Discussiones Mathematicae
Graph Theory 37 (2017) 1067–1078
doi:10.7151/dmgt.1980

ON *H*-IRREGULARITY STRENGTH OF GRAPHS

Faraha Ashraf

Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan

e-mail: faraha27@gmail.com

Martin Bača[1]

Marcela Lascsáková

and

Andrea Semaničová-Feňovčíková

Department of Applied Mathematics and Informatics
Technical University, Košice, Slovakia

e-mail: martin.baca@tuke.sk
marcela.lascsakova@tuke.sk
andrea.fenovcikova@tuke.sk

Abstract

New graph characteristic, the total H-irregularity strength of a graph, is introduced. Estimations on this parameter are obtained and for some families of graphs the precise values of this parameter are proved.

Keywords: H-covering, H-irregular labeling, H-irregularity strength.

2010 Mathematics Subject Classification: 05C78, 05C70.

1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A *labeling* of a graph is a map that carries graph elements to the numbers (usually to the positive or non-negative integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings*

[1]Corresponding author.

or *edge labelings*. If the domain is $V(G) \cup E(G)$ then we call the labeling *total labeling*. The most complete recent survey of graph labelings is [12].

For an edge k-labeling $\delta : E(G) \to \{1, 2, \dots, k\}$ the associated weight of a vertex $x \in V(G)$ is $w_\delta(x) = \sum_{xy \in E(G)} \delta(xy)$, where the sum is over all vertices y adjacent to x.

Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba in [9] introduced edge k-labeling δ of a graph G such that $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings are called *irregular assignments* and the *irregularity strength* $\mathrm{s}(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength $\mathrm{s}(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [2, 3, 4, 7, 10, 11, 17, 18, 19].

Motivated by irregularity strengths, Bača, Jendrol', Miller and Ryan in [5] defined the total labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G one has

$$wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \neq wt_\varphi(x'y') = \varphi(x') + \varphi(x'y') + \varphi(y').$$

The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of the graph G, $\mathrm{tes}(G)$. The total edge irregularity strength is an invariant analogous to the irregularity strength.

A lower bound on the total edge irregularity strength of a graph G is given in [5]

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G.

Ivančo and Jendrol' [14] posed a conjecture that for an arbitrary graph G different from K_5 and with maximum degree $\Delta(G)$, $\mathrm{tes}(G) = \max\{\lceil (|E(G)| + 2)/3 \rceil, \lceil (\Delta(G) + 1)/2 \rceil\}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [15] and [16], for the categorical product of two cycles in [1], for generalized Petersen graphs in [13], for generalized prisms in [6], for corona product of a path with certain graphs in [20] and for large dense graphs with $(|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2$ in [8].

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*.

Let G be a graph admitting H-covering. For the subgraph $H \subseteq G$ under the total k-labeling φ, we define the associated H-weight as

$$wt_\varphi(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e).$$

A total k-labeling φ is called an *H-irregular total k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\varphi(H') \neq wt_\varphi(H'')$. The *total H-irregularity strength* of a graph G, denoted $\text{ths}(G, H)$, is the smallest integer k such that G has an H-irregular total k-labeling. If H is isomorphic to K_2, then the K_2-irregular total k-labeling is isomorphic to the edge irregular total k-labeling and thus the total K_2-irregularity strength of a graph G is equivalent to the total edge irregularity strength, that is $\text{ths}(G, K_2) = \text{tes}(G)$.

Analogously, we can define H-irregular edge k-labeling and H-irregular vertex k-labeling.

Let G be a graph admitting H-covering. For the subgraph $H \subseteq G$ under the edge k-labeling β, $\beta : E(G) \to \{1, 2, \dots, k\}$, we define the associated H-weight as

$$wt_\beta(H) = \sum_{e \in E(H)} \beta(e).$$

An edge k-labeling β is called an *H-irregular edge k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\beta(H') \neq wt_\beta(H'')$. The *edge H-irregularity strength* of a graph G, denoted $\text{ehs}(G, H)$, is the smallest integer k such that G has an H-irregular edge k-labeling.

