^*_{mt}(W_n) \leq w(f) = \begin{cases} \lceil \frac{n+3}{2} \rceil & n \equiv 2 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & n \equiv 0 \pmod 4, \\ \lceil \frac{n+2}{2} \rceil & \text{otherwise.} \end{cases}$$

Now the result follows from Lemma 33. ■

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