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PER-SPECTRAL CHARACTERIZATIONS OF SOME BIPARTITE GRAPHS

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Abstract

A graph is said to be characterized by its permanental spectrum if there is no other non-isomorphic graph with the same permanental spectrum. In this paper, we investigate when a complete bipartite graph $K_{p,p}$ with some edges deleted is determined by its permanental spectrum. We first prove that a graph obtained from $K_{p,p}$ by deleting all edges of a star $K_{1,l}$, provided $l < p$, is determined by its permanental spectrum. Furthermore, we show that all graphs with a perfect matching obtained from $K_{p,p}$ by removing five or fewer edges are determined by their permanental spectra.

Keywords: permanent, permanental polynomial, per-spectrum, cospectral.

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1. Introduction

By a *graph* we always mean a simple undirected graph G with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote by $\overline{G}$

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the complement of G. The *degree* of a vertex $v \in V(G)$ is denoted by $d_G(v)$, abbreviated as $d(v)$. Let $G - E(H)$ be a graph obtained from G by deleting the edges of H, where H is a subgraph of G. Let $G \cup H$ be the union of two graphs G and H which have no common vertices. For any positive integer l, let lG denote the union of l disjoint copies of graph G. The complete graph, path, cycle and star of order n are denoted by K_n, P_n, C_n and $K_{1,n-1}$, respectively. Let $c_i(G)$ and $p_i(G)$ denote respectively the number of i-cycles and i-vertex paths in G.

An *r-matching* in G is a set of r pairwise non-adjacent edges. The number of r-matchings in G is denoted by $q(G, r)$. For an r-matching M in G, if G has no r'-matching such that $r' > r$, then M is called a *maximum matching* of G. The number $\nu(G)$ of edges in a maximum matching is called the *matching number* of G.

The *permanent* of an $n \times n$ matrix X with entries x_{ij} $(i, j = 1, 2, \dots, n)$ is defined by

$$\mathrm{per}(X) = \sum_{\sigma} \prod_{i=1}^{n} x_{i\sigma(i)},$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$. Valiant [19] has shown that computing the permanent is #P-complete even when restricted to $(0, 1)$-matrices.

Let $A(G)$ be the adjacency matrix of G. The polynomial $\phi(G, x) = \det(xI - A(G))$, where I is the identity matrix, is called the *characteristic polynomial* of graph G. The *adjacency spectrum* of graph G consists of the eigenvalues of $A(G)$ together with their multiplicities. Similarly, the *permanental polynomial* of G, denoted by $\pi(G, x)$, is defined as $\pi(G, x) = \mathrm{per}(xI - A(G))$, where I is the identity matrix. The *permanental spectrum* (*per-spectrum for short*) of G is the collection of all roots (together with their multiplicities) of $\pi(G, x)$. The multiplicity of zeroes in the per-spectrum of G is called *permanental nullity* of G, denoted by $\eta_{per}(G)$.

The permanental polynomials of graphs was systematically introduced in mathematical and chemical literature almost simultaneously by Merris *et al.* [18] and Kasum *et al.* [14]. For a period of time, little about the study of permanental polynomials seems to have been published. This may be due to the difficulty of computing $\mathrm{per}(xI - A(G))$. However, permanental polynomials and their applications have received a lot of attention from researchers in recent years. See, for example, [1, 2, 3, 5–8, 12, 21, 23, 24, 25], and the references therein.

Two graphs are *cospectral* (respectively *per-cospectral*) if they share the same adjacency spectrum (respectively per-spectrum). A graph G is said to be *determined by its per-spectrum* (DPS for short) if every graph per-cospectral with G is isomorphic to G.

For any graph polynomial, it is of interest to determine its ability to characterize graphs, see [9, 10]. Merris *et al.* [18] first found that the per-spectrum distinguishes the five cospectral graphs of [13]. And they stated that the per-spectrum seems a little better than the adjacency spectrum when it comes to distinguishing graphs which are not trees. Motivated by the Merris *et al.*'s statement, Liu and Zhang [15, 16] investigated paths, stars, cycles and lollipop graphs which are DPS. And they stated that graphs determined by the adjacency spectra are not necessarily determined by the permanental spectra. Up to now, only a few types of graphs with very special structures have been proved to be DPS, such as, all graphs which are obtained from a complete graph by removing six or fewer edges [22, 26], and complete bipartite graphs [20]. Furthermore, Borowiecki [4] showed that if G_1 and G_2 are bipartite graphs without cycles of length k, $k \equiv 0 \pmod 4$, then G_1 and G_2 are per-cospectral if and only if G_1 and G_2 are cospectral. Yan and Zhang [23] gave a method to construct infinitely many pairs of 2-connected bipartite graphs which are per-cospectral.

In this paper, we intend to investigate when a complete regular bipartite graph with some edges deleted is DPS. And we obtain the results as follows. We show that $K_{p,p} - E(K_{1,l})$ is DPS, and prove that all graphs with a perfect matching obtained from $K_{p,p}$ by removing at most five edges are DPS. If the restriction "a perfect matching" is canceled in some graphs above, then these graphs are not necessarily determined by their per-spectra.

2. Preliminaries

Zhang *et al.* [26] enumerated all graphs with at most five edges and no isolated vertices. It is not difficult to check that there exist exactly 37 non-isomorphic bipartite graphs in these graphs. Thus, up to isomorphism there exist exactly 37 bipartite graphs obtained from $K_{p,p}$ by removing five or fewer edges, where $p \geq 5$, which are labeled by G_i, $1 \leq i \leq 37$, and depicted in Figure 1.

Lovász gave a formula about the relation between $q(G, r)$ and $q(\overline{G}, i)$, which will play a key role in the proofs of our main results.

Lemma 2.1 [17]**.** *Let G be a simple graph with n vertices and $\overline{G}$ the complement of G. Then*

$$q(G,r) = \sum_{i=0}^{r} (-1)^i \binom{n-2i}{2r-2i} (2r-2i-1)!!\, q(\overline{G}, i),$$

where $s!! = s \times (s-2)!!$, and $(-1)!! = 0!! = 1$.

Lemma 2.2 [11]**.** *Let G be a bipartite graph with n vertices and m edges. Then*
(i) $q(G,2) = \binom{m}{2} - \sum_{v \in V(G)} \binom{d(v)}{2}$,
(ii) $q(G,3) = \binom{m}{3} - (m-2) \sum_{v \in V(G)} \binom{d(v)}{2} + 2 \sum_{v \in V(G)} \binom{d(v)}{3} + \sum_{uv \in E(G)} (d(u)-1)(d(v)-1)$.