Let G be a graph admitting H-covering. For the subgraph $H \subseteq G$ under the vertex k-labeling α, $\alpha : V(G) \to \{1, 2, \dots, k\}$, we define the associated H-weight as

$$wt_\alpha(H) = \sum_{v \in V(H)} \alpha(v).$$

A vertex k-labeling α is called an *H-irregular vertex k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H there is $wt_\alpha(H') \neq wt_\alpha(H'')$. The *vertex H-irregularity strength* of a graph G, denoted $\text{vhs}(G, H)$, is the smallest integer k such that G has an H-irregular vertex k-labeling. Note that $\text{vhs}(G, H) = \infty$ if there exist two subgraphs in G isomorphic to H that have the same vertex sets. Evidently, if there exist two subgraphs H_i, H_j, $i \neq j$, such that $V(H_i) = V(H_j)$ then

$$wt_\alpha(H_i) = \sum_{v \in V(H_i)} \alpha(v) = \sum_{v \in V(H_j)} \alpha(v) = wt_\alpha(H_j).$$

In the paper, we estimate the bounds of the parameter $\text{ths}(G, H)$ and determine the exact values of the total H-irregularity strength for several families of graphs, namely, paths, ladders and fans.

2. Results

Our first result gives a lower bound of the total H-irregularity strength.

Theorem 1. *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Then*

$$\mathrm{ths}(G,H) \geq \left\lceil 1 + \frac{t-1}{|V(H)|+|E(H)|} \right\rceil .$$

Proof. Let G be a graph that admits an H-covering given by t subgraphs isomorphic to H. Assume that φ is an H-irregular total k-labeling of a graph G with $\mathrm{ths}(G,H) = k$. The smallest weight of a subgraph H under the total k-labeling is at least $|V(H)|+|E(H)|$ and the largest H-weight admits the value at most $(|V(H)|+|E(H)|)k$. Since H-covering of G is given by t subgraphs, we get

$$|V(H)|+|E(H)|+t-1 \leq (|V(H)|+|E(H)|)k$$

and

$$k \geq \left\lceil 1 + \frac{t-1}{|V(H)|+|E(H)|} \right\rceil .$$

∎

If H is isomorphic to K_2, then immediately from Theorem 1 it follows the lower bound on the total edge irregularity strength given in [5].

Corollary 2. *Let $G = (V,E)$ be a graph having non-empty edge set. Then*

$$\mathrm{ths}(G,K_2) = \mathrm{tes}(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil .$$

The lower bound in Theorem 1 is tight as can be seen from the following theorems which determine the exact values of the total H-irregularity strength for paths and ladders.

Theorem 3. *Let n, m, $2 \leq m \leq n$, be positive integers. Then*

$$\mathrm{ths}(P_n,P_m) = \left\lceil \frac{m+n-1}{2m-1} \right\rceil .$$

Proof. Let P_n be a path with the vertex set $V(P_n) = \{v_i : i = 1,2,\ldots,n\}$ and the edge set $E(P_n) = \{v_iv_{i+1} : i = 1,2,\ldots,n-1\}$. Clearly, for every m, $2 \leq m \leq n$, the path P_n admits a P_m-covering with exactly $n-m+1$ subpaths. Put $k = \left\lceil \frac{m+n-1}{2m-1} \right\rceil$. According to Theorem 1, k is the lower bound of $\mathrm{ths}(P_n,P_m)$.

In order to show the converse inequality, it only remains to describe a P_m-irregular total k-labeling $\varphi : V(P_n) \cup E(P_n) \to \{1, 2, \dots, k\}$ as follows

$$\varphi(v_i) = \left\lceil \frac{m-1+i}{2m-1} \right\rceil, \qquad \text{for } i = 1, 2, \dots, n,$$
$$\varphi(v_i v_{i+1}) = \left\lceil \frac{i}{2m-1} \right\rceil, \qquad \text{for } i = 1, 2, \dots, n-1.$$

We can see that all vertex and edge labels are at most k. Every subpath P_m in P_n is of the form $P_m^j = v_j v_{j+1} \cdots v_{m+j-1}$, where $j = 1, 2, \dots, n-m+1$. For the P_m-weight of the path P_m^j, $j = 1, 2, \dots, n-m+1$, under the total labeling φ we get

$$wt_\varphi(P_m^j) = \sum_{v \in V(P_m^j)} \varphi(v) + \sum_{e \in E(P_m^j)} \varphi(e). \tag{2}$$

Since vertex labels and edge labels form non-decreasing sequences, it is enough to prove that $wt_\varphi(P_m^j) < wt_\varphi(P_m^{j+1})$, $j = 1, 2, \dots, n-m$.

In fact, with respect to (2), we get

$$wt_\varphi(P_m^j) = \varphi(v_j) + \varphi(v_j v_{j+1}) + \sum_{i=j+1}^{m+j-1} \varphi(v_i) + \sum_{i=j+1}^{m+j-2} \varphi(v_i v_{i+1}) \tag{3}$$

and

$$wt_\varphi(P_m^{j+1}) = \sum_{i=j+1}^{m+j-1} \varphi(v_i) + \sum_{i=j+1}^{m+j-2} \varphi(v_i v_{i+1}) + \varphi(v_{m+j}) + \varphi(v_{m+j-1} v_{m+j}). \tag{4}$$

Because for every $j = 1, 2, \dots, n-m$

$$\begin{aligned}
\varphi(v_{m+j}) + \varphi(v_{m+j-1} v_{m+j}) &= \left\lceil \frac{2m-1+j}{2m-1} \right\rceil + \left\lceil \frac{m-1+j}{2m-1} \right\rceil \\
&= 1 + \left\lceil \frac{j}{2m-1} \right\rceil + \left\lceil \frac{m-1+j}{2m-1} \right\rceil \\
&= 1 + \varphi(v_j v_{j+1}) + \varphi(v_j),
\end{aligned}$$

then $wt_\varphi(P_m^j) < wt_\varphi(P_m^{j+1})$ and we are done. ■

Theorem 4. *Let $L_n \cong P_n \square P_2$, $n \geq 3$, be a ladder admitting a C_m-covering, $m = 4, 6$. Then*

$$\text{ths}(L_n, C_m) = \left\lceil \frac{3m+2n}{4m} \right\rceil.$$

Proof. Let $L_n \cong P_n \Box P_2$, $n \geq 3$, be a ladder with the vertex set $V(L_n) = \{v_i, u_i : i = 1, 2, \dots, n\}$ and the edge set $E(L_n) = \{v_i v_{i+1}, u_i u_{i+1} : i = 1, 2, \dots, n-1\} \cup \{v_i u_i : i = 1, 2, \dots, n\}$. The ladder L_n, $n \geq 3$, admits a C_4-covering with exactly $n-1$ cycles C_4 and a C_6-covering with exactly $n-2$ cycles C_6. With respect to Theorem 1 we have $\text{ths}(L_n, C_m) \geq \lceil \frac{3m+2n}{4m} \rceil$. Put $k = \lceil \frac{3m+2n}{4m} \rceil$. To show that k is an upper bound for redthe total C_m-irregularity strength of L_n we define a C_m-irregular total k-labeling $\varphi_m : V(L_n) \cup E(L_n) \to \{1, 2, \dots, k\}$, $m = 4, 6$, in the following way:

$$\begin{aligned}
\varphi_4(v_i) &= \left\lceil \frac{i+6}{8} \right\rceil, && \text{for } i = 1, 2, \dots, n,\\
\varphi_4(u_i) &= \left\lceil \frac{i+2}{8} \right\rceil, && \text{for } i = 1, 2, \dots, n,\\
\varphi_4(v_i v_{i+1}) &= \left\lceil \frac{i+1}{8} \right\rceil, && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_4(u_i u_{i+1}) &= \left\lceil \frac{i}{8} \right\rceil, && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_4(v_i u_i) &= \left\lceil \frac{i+4}{8} \right\rceil, && \text{for } i = 1, 2, \dots, n,
\end{aligned}$$

and

$$\begin{aligned}
\varphi_6(v_i) &= \left\lceil \frac{i+10}{13} \right\rceil, && \text{for } i = 1, 2, \dots, n,\\
\varphi_6(u_i) &= \left\lceil \frac{i+7}{13} \right\rceil, && \text{for } i = 1, 2, \dots, n,\\
\varphi_6(v_i v_{i+1}) &= \left\lceil \frac{i+5}{13} \right\rceil, && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_6(u_i u_{i+1}) &= \left\lceil \frac{i+3}{13} \right\rceil, && \text{for } i = 1, 2, \dots, n-1,\\
\varphi_6(v_i u_i) &= \left\lceil \frac{i}{13} \right\rceil, && \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