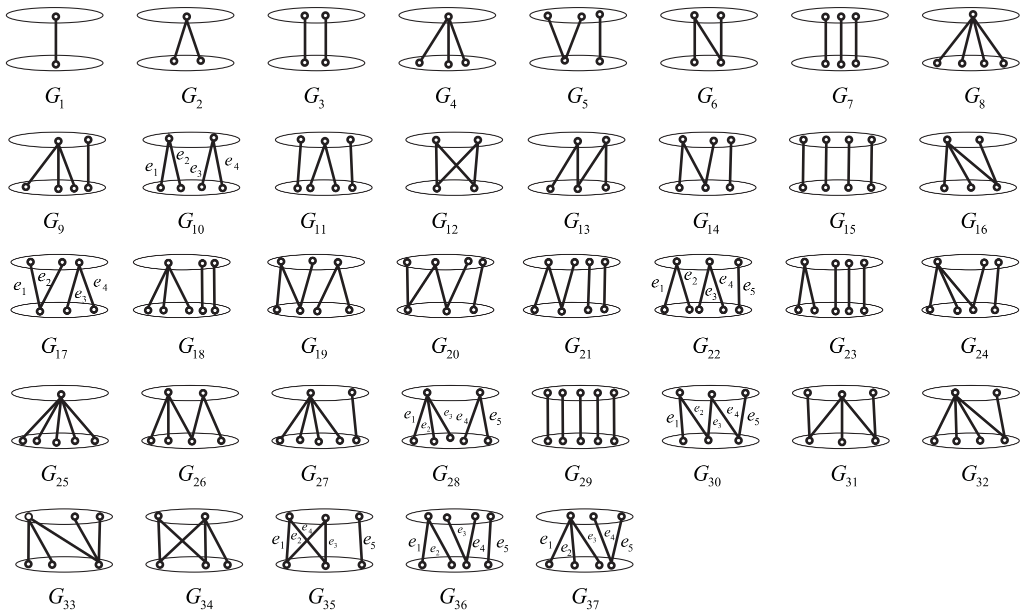

Figure 1. All graphs obtained from $K_{p,p}$ by deleting five or fewer edges drawn.

A subgraph H of a graph G is said to be a *Sachs graph* if each component of H is either a single edge or a cycle.

Lemma 2.3 [18]**.** *Let G be a graph with $\pi(G,x)=\sum_{k=0}^{n} b_k(G)x^{n-k}$. Then*

$$b_k(G)=(-1)^k\sum_{H} 2^{c(H)},\ 1\le k\le n,$$

where the sum is taken over all Sachs subgraphs H of G on k vertices, and $c(H)$ is the number of cycles in H.

Lemma 2.4 [15]**.** *Let G be a graph with n vertices and m edges, and let $(d_1,d_2,\ldots,d_n)$ be the degree sequence of G. Then*

(i) $b_0(G)=1$,
(ii) $b_1(G)=0$,
(iii) $b_2(G)=m$,
(iv) $b_3(G)=-2c_3(G)$,
(v) $b_4(G)=\binom{m}{2}-\sum_{i=1}^{n}\binom{d_i}{2}+2c_4(G)$.

Lemma 2.5. *Let G be a bipartite graph with m edges, and let $d_G(Q_i)$ denote the degree sum of four vertices which are on the ith quadrangle in G. Then*

$$b_6(G)=q(G,3)+2\left(\sum_{i=1}^{c_4(G)}(m+4-d_G(Q_i))+c_6(G)\right).$$

Proof. By the definition of Sachs subgraph, we get that the Sachs subgraphs of G on six vertices are of three kinds: $3K_2$, $C_4 \cup K_2$ and C_6. The number of $3K_2$ in G is equal to $q(G,3)$. For the ith quadrangle in G, there exist exactly $m+4-d_G(Q_i)$ edges each which is not incident to any vertex of the ith quadrangle. So, the number of $C_4 \cup K_2$ in G is equal to $\sum_{i=1}^{c_4(G)}(m+4-d_G(Q_i))$. It follows, by Lemma 2.3, that $b_6(G) = q(G,3) + 2\left(\sum_{i=1}^{c_4(G)}(m+4-d_G(Q_i)) + c_6(G)\right)$. ■

Lemma 2.6 [15]**.** *The following parameters and properties of a graph G can be deduced from the per-spectrum.*

(i) *The number of vertices.*
(ii) *The number of edges.*
(iii) *The number of triangles.*
(iv) *Whether G is bipartite.*

Lemma 2.7 [20]**.** *Let G be a bipartite graph with n vertices. Then $\eta_{per}(G) = n - 2\nu(G)$.*

Remark 2.8. Lemma 2.7 implies that if the matching numbers of two bipartite graphs are not equal, then the two bipartite graphs are not per-cospectral.

Zhang *et al.* [26] gave a formula to compute the number of 4-cycles in Lemma 2.7. From the proof of Lemma 2.7 of [26], we can easily obtain the following lemma.

Lemma 2.9. *Let $H \subseteq K_{p,p}$ be a bipartite graph with l edges and let $G = K_{p,p} - E(H)$. Then*

$$c_4(G) = \left[\binom{p}{2}\right]^2 - l(p-1)^2 + \binom{l}{2} + (p-2)\sum_{v\in V(H)}\binom{d(v)}{2} - p_4(H) + c_4(H).$$

By Lemma 2.9, we calculate the number of quadrangles of all graphs except for G_1 in Figure 1, as shown in Table 1.

Lemma 2.10. *Let $H \subseteq K_{p,p}$ be a bipartite graph with l edges, and let $G = K_{p,p} - E(H)$. Then*

$$\begin{aligned} c_6(G) = {} & 6\left[\binom{p}{3}\right]^2 - 4l\left[\binom{p-1}{2}\right]^2 + 3(p-2)^2\left(\binom{l}{2} - z(H)\right) + 2z(H)(p-2) \\ & \times \binom{p-1}{2} - p_4(H)(p-2)^2 - 2(p-2)\left(\sum_{i=1}^{z(H)}(l+2-d_i(P_3))\right) \qquad (1) \\ & - 2q(H,3) + 2y_1(H) + y_2(H) + p_5(H)(p-2) - p_6(H) + c_6(H), \end{aligned}$$

where $d_j(P_3)$ *denotes the degree sum of three vertices on the* jth P_3 *in* H, $z(H) = \sum_{v\in V(H)} \binom{d(v)}{2}$, $y_2(H)$ *denotes the number of* $P_4 \cup K_2$ *in* H, *and* $y_1(H)$ *denotes the number of* $2P_3$ *in* H *such that the vertices of degree two in* $2P_3$ *belong to different partite set of* H.

Graph	$c_4(G)$	Graph	$c_4(G)$	Graph	$c_4(G)$
G_2	$\binom{p}{2}^2 - 2p^2 + 5p - 3$	G_3	$\binom{p}{2}^2 - 2p^2 + 4p - 1$	G_4	$\binom{p}{2}^2 - 3p^2 + 9p - 6$
G_5	$\binom{p}{2}^2 - 3p^2 + 7p - 2$	G_6	$\binom{p}{2}^2 - 3p^2 + 8p - 5$	G_7	$\binom{p}{2}^2 - 3p^2 + 6p$
G_8	$\binom{p}{2}^2 - 4p^2 + 14p - 10$	G_9	$\binom{p}{2}^2 - 4p^2 + 11p - 4$	G_{10},G_{17}	$\binom{p}{2}^2 - 4p^2 + 10p - 2$
G_{11}	$\binom{p}{2}^2 - 4p^2 + 9p$	G_{12}	$\binom{p}{2}^2 - 4p^2 + 12p - 9$	G_{13}	$\binom{p}{2}^2 - 4p^2 + 11p - 6$
G_{14}	$\binom{p}{2}^2 - 4p^2 + 10p - 3$	G_{15}	$\binom{p}{2}^2 - 4p^2 + 8p + 2$	G_{16}	$\binom{p}{2}^2 - 4p^2 + 12p - 8$
G_{18}	$\binom{p}{2}^2 - 5p^2 + 13p - 1$	G_{19}	$\binom{p}{2}^2 - 5p^2 + 13p - 2$	G_{20}	$\binom{p}{2}^2 - 5p^2 + 13p - 3$
G_{21}	$\binom{p}{2}^2 - 5p^2 + 12p$	G_{22},G_{36}	$\binom{p}{2}^2 - 5p^2 + 12p + 1$	G_{23}	$\binom{p}{2}^2 - 5p^2 + 11p + 3$
G_{24}	$\binom{p}{2}^2 - 5p^2 + 14p - 5$	G_{25}	$\binom{p}{2}^2 - 5p^2 + 20p - 15$	G_{26}	$\binom{p}{2}^2 - 5p^2 + 15p - 8$
G_{27}	$\binom{p}{2}^2 - 5p^2 + 16p - 7$	G_{28},G_{37}	$\binom{p}{2}^2 - 5p^2 + 14p - 3$	G_{29}	$\binom{p}{2}^2 - 5p^2 + 10p + 5$
G_{30},G_{35}	$\binom{p}{2}^2 - 5p^2 + 14p - 6$	G_{31}	$\binom{p}{2}^2 - 5p^2 + 15p - 9$	G_{32}	$\binom{p}{2}^2 - 5p^2 + 17p - 12$
G_{33}	$\binom{p}{2}^2 - 5p^2 + 16p - 11$	G_{34}	$\binom{p}{2}^2 - 5p^2 + 16p - 12$		

Table 1. The number of quadrangles of all graphs except for G_1 in Figure 1.

Proof. Let $E(H) = \{e_1, e_2, \ldots, e_l\}$. For each $i = 1, 2, \ldots, l$, let J_i denote the set of hexagons (6-cycles) of $K_{p,p}$ containing e_i. We know that $K_{p,p}$ contains $6\left[\binom{p}{3}\right]^2$ hexagons. By the Inclusion-Exclusion Principle, we have

$$
\begin{aligned}
c_6(G) \;=\; & 6\left[\binom{p}{3}\right]^2 - \sum_{i=1}^{l}|J_i| + \sum_{i<j}|J_i\cap J_j| - \sum_{i<j<k}|J_i\cap J_j\cap J_k| \\
& + \sum_{i<j<k<r}|J_i\cap J_j\cap J_k\cap J_r| - \sum_{i<j<k<r<s}|J_i\cap J_j\cap J_k\cap J_r\cap J_s| \\
& + \sum_{i<j<k<r<s<t}|J_i\cap J_j\cap J_k\cap J_r\cap J_s\cap J_t|.
\end{aligned}
\tag{2}
$$

Since each edge of $K_{p,p}$ is contained in $4\left[\binom{p-1}{2}\right]^2$ hexagons, we have $\sum_{i=1}^{l}|J_i| = 4l\left[\binom{p-1}{2}\right]^2$.

For $i \neq j$, $|J_i \cap J_j| = \begin{cases} 2\binom{p-2}{1}\binom{p-1}{2}, & \text{if } e_i \text{ is adjacent to } e_j, \\ 3(p-2)^2, & \text{otherwise.} \end{cases}$

For any graph H, it contains exactly $\binom{l}{2} - \sum_{v\in V(H)}\binom{d(v)}{2}$ pairs of pairwise disjoint edges. On the other hand, the number of P_3 in H equals $z(H) = \sum_{v\in V(H)}\binom{d(v)}{2}$. It follows that $\sum_{i<j}|J_i\cap J_j| = 3(p-2)^2\left(\binom{l}{2} - z(H)\right) + 2z(H)(p-2)\binom{p-1}{2}$.

Note that any three edges in a C_6 induce a P_4, $3K_2$ or $P_3 \cup K_2$. Observe that any P_3 in H is contained in $l+2-d(P_3)$ disjoint unions of P_3 and K_2 in H. Further, exactly $2(p-2)(l+2-d_i(P_3))$ hexagons in $K_{p,p}$ contain the disjoint unions of the ith P_3 and K_2 in H. We again note that any $3K_2$ in H is contained exactly in two 6-cycles which are spanned by $3K_2$ in $K_{p,p}$, and the number of $3K_2$ in H equals $q(H,3)$. Hence, there exist $2q(H,3)$ hexagons in $K_{p,p}$ containing all $3K_2$ in H. Additionally, any P_4 is contained in $(p-2)^2$ 6-cycles in $K_{p,p}$. It follows that $\sum_{i<j<k} |J_i \cap J_j \cap J_k| = p_4(H)(p-2)^2 + 2(p-2)\left(\sum_{i=1}^{z(H)}(l+2-d_i(P_3))\right) + 2q(H,3)$.

Similarly, any four edges in a C_6 induce a P_5, $2P_3$ or $P_4 \cup K_2$, where the vertices of degree two in $2P_3$ must belong to different partite in H. It can be seen that any $2P_3$ is contained in two 6-cycles which are spanned by $2P_3$ in $K_{p,p}$. For a P_5, there exist exactly $p-2$ hexagons in $K_{p,p}$ containing it. It follows that $\sum_{i<j<k<r} |J_i \cap J_j \cap J_k \cap S_s| = 2y_1(H) + y_2(H) + p_5(H)(p-2)$.

Since every five edges in a C_6 induce a P_6, we have $\sum_{i<j<k<r<s} |J_i \cap J_j \cap J_k \cap J_r \cap J_s| = p_6(H)$. Similarly, $\sum_{i<j<k<r<s<t} |J_i \cap J_j \cap J_k \cap J_r \cap J_s \cap J_t| = c_6(H)$. Substituting such equations into the expression (2), we obtain equation (1). ■

Let $Q_4(G)$ be the set of all 4-cycles of G. For each $Q \in Q_4(G)$, define $d_G(Q) = \sum_{v \in V(Q)} d_G(v)$ and $D(G) = \sum_{Q \in Q_4(G)} d_G(Q)$.

Lemma 2.11.

$$D(G) = \begin{cases} 4p\left[\binom{p}{2}\right]^2 - 20p^3 + 60p^2 - 24p - 28, & \textit{if } G = G_{10}, G_{17} \textit{ with } p \geq 4, \\ 4p\left[\binom{p}{2}\right]^2 - 25p^3 + 72p^2 - p - 64, & \textit{if } G = G_{22}, G_{36} \textit{ with } p \geq 5, \\ 4p\left[\binom{p}{2}\right]^2 - 25p^3 + 84p^2 - 40p - 46, & \textit{if } G = G_{28}, G_{37} \textit{ with } p \geq 5, \\ 4p\left[\binom{p}{2}\right]^2 - 25p^3 + 84p^2 - 55p - 24, & \textit{if } G = G_{30} \textit{ with } p \geq 3, \\ 4p\left[\binom{p}{2}\right]^2 - 25p^3 + 84p^2 - 57p - 18, & \textit{if } G = G_{35} \textit{ with } p \geq 3. \end{cases}$$

Proof. We only consider the case $G = G_{30}$. The proof of other cases is quite similar to G_{30} and is thus omitted.