It is a routine matter to verify that under the labelings φ_4 and φ_6 all vertex and edge labels are at most k. For the C_m-weight of the cycle C_m^j, $j = 1, 2, \dots, n - \frac{m}{2} + 1$, under the total labeling φ_m, $m = 4, 6$, we get

$$wt_{\varphi_m}(C_m^j) = \sum_{v \in V(C_m^j)} \varphi_m(v) + \sum_{e \in E(C_m^j)} \varphi_m(e). \tag{5}$$

One can see that vertex labels and edge labels form non-decreasing sequences, therefore it is enough to prove that $wt_{\varphi_m}(C_m^j) < wt_{\varphi_m}(C_m^{j+1})$, $j = 1, 2, \dots, n - \frac{m}{2}$.

For every $j = 1, 2, \dots, n-2$, we have

$$\begin{aligned}
&\varphi_4(v_{j+1}v_{j+2}) + \varphi_4(v_{j+2}) + \varphi_4(u_{j+1}u_{j+2}) + \varphi_4(u_{j+2}) + \varphi_4(v_{j+2}u_{j+2}) \\
&\quad = \left\lceil \frac{j+2}{8} \right\rceil + \left\lceil \frac{j+8}{8} \right\rceil + \left\lceil \frac{j+1}{8} \right\rceil + \left\lceil \frac{j+4}{8} \right\rceil + \left\lceil \frac{j+6}{8} \right\rceil \\
&\quad = \varphi_4(u_j) + 1 + \varphi_4(u_ju_{j+1}) + \varphi_4(v_jv_{j+1}) + \varphi_4(v_ju_j) + \varphi_4(v_j),
\end{aligned}$$

thus with respect to (5) $wt_{\varphi_4}(C_4^{j+1}) = 1 + wt_{\varphi_4}(C_4^j)$.

Because for every $j = 1, 2, \dots, n-3$,

$$\begin{aligned}
&\varphi_6(v_{j+2}v_{j+3}) + \varphi_6(v_{j+3}) + \varphi_6(u_{j+2}u_{j+3}) + \varphi_6(u_{j+3}) + \varphi_6(v_{j+3}u_{j+3}) \\
&\quad = \left\lceil \frac{j+7}{13} \right\rceil + \left\lceil \frac{j+13}{13} \right\rceil + \left\lceil \frac{j+5}{13} \right\rceil + \left\lceil \frac{j+10}{13} \right\rceil + \left\lceil \frac{j+3}{13} \right\rceil \\
&\quad = \varphi_6(u_j) + 1 + \varphi_6(v_ju_j) + \varphi_6(v_jv_{j+1}) + \varphi_6(v_j) + \varphi_6(u_ju_{j+1}),
\end{aligned}$$

then by (5) $wt_{\varphi_6}(C_6^{j+1}) = 1 + wt_{\varphi_6}(C_6^j)$.

Thus, the labelings φ_m, for $m = 4, 6$, are desired C_m-irregular total k-labelings. ∎

Let G be a graph admitting H-covering. By the symbol $\mathbb{H}_m^S = \left(H_1^S, H_2^S, \dots, H_m^S\right)$ we denote the set of all subgraphs of G isomorphic to H such that the graph S, $S \ncong H$, is their maximum common subgraph. Thus $V(S) \subset V\left(H_i^S\right)$ and $E(S) \subset E\left(H_i^S\right)$ for every $i = 1, 2, \dots, m$. Next theorem gives another lower bound of the total H-irregularity strength.