We use the notation in Figure 1 for G_{30} and let H be a path of order 6 as a subgraph of $K_{p,p}$. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_5$ and $e_5 = v_5v_6$ denote the edges of H. Direct computation yields $D(K_{p,p}) = 4p\left[\binom{p}{2}\right]^2$. We will compute $D(G_{30})$ by deleting the edges $e_1, \ldots, e_5$ one edge in turn.

Step 1. We observe that $K_{p,p}$ has $(p-1)^2$ quadrangles containing e_1, and these quadrangles will be destroyed in $K_{p,p} - e_1$. We also note that $K_{p,p} - e_1$ has $2(p-1)\binom{p-1}{2}$ quadrangles containing exactly one endpoint of e_1 and three vertices in $V(K_{p,p}) - \{v_1, v_2\}$. For each of such 4-cycle, its degree sum in $K_{p,p} - e_1$ will decrease by 1. As each quadrangle in $K_{p,p}$ has degree sum $4p$, it follows that

$$D(K_{p,p}) - D(K_{p,p} - e_1) = 4p(p-1)^2 + 2(p-1)\binom{p-1}{2} = 5p^3 - 12p^2 + 9p - 2. \tag{3}$$

Step 2. Note that $K_{p,p} - e_1$ has $(p-1)(p-2)$ quadrangles containing e_2, and these quadrangles will be destroyed in $K_{p,p} - \{e_1, e_2\}$, and that the degree sum of each such quadrangle in $K_{p,p} - e_1$ is $4p - 1$. We also note that $K_{p,p} - e_1$ has $(p-1)\binom{p-2}{2} + (p-2)\binom{p-1}{2}$ quadrangles containing exactly one endpoint of e_2 and three vertices in $V(K_{p,p}) - \{v_1, v_2, v_3\}$, and for each of such 4-cycles, its degree sum in $K_{p,p} - \{e_1, e_2\}$ will decrease by 1 from its degree sum in $K_{p,p} - e_1$. Moreover, $K_{p,p} - e_1$ has $\binom{p-1}{2}$ quadrangles containing vertices v_1 and v_3 but not v_2, and for each of such 4-cycles, its degree sum in $K_{p,p} - \{e_1, e_2\}$ will decrease by 1 from its degree sum in $K_{p,p} - e_1$. Thus, after deleting e_2 in $K_{p,p} - e_1$, we have

$$\begin{aligned} & D(K_{p,p} - e_1) - D(K_{p,p} - e_1 - e_2) \\ & = (p-1)\left((p-2)(4p-1) + \binom{p-2}{2} + \binom{p-1}{2}\right) \\ & = 5p^3 - 18p^2 + 19p - 6. \end{aligned} \tag{4}$$

Step 3. Again $K_{p,p} - e_1 - e_2$ has $(p-2)^2 + (p-2)$ quadrangles containing e_3, and these quadrangles will be destroyed in $K_{p,p} - \{e_1, e_2, e_3\}$. Among these quadrangles, there exist $(p-2)$ 4-cycles each of which contains vertices $\{v_1, v_4, v_3\}$ and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4\}$, and has degree sum $4p - 2$ in $K_{p,p} - e_1 - e_2$; and each of the $(p-2)^2$ others has degree sum $4p-1$ in $K_{p,p} - e_1 - e_2$, and each of such 4-cycles contains two endpoints of e_3 and two vertices in $V(K_{p,p}) - \{v_1, v_2\}$. Moreover, $K_n - e_1 - e_2$ has $2(p-2)\binom{p-2}{2}$ 4-cycles each of which contains exactly one endpoint of e_3 and three vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4\}$; and has $2\binom{p-2}{2} + (p-2)^2$ 4-cycles each of which contains exactly one of vertex pairs in $\{(v_1, v_3), (v_1, v_4), (v_2, v_4)\}$ and two vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4\}$. The degree sum of each of these $(2p-2)\binom{p-2}{2} + (p-2)^2$ quadrangles in $K_{p,p} - \{e_1, e_2\}$ will be decreased by 1 in $K_{p,p} - \{e_1, e_2, e_3\}$. Thus

$$\begin{aligned} & D(K_{p,p} - e_1 - e_2) - D(K_{p,p} - e_1 - e_2 - e_3) \\ & = 4p(p-2)^2 + (p-2)(4p-2) + (2p-2)\binom{p-2}{2} \\ & = 5p^3 - 18p^2 + 17p - 2. \end{aligned} \tag{5}$$

Step 4. We note that $K_{p,p} - e_1 - e_2 - e_3$ has $(p-2)(p-3) + 2p - 5$ quadrangles containing e_4. Among these quadrangles, there exist $(p-2)$ 4-cycles each of which contains three vertices $\{v_1, v_4, v_5\}$ and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5\}$, and has degree sum $4p - 2$ in $K_{p,p} - e_1 - e_2 - e_3$; there exist $(p-3)$ 4-cycles each of which contains three vertices $\{v_2, v_4, v_5\}$ and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5\}$, and has degree sum $4p - 3$ in $K_{p,p} - e_1 - e_2 - e_3$; and each of $(p-2)(p-3)$ others has degree sum $4p-1$ in $K_{p,p} - e_1 - e_2 - e_3$, and each of such 4-cycles contains two endpoints of e_4 and two vertices in $V(K_{p,p}) - \{v_1, v_2, v_3\}$. All these 4-cycles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4\}$. Furthermore, $K_{p,p} -$

$e_1 - e_2 - e_3$ has $(p-2)\binom{p-3}{2} + (p-3)\binom{p-2}{2}$ 4-cycles each of which contains exactly one endpoint of e_4 and three vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5\}$; and has $2(p-2)(p-3) + 2\binom{p-2}{2} + \binom{p-3}{2}$ 4-cycles each of which contain exactly one of vertex pairs in $\{(v_1, v_4), (v_2, v_4), (v_1, v_5), (v_2, v_5), (v_3, v_5)\}$ and two vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5\}$. The degree sum of each of these $2(p-2)(p-3) + (p-1)\binom{p-3}{2} + (p-1)\binom{p-2}{2}$ quadrangles in $K_{p,p} - \{e_1, e_2, e_3\}$ will be decreased by 1 in $K_{p,p} - \{e_1, e_2, e_3, e_4\}$. Thus, after deleting e_4 in $K_{p,p} - e_1 - e_2 - e_3$, we have

$$\begin{aligned} &D(K_{p,p} - e_1 - e_2 - e_3) - D(K_{p,p} - e_1 - e_2 - e_3 - e_4) \\ &= 4p^3 - 11p^2 - 6p + 19 + (p-1)(p-3)^2 = 5p^3 - 18p^2 + 9p + 10. \end{aligned} \tag{6}$$

Step 5. We observe that $K_{p,p} - e_1 - e_2 - e_3 - e_4$ has $(p-3)^2 + 3(p-3)$ quadrangles containing e_5, and these quadrangles will be destroyed in $K_n - \{e_1, e_2, e_3, e_4, e_5\}$. Among these quadrangles, there exist $(p-3)$ 4-cycles each of which contains three vertices $\{v_1, v_5, v_6\}$ and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and has degree sum $4p-2$ in $K_{p,p} - e_1 - e_2 - e_3 - e_4$; there exist $2(p-3)$ 4-cycles each of which contains one of vertex pairs in $\{(v_2, v_5, v_6), (v_3, v_5, v_6)\}$ and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and has degree sum $4p-3$ in $K_{p,p} - e_1 - e_2 - e_3 - e_4$; and each of the $(p-3)^2$ others has degree sum $4p-1$ in $K_{p,p} - e_1 - e_2 - e_3 - e_4$, and each of such 4-cycles contains two endpoints of e_5 and two vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4\}$. Moreover, $K_n - e_1 - e_2 - e_3 - e_4$ has $2(p-3)\binom{p-3}{2}$ 4-cycles each of which contains exactly one endpoint of e_5 and three vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$; it has $4\binom{p-3}{2} + 3(p-3)^2$ quadrangles each of which contains one of vertex pairs in $\{(v_1, v_5), (v_2, v_5), (v_3, v_5), (v_1, v_6), (v_2, v_6), (v_3, v_6), (v_4, v_6)\}$ and two vertices in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$; and it has $2(p-3)$ quadrangles each of which contains three vertices either $\{v_1, v_3, v_6\}$ or $\{v_1, v_4, v_6\}$, and one vertex in $V(K_{p,p}) - \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The degree sum of each of these $2(p-1)\binom{p-3}{2} + 3(p-3)^2 + 2(p-3)$ quadrangles in $K_{p,p} - \{e_1, e_2, e_3, e_4\}$ will be decreased by 1 in $K_{p,p} - \{e_1, e_2, e_3, e_4, e_5\}$. Thus, after deleting e_5 in $K_{p,p} - e_1 - e_2 - e_3 - e_4$, we have

$$\begin{aligned} &D(K_{p,p} - e_1 - e_2 - e_3 - e_4) - D(K_{p,p} - e_1 - e_2 - e_3 - e_4 - e_5) \\ &= (4p+2)(p-3)^2 + 2(p-1)\binom{p-3}{2} + 12p^2 - 42p + 18 \\ &= 5p^3 - 18p^2 + p + 24. \end{aligned} \tag{7}$$

Combining equations (3)–(7), we have $D(G_{30}) = 4p\left[\binom{p}{2}\right]^2 - 25p^3 + 84p^2 - 55p - 24$. ∎

3. Main Results

Theorem 3.1. *The graph $K_{p,p} - E(K_{1,l})$ is DPS, where $l < p$.*

Proof. We directly verify that if $p \leq 2$ then $K_{p,p} - E(K_{1,l})$ is DPS. Assume $p \geq 3$. Let G be a graph per-cospectral with $K_{p,p} - E(K_{1,l})$. By Lemma 2.6, we know that G is a bipartite graph with $2p$ vertices and $p^2 - l$ edges. Moreover, by Lemma 2.7, we have $\nu(G) = p$. Thus, G must be isomorphic to some $K_{p,p} - E(H)$ for a subgraph H of $K_{p,p}$ with $|E(H)| = l$. By Lemma 2.1, we have

$$\begin{aligned} &q(K_{p,p} - E(H), 2) \\ &= 3!!\binom{2p}{4} - \left(2\binom{p}{2} + l\right)\binom{2p-2}{2} + 2q(K_p, 2) + \left[\binom{p}{2}\right]^2 + 2l\binom{p-1}{2} \\ &\quad + \binom{l}{2} - \sum_{v \in V(H)} \binom{d(v)}{2}. \end{aligned} \tag{8}$$

By Lemmas 2.4(v) and 2.9, and equation (8), we have

$$\begin{aligned} &b_4(K_{p,p} - E(K_{1,l})) - b_4(K_{p,p} - E(H)) \\ &= (2p-5)\left(\sum_{v \in V(K_{1,l})} \binom{d(v)}{2} - \sum_{v \in V(H)} \binom{d(v)}{2}\right) + 2p_4(H) - 2c_4(H). \end{aligned}$$

As $\sum_{v \in V(H)} \binom{d(v)}{2}$ equals the number of P_3's in H, $\sum_{v \in V(H)} \binom{d(v)}{2} \leq \binom{l}{2}$, which implies that if $H \ncong K_{1,l}$ then $\sum_{v \in V(H)} \binom{d(v)}{2} < \sum_{v \in V(K_{1,l})} \binom{d(v)}{2}$. It can be shown that every 4-cycle yields four distinct P_4's, which indicates that $p_4(H) - c_4(H) \geq 0$. It follows that $b_4(K_{p,p} - E(K_{1,l})) - b_4(K_{p,p} - E(H)) > 0$, a contradiction. Hence, $H \cong K_{1,l}$. ∎

By Theorem 3.1, we obtain the following immediate consequence.

Corollary 3.2. *If $l < p$, then graphs G_1, G_2, G_4, G_8 and G_{25} are DPS.*

Theorem 3.3. *Graph G_3 is DPS.*

Proof. There exist exactly two non-isomorphic graphs obtained from $K_{p,p}$ by deleting two edges, that is, G_2 and G_3. By Theorem 3.1, G_2 is DPS when $p > 2$. If $p = 2$, then $1 = \nu(G_2) \neq \nu(G_3) = 2$. By Lemma 2.7, G_2 and G_3 are not per-cospectral. This indicates that G_3 is DPS. ∎

Theorem 3.4. *Let $\mathcal{G}_1$ be a set of graphs obtained from $K_{p,p}$ by removing three edges. If the matching number of $G \in \mathcal{G}_1$ equals p, then G is DPS.*

Proof. We know that $\mathcal{G}_1 = \{G_4, G_5, G_6, G_7\}$. By Theorem 3.1, we know that G_4 is DPS when $p \neq 3$. Additionally, we find that if $p = 3$, then $\nu(G_4) = 2$ and the matching number of every graph in $\mathcal{G}_1 - \{G_4\}$ equals 3. This indicates, by Lemma 2.7, that G_4 is not per-cospectral with every graph in $\mathcal{G}_1 - \{G_4\}$.