Theorem 5. *Let G be a graph admitting an H-covering. Let S_i, $i = 1, 2, \dots, z$, be all subgraphs of G such that S_i is a maximum common subgraph of m_i, $m_i \geq 2$, subgraphs of G isomorphic to H. Then*

$$\operatorname{ths}(G, H) \geq \max\left\{ \left\lceil 1 + \tfrac{m_1 - 1}{|V(H/S_1)| + |E(H/S_1)|} \right\rceil, \dots, \left\lceil 1 + \tfrac{m_z - 1}{|V(H/S_z)| + |E(H/S_z)|} \right\rceil \right\}.$$

Proof. Let G be a graph admitting an H-covering. Suppose $\mathbb{H}_{m_i}^{S_i}$, $i = 1, 2, \dots, z$, is the set of all subgraphs $H_1^{S_i}, H_2^{S_i}, \dots, H_{m_i}^{S_i}$, where each of them is isomorphic to H, and S_i is their maximum common subgraph. Let ψ be an optimal total labeling of G. The H-weights of the graphs $H_1^{S_i}, H_2^{S_i}, \dots, H_{m_i}^{S_i}$

$$wt\left(H_j^{S_i}\right) = \sum_{v \in V(S_i)} \psi(v) + \sum_{e \in E(S_i)} \psi(e) + \sum_{v \in V\left(H_j^{S_i}/S_i\right)} \psi(v) + \sum_{e \in E\left(H_j^{S_i}/S_i\right)} \psi(e),$$

$j = 1, 2, \dots, m_i$, are all distinct. Moreover, each of them contains the value $\sum_{v \in V(S_i)} \psi(v) + \sum_{e \in E(S_i)} \psi(e)$. The largest among these H-weights must be at least

$$\sum_{v \in V(S_i)} \psi(v) + \sum_{e \in E(S_i)} \psi(e) + |V(H/S_i)| + |E(H/S_i)| + m_i - 1.$$

This weight is the sum of at most $|V(H/S_i)| + |E(H/S_i)|$ labels (without labels from the set $\{\psi(x) : x \in V(S_i) \cup E(S_i)\}$). So at least one label has the value at least $\lceil 1 + (m_i - 1)/(|V(H/S_i)| + |E(H/S_i)|)\rceil$, for $i = 1, 2, \ldots, z$. Thus for the total H-irregularity strength of graph G we have

$$\text{ths}(G, H) \geq \max\left\{\left\lceil 1 + \frac{m_1-1}{|V(H/S_1)|+|E(H/S_1)|}\right\rceil, \ldots, \left\lceil 1 + \frac{m_z-1}{|V(H/S_z)|+|E(H/S_z)|}\right\rceil\right\}. \blacksquare$$

If H is isomorphic to K_2 then from Theorem 5 it follows the lower bound on the total edge irregularity strength given in [5].

Corollary 6. *Let $G = (V, E)$ be a graph with maximum degree $\Delta(G)$. Then*

$$\text{ths}(G, K_2) = \text{tes}(G) \geq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil.$$

The lower bound in Theorem 5 is tight as can be seen from the next theorem.

Theorem 7. *Let F_n, $n \geq 2$, be a fan on $n + 1$ vertices. Then*

$$\text{ths}(F_n, C_3) = \left\lceil \frac{n+3}{5} \right\rceil.$$

Proof. A *fan* F_n, $n \geq 2$, is a graph obtained by joining all vertices of path P_n to a new vertex, called the centre. Thus F_n contains $n+1$ vertices, say, $w, v_1, v_2, \ldots, v_n$ and $2n - 1$ edges wv_i, $i = 1, 2, \ldots, n$, and v_iv_{i+1}, $i = 1, 2, \ldots, n - 1$. The fan F_n admits a C_3-covering with exactly $n - 1$ cycles C_3. In view of the lower bound from Theorem 5 it suffices to prove the existence of a C_3-irregular total labeling $\psi : V(F_n) \cup E(F_n) \to \{1, 2, \ldots, \lceil (n + 3)/5\rceil\}$ such that $wt_\psi(C_3^j) \neq wt_\psi(C_3^i)$ for every $i, j = 1, 2, \ldots, n - 1$, $j \neq i$. We describe the irregular total labeling ψ in the following way:

$$\begin{aligned}
\psi(v_i) &= \left\lceil \frac{i+3}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n,\\
\psi(v_iv_{i+1}) &= \left\lceil \frac{i+2}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n-1,\\
\psi(wv_i) &= \left\lceil \frac{i}{5} \right\rceil, && \text{for } i = 1, 2, \ldots, n,\\
\psi(w) &= 1.
\end{aligned}$$

Under the labeling ψ all vertex labels and edge labels are at most $\lceil (n + 3)/5\rceil$ and for C_3-weight of the cycle $C_3^j = v_jv_{j+1}w$, $j = 1, 2, \ldots, n - 1$, we have

$$wt_\psi(C_3^j) = \psi(v_j) + \psi(v_jv_{j+1}) + \psi(v_{j+1}) + \psi(wv_j) + \psi(wv_{j+1}) + \psi(w). \tag{6}$$

Since under the labeling ψ vertex labels and edge labels form non-decreasing sequences and for every $j = 1, 2, \dots, n-2$,

$$\begin{aligned}\psi(v_{j+1}v_{j+2}) + \psi(v_{j+2}) + \psi(wv_{j+2}) &= \left\lceil \frac{j+3}{5} \right\rceil + \left\lceil \frac{j+5}{5} \right\rceil + \left\lceil \frac{j+2}{5} \right\rceil \\ &= \psi(v_j) + 1 + \psi(wv_j) + \psi(v_jv_{j+1}),\end{aligned}$$

with respect to (6) we get $wt_\psi(C_3^{j+1}) = 1 + wt_\psi(C_3^j)$. It proves that the irregular total labeling ψ has the required properties. ■

Next we will introduce an upper bound for the parameter $\text{ths}(G, H)$.

Theorem 8. *Let G be a graph admitting an H-covering. Then*

$$\text{ths}(G, H) \leq 2^{|E(G)|-1}.$$

Proof. Let G be a graph admitting H-covering given by subgraphs $H_1, H_2, \dots, H_t$. Let us denote the edges of G arbitrarily by the symbols $e_1, e_2, \dots, e_{|E(G)|}$. We define a total $2^{|E(G)|-1}$-labeling f of G in the following way:

$$\begin{aligned} f(v) &= 1, && \text{for } v \in V(G), \\ f(e_i) &= 2^{i-1}, && \text{for } i = 1, 2, \dots, |E(G)|. \end{aligned}$$

Let us define the labeling θ such that

$$\theta_{i,j} = \begin{cases} 1, & \text{if } e_i \in E(H_j), \\ 0, & \text{if } e_i \notin E(H_j), \end{cases}$$

where $i = 1, 2, \dots, |E(G)|$, $j = 1, 2, \dots, t$.

The H-weights are the sums of all vertex labels and edge labels of vertices and edges in the given subgraph. Thus, for $j = 1, 2, \dots, t$, we have

$$\begin{aligned} wt_f\left(H^j\right) &= \sum_{v \in V(H^j)} f(v) + \sum_{e \in E(H^j)} f(e) = \sum_{v \in V(H^j)} 1 + \sum_{e_i \in E(H^j)} 2^{i-1} \\ &= |V(H_j)| + \sum_{i=1}^{|E(G)|} \theta_{i,j} 2^{i-1}. \end{aligned} \tag{7}$$

As $|V(H_j)| = |V(H)|$ for every $j = 1, 2, \dots, t$, for proving that the H-weights are all distinct it is enough to show that the sums $\sum_{i=1}^{|E(G)|} \theta_{i,j} 2^{i-1}$ are distinct for every $j = 1, 2, \dots, t$. However, this is evident if we note that the ordered $|E(G)|$-tuple $(\theta_{|E(G)|,j}\theta_{|E(G)|-1,j} \cdots \theta_{2,j}\theta_{1,j})$ corresponds to binary code representation of the sum (7). As different subgraphs isomorphic to H cannot have the same edge sets, we immediately get that the $|E(G)|$-tuples are different for different subgraphs. ■

In certain cases we can decrease the upper bound of $\text{ths}(G,H)$ from Theorem 8 as follows.