By the above argument, we note that Theorem 3.4 holds only when G_5, G_6 and G_7 are not pairwise per-cospectral. By Lemma 2.4(v) and Table 1, we have $b_4(G_6)-b_4(G_7)=\sum_{i=1}^{2p}\binom{d_{G_7}(v_i)}{2}-\sum_{i=1}^{2p}\binom{d_{G_6}(v_i)}{2}+2(c_4(G_6)-c_4(G_7))=4p-12$. Hence $b_4(G_6)-b_4(G_7)=0$ only when $p=3$. Using Maple 12.0, we compute the permanental polynomials of G_6 and G_7 for $p=3$, respectively, as $\pi(G_6,x)=x^6+6x^4+9x^2+1$ and $\pi(G_7,x)=x^6+6x^4+9x^2+4$. So, G_6 and G_7 are not per-cospectral. Similarly, we have $b_4(G_5)-b_4(G_6)=-2p+7$ and $b_4(G_5)-b_4(G_7)=2p-5$. There exists no integer p such that $b_4(G_5)-b_4(G_6)=0$ or $b_4(G_5)-b_4(G_7)=0$. These indicate that neither the pair G_5 and G_6, nor G_5 and G_7 are per-cospectral. ■

Theorem 3.5. *Let $\mathcal{G}_2$ denote the set of graphs obtained from $K_{p,p}$ by removing four edges. If the matching number of $G\in\mathcal{G}_2$ equals p, then G is DPS.*

Proof. It can be shown that $\mathcal{G}_2=\{G_8,G_9,\ldots,G_{17}\}$. By Theorem 3.1, we know that if $p\neq 4$, then G_8 is DPS. Checking every graph in $\mathcal{G}_2$, we note that the matching number of all graphs in $\mathcal{G}_2-\{G_8\}$ equals 4 and $\nu(G_8)=3$ when $p=4$. So, we obtain, by Lemma 2.7, that if $p=4$, then G_8 is not per-cospectral with any graph in $\mathcal{G}_2-\{G_8\}$. In what follows we show that G_9, G_{10}, …, G_{17} are not pairwise per-cospectral. First, by Lemma 2.4(v) and Table 1, we compute the subtractions of 4th coefficients of permanental polynomials of any two graphs in $\mathcal{G}_2-\{G_8\}$, as shown in Table 2.

By Table 2, we note that $b_4(G_{10})=b_4(G_{17})$. By Lemma 2.2(ii), we have $q(G_{10},3)=q(G_{17},3)=\binom{p^2-4}{3}-(p^2-6)\left((2p-6)\binom{p}{2}+4\binom{p-1}{2}+2\binom{p-2}{2}\right)+2\left((2p-6)\binom{p}{3}+4\binom{p-1}{3}+2\binom{p-2}{3}\right)+p^4-2p^3-11p^2+28p-8$. Additionally, by Lemma 2.10, we have $c_6(G_{10})=6\left[\binom{p}{3}\right]^2-16\left[\binom{p-1}{2}\right]^2+12(p-2)^2+4(p-2)\binom{p-1}{2}-8(p-2)$ and $c_6(G_{17})=6\left[\binom{p}{3}\right]^2-16\left[\binom{p-1}{2}\right]^2+12(p-2)^2+4(p-2)\binom{p-1}{2}-8(p-2)+2$. By Lemmas 2.5 and 2.11, we have $b_6(G_{10})-b_6(G_{17})=-4\neq 0$, which implies that G_{10} and G_{17} are not per-cospectral.

By Table 2, we see that $b_4(G_{14})-b_4(G_{15})=4p-12$, $b_4(G_{15})-b_4(G_{16})=-8p+24$ and $b_4(G_{14})-b_4(G_{16})=-4p+12$. Hence $b_4(G_{14})-b_4(G_{15})=0$, $b_4(G_{15})-b_4(G_{16})=0$ and $b_4(G_{14})-b_4(G_{16})=0$ only when $p=3$. However, by the definition of G_{15}, we have $p\geq 4$ in G_{15}. Thus, neither the pair G_{14} and G_{15}, nor G_{15} and G_{16} are per-cospectral. Additionally, examining G_{14} and G_{16}, we note that $\nu(G_{14})=3$ and $\nu(G_{16})=2$ when $p=3$. This implies, by Lemma 2.7, that G_{14} and G_{16} are not per-cospectral. Similarly, we also note that $b_4(G_{10})-b_4(G_{12})=-4p+16$ and $b_4(G_{12})-b_4(G_{17})=4p-16$. Hence $b_4(G_{10})-b_4(G_{12})=0$ and $b_4(G_{12})-b_4(G_{17})=0$ only when $p=4$. Employing Maple 12.0, we compute $\pi(G_{10},x)$, $\pi(G_{12},x)$ and $\pi(G_{17},x)$ with $p=4$, respectively, as $\pi(G_{10},x)=x^8+12x^6+60x^4+112x^2+64$, $\pi(G_{12},x)=x^8+12x^6+60x^4+96x^2+16$ and $\pi(G_{17},x)=x^8+12x^6+60x^4+116x^2+64$. These imply that neither the pair G_{10} and G_{12}, nor G_{12} and G_{17} are per-cospectral.

Finally, checking Table 2, we note that there exists no integer p such that the subtractions of 4th coefficients of permanental polynomials of any two graphs in $\mathcal{G}_2 - \{G_8\}$ equal 0 excepting the cases as above, which imply that these graphs are not pairwise per-cospectral. So, the theorem is proved. ■

	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}	G_{17}
G_9	$2p-5$	$4p-10$	$-2p+11$	4	$2p-3$	$6p-15$	$-2p+9$	$2p-5$
G_{10}		$2p-5$	$-4p+16$	$-2p+9$	2	$4p-10$	$-4p+14$	0
G_{11}			$-6p+21$	$-4p+14$	$-2p+7$	$2p-5$	$-6p+19$	$-2p+5$
G_{12}				$2p-7$	$4p-14$	$8p-26$	-2	$4p-16$
G_{13}					$2p-7$	$6p-19$	$-2p+5$	$2p-9$
G_{14}						$4p-12$	$-4p+12$	-2
G_{15}							$-8p+24$	$-4p+10$
G_{16}								$4p-14$

Table 2. The subtractions of 4th coefficients of permanental polynomials of any two graphs in $\mathcal{G}_2 - \{G_8\}$.

Theorem 3.6. *Let $\mathcal{G}_3$ be a set of graphs obtained from $K_{p,p}$ by removing five edges. If the matching number of $G \in \mathcal{G}_3$ equals p, then G is DPS.*

Proof. We know that $\mathcal{G}_3 = \{G_{18}, G_{19}, \ldots, G_{37}\}$. By Theorem 3.1, if $p \neq 5$, then G_{25} is DPS. Checking every graph in $\mathcal{G}_3$, we note that the matching number of all graphs in $\mathcal{G}_3 - \{G_{25}\}$ equals 5 and $\nu(G_{25}) = 4$ when $p = 5$. By Lemma 2.7, G_{25} is not per-cospectral with any graph in $\mathcal{G}_3 - \{G_{25}\}$. In what follows we show that any two graphs in $\mathcal{G}_3 - \{G_{25}\}$ are not per-cospectral. We first compute the subtractions of 4th coefficients of permanental polynomials of any two graphs in $\mathcal{G}_3 - \{G_{25}\}$, see Table 3.