Theorem 9. *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. If every subgraph H_i, $i = 1, 2, \dots, t$, isomorphic to H contains at least one edge e such that $e \notin E(H_j)$ for every $j = 1, 2, \dots, t$, $j \neq i$, then*

$$\text{ths}(G,H) \leq t.$$

Proof. Let G be a graph admitting H-covering given by subgraphs $H_1, H_2, \dots, H_t$. Let us denote by e_i, $i = 1, 2, \dots, t$, the edge of H_i such that $e_i \notin E(H_j)$ for every $j = 1, 2, \dots, t$, $j \neq i$.

We define a total t-labeling f of G in the following way:

$$\begin{aligned} f(v) &= 1, && \text{for } v \in V(G),\\ f(e) &= 1, && \text{for } e \in E(G) \setminus \{e_1, e_2, \dots, e_t\},\\ f(e_i) &= i, && \text{for } i = 1, 2, \dots, t. \end{aligned}$$

For the H-weight of the subgraph H_j, $j = 1, 2, \dots, t$, we obtain

$$\begin{aligned} wt_f(H^j) &= \sum_{v \in V(H^j)} f(v) + \sum_{e \in E(H^j)} f(e)\\ &= \sum_{v \in V(H^j)} f(v) + \sum_{e \in E(H^j) \setminus \{e_j\}} f(e) + f(e_j)\\ &= \sum_{v \in V(H^j)} 1 + \sum_{e \in E(H^j) \setminus \{e_j\}} 1 + j = |V(H_j)| + (|E(H_j)| - 1) + j. \end{aligned}$$

As $|V(H_j)| = |V(H)|$ and $|E(H_j)| = |E(H)|$ for every $j = 1, 2, \dots, t$, we get

$$wt_f(H^j) = |V(H)| + |E(H)| - 1 + j,$$

which means that all H-weights are distinct. This concludes the proof. ■

3. Conclusion

In this paper we introduced a new graph parameter, the total H-irregularity strength, $\text{ths}(G,H)$, as a generalization of the well-known total edge irregularity strength. We proved that for every graph G admitting an H-covering given by t subgraphs isomorphic to H, $\text{ths}(G,H) \geq \lceil 1 + (t-1)/(|V(H)| + |E(H)|) \rceil$ and the sharpness of this bound is reached for the following graphs: the path P_n covered by paths P_m, $m \leq n$, and the ladder covered by a cycle.

Further, we proved that if S_i, $i = 1, 2, \ldots, z$, are all subgraphs of a graph G admitting an H-covering such that S_i is a maximum common subgraph of m_i, $m_i \geq 2$, subgraphs of G isomorphic to H, then $\text{ths}(G, H) \geq \max\{\lceil 1 + (m_1 - 1)/(|V(H/S_1)| + |E(H/S_1)|)\rceil, \ldots, \lceil 1 + (m_z - 1)/(|V(H/S_z)| + |E(H/S_z)|)\rceil\}$. The tightness of this bound was proved for the fan F_n covered by cycles C_3.

We conclude with the following conjecture which is a generalization of the conjecture posed by Ivančo and Jendrol' [14].

Conjecture 10. *Let S_i, $i = 1, 2, \ldots, z$, be all subgraphs of G such that S_i is a maximum common subgraph of m_i, $m_i \geq 2$, subgraphs of G isomorphic to H. Then for every graph G admitting an H-covering given by t subgraphs isomorphic to H, except when G is isomorphic to K_5 and H is isomorphic to K_2, it holds*

$$\text{ths}(G, H) = \max\left\{ \left\lceil 1 + \tfrac{t-1}{|V(H)|+|E(H)|} \right\rceil, \left\lceil 1 + \tfrac{m_1-1}{|V(H/S_1)|+|E(H/S_1)|} \right\rceil, \ldots, \left\lceil 1 + \tfrac{m_z-1}{|V(H/S_z)|+|E(H/S_z)|} \right\rceil \right\}.$$

Acknowledgement

The research for this article was supported by APVV-15-0116.

References

[1] A. Ahmad, M. Bača and M.K. Siddiqui, *On edge irregular total labeling of categorical product of two cycles*, Theory Comput. Syst. **54** (2014) 1–12.
doi:10.1007/s00224-013-9470-3

[2] M. Aigner and E. Triesch, *Irregular assignments of trees and forests*, SIAM J. Discrete Math. **3** (1990) 439–449.
doi:10.1137/0403038

[3] D. Amar and O. Togni, *Irregularity strength of trees*, Discrete Math. **190** (1998) 15–38.
doi:10.1016/S0012-365X(98)00112-5

[4] M. Anholcer and C. Palmer, *Irregular labellings of circulant graphs*, Discrete Math. **312** (2012) 3461–3466.
doi:10.1016/j.disc.2012.06.017

[5] M. Bača, S. Jendrol', M. Miller and J. Ryan, *On irregular total labellings*, Discrete Math. **307** (2007) 1378–1388.
doi:10.1016/j.disc.2005.11.075

[6] M. Bača and M.K. Siddiqui, *Total edge irregularity strength of generalized prism*, Appl. Math. Comput. **235** (2014) 168–173.
doi:10.1016/j.amc.2014.03.001

[7] T. Bohman and D. Kravitz, *On the irregularity strength of trees*, J. Graph Theory **45** (2004) 241–254.
doi:10.1002/jgt.10158

[8] S. Brandt, J. Miškuf and D. Rautenbach, *On a conjecture about edge irregular total labellings*, J. Graph Theory **57** (2008) 333–343.
doi:10.1002/jgt.20287

[9] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, *Irregular networks*, Congr. Numer. **64** (1988) 187–192.

[10] R.J. Faudree, M.S. Jacobson, J. Lehel and R.H. Schlep, *Irregular networks, regular graphs and integer matrices with distinct row and column sums*, Discrete Math. **76** (1988) 223–240.
doi:10.1016/0012-365X(89)90321-X

[11] A. Frieze, R.J. Gould, M. Karoński and F. Pfender, *On graph irregularity strength*, J. Graph Theory **41** (2002) 120–137.
doi:10.1002/jgt.10056

[12] J.A. Gallian, *Graph labeling*, Electron. J. Combin. **17** (2014) 1–389.

[13] K.M.M. Haque, *Irregular total labellings of generalized Petersen graphs*, Theory Comput. Syst. **50** (2012) 537–544.
doi:10.1007/s00224-011-9350-7

[14] J. Ivančo and S. Jendrol', *Total edge irregularity strength of trees*, Discuss. Math. Graph Theory **26** (2006) 449–456.
doi:10.7151/dmgt.1337

[15] S. Jendrol', J. Miškuf and R. Soták, *Total edge irregularity strength of complete and complete bipartite graphs*, Electron. Notes Discrete Math. **28** (2007) 281–285.
doi:10.1016/j.endm.2007.01.041

[16] S. Jendrol', J. Miškuf and R. Soták, *Total edge irregularity strength of complete graphs and complete bipartite graphs*, Discrete Math. **310** (2010) 400–407.
doi:10.1016/j.disc.2009.03.006

[17] M. Kalkowski, M. Karoński and F. Pfender, *A new upper bound for the irregularity strength of graphs*, SIAM J. Discrete Math. **25** (2011) 1319–1321.
doi:10.1137/090774112

[18] P. Majerski and J. Przybyło, *On the irregularity strength of dense graphs*, SIAM J. Discrete Math. **28** (2014) 197–205.
doi:10.1137/120886650

[19] T. Nierhoff, *A tight bound on the irregularity strength of graphs*, SIAM J. Discrete Math. **13** (2000) 313–323.
doi:10.1137/S0895480196314291

[20] Nurdin, A.N.M. Salman and E.T. Baskoro, *The total edge-irregular strengths of the corona product of paths with some graphs*, J. Combin. Math. Combin. Comput. **65** (2008) 163–175.

Received 2 May 2016
Accepted 1 September 2016