By Table 3, we observe that $b_4(G_{30}) = b_4(G_{35})$. By Lemma 2.2(ii), we have $q(G_{30}, 3) = \binom{p^2-5}{3} - (p^2-7)\left((2p-6)\binom{p}{2} + 2\binom{p-1}{2} + 4\binom{p-2}{2}\right) + 2\left((2p-6)\binom{p}{3} + 2\binom{p-1}{3} + 4\binom{p-2}{3}\right) + p^4 - 2p^3 - 14p^2 + 38p - 14$ and $q(G_{35}, 3) = \binom{p^2-5}{3} - (p^2-7)\left((2p-6)\binom{p}{2} + 2\binom{p-1}{2} + 4\binom{p-2}{2}\right) + 2\left((2p-6)\binom{p}{3} + 2\binom{p-1}{3} + 4\binom{p-2}{3}\right) + p^4 - 2p^3 - 14p^2 + 38p - 15$. Additionally, by Lemma 2.10, we have $c_6(G_{30}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 15(p-2)^2 + 8(p-2)\binom{p-1}{2} - 10(p-2) + 1$ and $c_6(G_{35}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 14(p-2)^2 + 8(p-2)\binom{p-1}{2} - 8(p-2)$. By Lemmas 2.5 and 2.11, we obtain that $b_6(G_{30}) - b_6(G_{35}) = 2p^2 - 16p + 31$. Solving $2p^2 - 16p + 31 = 0$, we have $p = \frac{8 \pm \sqrt{2}}{2}$. This contradicts that p is an integer. So, $\pi(G_{30}, x) \neq \pi(G_{35}, x)$.

Similarly, we note that $b_4(G_{22}) = b_4(G_{36})$. By Lemma 2.2(ii), we have $q(G_{22}, 3) = q(G_{36}, 3) = \binom{p^2-5}{3} - (p^2-7)\left((2p-8)\binom{p}{2} + 6\binom{p-1}{2} + 2\binom{p-2}{2}\right) + 2((2p-8)\binom{p}{3} + 6\binom{p-1}{3} + 2\binom{p-2}{3}) + p^4 - 2p^3 - 14p^2 + 34p - 3$. By Lemma 2.10, we have

$c_6(G_{22}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 24(p-2)^2 + 4(p-2)\binom{p-1}{2} - 12(p-2) - 8$ and $c_6(G_{36}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 24(p-2)^2 + 4(p-2)\binom{p-1}{2} - 12(p-2) - 6$. By Lemmas 2.5 and 2.11, we obtain that $b_6(G_{22}) - b_6(G_{36}) = -4 \neq 0$, which indicates that G_{22} and G_{36} are not per-cospectral.

By Table 3, we again observe that $b_4(G_{28}) = b_4(G_{37})$. By Lemma 2.2(ii), we have $q(G_{28},3) = q(G_{37},3) = \binom{p^2-5}{3} - (p^2-7)\left((2p-7)\binom{p}{2} + 5\binom{p-1}{2} + \binom{p-2}{2} + \binom{p-3}{2}\right) + 2\left((2p-7)\binom{p}{3} + 5\binom{p-1}{3} + \binom{p-2}{3} + \binom{p-3}{3}\right) + p^4 - 2p^3 - 14p^2 + 38p - 11$. By Lemma 2.10, we have $c_6(G_{28}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 18(p-2)^2 + 8(p-2)\binom{p-1}{2} - 18(p-2)$ and $c_6(G_{36}) = 6\left[\binom{p}{3}\right]^2 - 20\left[\binom{p-1}{2}\right]^2 + 18(p-2)^2 + 8(p-2)\binom{p-1}{2} - 18(p-2) + 6$. By Lemmas 2.5 and 2.11, we have $b_6(G_{28}) - b_6(G_{37}) = -12 \neq 0$, which indicates that G_{28} and G_{37} are not per-cospectral.

By Table 3, note that $b_4(G_{19}) - b_4(G_{31}) = -4p + 16$, $b_4(G_{24}) - b_4(G_{34}) = -4p + 16$, $b_4(G_{30}) - b_4(G_{36}) = 4p - 16$, and $b_4(G_{35}) - b_4(G_{36}) = 4p - 16$. So, only for $p = 4$ $b_4(G_{19}) - b_4(G_{31}) = 0$, $b_4(G_{24}) - b_4(G_{34}) = 0$, $b_4(G_{30}) - b_4(G_{36}) = 0$ and $b_4(G_{35}) - b_4(G_{36}) = 0$, hold. Using Maple 12.0, we compute the permanental polynomials of G_{19}, G_{24}, G_{30}, G_{31}, G_{34}, G_{35} and G_{36} for $p = 4$, respectively. We have $\pi(G_{19},x) = x^8+11x^6+46x^4+74x^2+36$, $\pi(G_{24},x) = x^8+11x^6+47x^4+74x^2+16$, $\pi(G_{30},x) = x^8+11x^6+45x^4+67x^2+25$, $\pi(G_{31},x) = x^8+11x^6+46x^4+65x^2+9$, $\pi(G_{34},x) = x^8+11x^6+47x^4+56x^2+4$, $\pi(G_{35},x) = x^8+11x^6+45x^4+68x^2+16$ and $\pi(G_{36},x) = x^8+11x^6+45x^4+76x^2+36$, which indicate that none of the pairs G_{19} and G_{31}, G_{24} and G_{34}, G_{30} and G_{36}, and G_{35} and G_{36} are per-cospectral. Similarly, we again note that $b_4(G_{28}) - b_4(G_{34}) = -4p+20$ and $b_4(G_{34}) - b_4(G_{37}) = 4p-20$. Hence $b_4(G_{28}) - b_4(G_{34}) = 0$ and $b_4(G_{34}) - b_4(G_{37}) = 0$ only when $p = 5$. Employing Maple 12.0, we compute the permanental polynomials of G_{28}, G_{34} and G_{37} with $p = 5$, respectively. We obtained that $\pi(G_{28},x) = x^{10} + 20x^8 + 210x^6 + 1116x^4 + 2376x^2 + 1296$, $\pi(G_{34},x) = x^{10} + 20x^8 + 210x^6 + 1092x^4 + 2040x^2 + 576$ and $\pi(G_{37},x) = x^{10} + 20x^8 + 210x^6 + 1128x^4 + 2424x^2 + 1296$, which imply that neither the pair G_{28} and G_{34}, nor G_{34} and G_{37} are per-cospectral.

By Table 3, we find that $b_4(G_{30}) - b_4(G_{33}) = -4p + 12$. Hence, $b_4(G_{30}) - b_4(G_{33}) = 0$ only when $p = 3$. Checking G_{30} and G_{33}, note that $\nu(G_{30}) = 3$ and $\nu(G_{32}) = 2$ when $p = 3$, which implies, by Lemma 2.7, that G_{30} and G_{30} are not per-cospectral. Similarly, we again note that $b_4(G_{33}) - b_4(G_{35}) = 4p - 12$. Checking G_{33} and G_{35}, we know that the matching number of G_{33} and G_{35} equals 2 when $p = 3$. This contradicts the assumption $p = 3$.

By Table 3, we see that $b_4(G_{18}) - b_4(G_{26}) = -4p + 16$, $b_4(G_{22}) - b_4(G_{30}) = -4p + 16$ and $b_4(G_{22}) - b_4(G_{35}) = -4p + 16$, respectively. Hence $b_4(G_{18}) - b_4(G_{26}) = 0$, $b_4(G_{22}) - b_4(G_{30}) = 0$ and $b_4(G_{22}) - b_4(G_{35}) = 0$ only when $p = 4$. However, by the definitions of G_{18} and G_{22}, we have $p \geq 5$ in G_{18} and G_{22}. By Lemma 2.6, these mean that none of the pairs G_{18} and G_{26}, G_{22} and G_{30}, and G_{22} and G_{35} are per-cospectral. Similarly, by Table 3, we see that $b_4(G_{21}) - b_4(G_{29}) =$

$4p - 12$ and $b_4(G_{24}) - b_4(G_{29}) = 8p - 24$. Hence $b_4(G_{21}) - b_4(G_{29}) = 0$ and $b_4(G_{24}) - b_4(G_{29}) = 0$ only when $p = 3$. These contradict the fact $p \geq 5$ in G_{29} by the definition of G_{29}. So, neither the pair G_{21} and G_{29}, nor G_{24} and G_{29} are per-cospectral.

Additionally, by Table 3, we also see that $b_4(G_{19}) - b_4(G_{23}) = 4p - 12$, $b_4(G_{19}) - b_4(G_{26}) = -6p + 18$, $b_4(G_{19}) - b_4(G_{32}) = -8p + 24$, $b_4(G_{20}) - b_4(G_{26}) = -4p + 12$, $b_4(G_{21}) - b_4(G_{24}) = -4p + 12$, $b_4(G_{21}) - b_4(G_{28}) = -4p + 8$, $b_4(G_{21}) - b_4(G_{37}) = -4p + 8$, $b_4(G_{23}) - b_4(G_{32}) = -12p + 36$, $b_4(G_{27}) - b_4(G_{30}) = 4p - 4$, $b_4(G_{27}) - b_4(G_{35}) = 4p - 4$ and $b_4(G_{31}) - b_4(G_{32}) = -4p + 8$. Observing these equations, it can be seen that only when $p = 1$, 2 or 3 these equations equal zero. By the definitions of G_{19}, G_{20}, G_{21}, G_{23}, G_{24}, G_{26}, G_{27}, G_{28}, G_{30}, G_{31}, G_{32} and G_{35}, we have $p \geq 4$ for these graphs. So, G_{19} and G_{23}, G_{19} and G_{26}, G_{19} and G_{32}, G_{20} and G_{26}, G_{21} and G_{24}, G_{21} and G_{28}, G_{21} and G_{37}, G_{23} and G_{32}, G_{27} and G_{30}, G_{27} and G_{35}, and G_{31} and G_{32} are not per-cospectral.

Finally, checking Table 3, we note that there exists no integer p such that the subtractions of 4th coefficients of permanental polynomials of any two graphs in $\mathcal{G}_2 - \{G_{25}\}$ equal 0 excepting the cases as above, which implies that these graphs are not pairwise per-cospectral. So, the theorem is proved. ■

4. Discussions

In this paper, we investigated which graphs obtained from $K_{p,p}$ by removing some edges are DPS. From the proofs of main theorems in Section 3, we know that the matching number plays a key role. If the matching number is less than p in these graphs as above, then we do not find a suitable method to show whether these graphs are DPS or not. But we find that $G_{33} = C_4 \cup 2K_1$ and $G_{35} = 2P_3$ are per-cospectral when $p = 3$, and the matching number of $C_4 \cup 2K_1$ and $2P_3$ equals 2. Furthermore, we can show that $C_4 \cup 2K_1$ and $2P_3$ is one of two pairs of minimum per-cospectral graphs (with the minimum number of edges), and the other pair is $K_{1,3} \cup K_2$ and $P_5 \cup K_1$.

Professor Haemers conjectured that almost all graphs are determined by their adjacency spectra. Thus, we propose an interesting question which can be thought of parallel question of considered in this paper, i.e.,

Question. Which graphs obtained from $K_{p,p}$ by removing five or fewer edges are determined by their adjacency spectra?

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	G_{19}	G_{20}	G_{21}	G_{22}	G_{23}	G_{24}	G_{26}	G_{27}	G_{28}	G_{29}	G_{30}	G_{31}	G_{32}	G_{33}	G_{34}	G_{35}	G_{36}	G_{37}
G_{18}	2	4	2p-3	2p-5	4p-10	-2p+9	-4p+16	-6p+15	-2p+5	6p-15	-2p+11	-4p+18	-8p+26	-6p+23	-6p+25	-2p+11	2p-5	-2p+5
G_{19}		2	2p-5	2p-7	4p-12	-2p+7	-4p+14	-6p+13	-2p+3	6p-17	-2p+9	-4p+16	-8p+24	-6p+21	-6p+23	-2p+9	2p-7	-2p+3
G_{20}			2p-7	2p-9	4p-14	-2p+5	-4p+12	-6p+11	-2p+1	6p-19	-2p+7	-4p+14	-8p+22	-6p+19	-6p+21	-2p+7	2p-9	-2p+1
G_{21}				-2	2p-7	-4p+12	-6p+19	-8p+18	-4p+8	4p-12	-4p+14	-6p+21	-10+29	-8p+26	-8p+28	-4p+14	-2	-4p+8
G_{22}					2p-5	-4p+14	-6p+21	-8p+20	-4p+10	4p-10	-4p+16	-6p+23	-10p+31	-8p+28	-8p+30	-4p+16	0	-4p+10
G_{23}						-6p+19	-8p+26	-10p+25	-6p+15	2p+5	-6p+21	-8p+28	-12p+36	-10+33	-10p+35	-6p+21	-2p+5	-6p+15
G_{24}							-2p+7	-4p+6	-4	8p-24	2	-2p+9	-6p+17	-4p+14	-4p+16	2	4p-14	-4
G_{26}								-2p-1	2p-11	10p-31	2p-5	2	-4p+10	-2p+7	-2p+9	2p-5	6p-21	2p-11
G_{27}									4p-10	12p-30	4p-4	2p+3	-2p+11	8	10	4p-4	8p-20	4p-10
G_{28}										8p-20	6	-2p+13	-6p+21	-4p+18	-4p+20	6	4p-10	0
G_{29}											-8p+26	-10p+33	-14p+41	-12p+38	-12p+40	-8p+26	-4p+10	-8p+20
G_{30}												-2p+7	-6p+15	-4p+12	-4p+14	0	4p-16	-6
G_{31}													-4p+8	-2p+5	-2p+7	2p-7	6p-23	2p-13
G_{32}														2p-3	2p-1	6p-15	10p-31	6p-21
G_{33}															2	4p-12	8p-28	4p-18
G_{34}																4p-14	8p-30	4p-20
G_{35}																	4p-16	-6
G_{36}																		-4p+10

Table 3. The subtractions of fourth coefficients of permanental polynomials of any two graphs in $\mathcal{G}_3 - \{G_{25}\}$.

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