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ON THE SPECTRAL CHARACTERIZATIONS OF GRAPHS

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Abstract

Several matrices can be associated to a graph, such as the adjacency matrix or the Laplacian matrix. The spectrum of these matrices gives some informations about the structure of the graph and the question "Which graphs are determined by their spectrum?" is still a difficult problem in spectral graph theory. Let $\mathscr{U}_p^{2q}$ be the set of graphs obtained from C_p by attaching two pendant edges to each of q $(q \leqslant p)$ vertices on C_p, whereas $\mathscr{V}_p^{2q}$ the subset of $\mathscr{U}_p^{2q}$ with odd p and its q vertices of degree 4 being nonadjacent to each other. In this paper, we show that each graph in $\mathscr{U}_p^{2q}$, p even and its q vertices of degree 4 being consecutive, is determined by its Laplacian spectrum. As well we show that if G is a graph without isolated vertices and adjacency cospectral with the graph in $\mathscr{V}_p^{p-1} = \{H\}$, then $G \cong H$.

Keywords: Laplacian spectrum, adjacency spectrum, cospectral graphs, spectral characterization.

2010 Mathematics Subject Classification: 05C50, 15A18.

1. Introduction

Throughout this paper, we only consider simple graph $G = (V_G, E_G)$, where $V_G = \{v_1, v_2, \ldots, v_n\}$ is the vertex set and E_G is the edge set. We call $n = |V_G|$ the *order* of G and $m = |E_G|$ the *size* of G. We follow the notation and terminology in [2] except if otherwise stated.

The *adjacency matrix* $A(G)$ of G is an $n \times n$ matrix with the (i, j)-entry equals to 1 if vertices i and j are adjacent and 0 otherwise. Let $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees, where d_i is the degree of v_i in G for $1 \leqslant i \leqslant n$. The *Laplacian matrix* of G is defined as $L(G) =$

$D(G) - A(G)$, while the *signless Laplacian matrix* is $Q(G) = D(G) + A(G)$. Since $A(G)$ and $L(G)$ are real and symmetric, their eigenvalues are real numbers. The eigenvalues of $A(G)$ and $L(G)$ are called the *eigenvalues* and the *Laplacian eigenvalues* of G, respectively. We denote them, respectively, by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$. The spectrum of $A(G)$ and $L(G)$ are called the *adjacency spectrum* and the *Laplacian spectrum* of G, respectively. Two graphs are *A-cospectral* (respectively *L-cospectral*) if they have the same adjacency spectrum (respectively Laplacian spectrum). A graph is said to be *determined by the adjacency* (respectively *Laplacian*) *spectrum* if there is no other non-isomorphic graph with the same adjacency spectrum (respectively Laplacian spectrum).

Let $C_n, P_n, K_{1,n-1}$ and K_n be the cycle, the path, the star and the complete graph of order n, respectively. Let U_p^{2p} be the graph obtained from C_p by attaching two pendant edges to each vertex of the C_p, while $\mathscr{U}_p^{2q}$ is the set of graphs which are obtained from C_p by attaching two pendant edges to each of q ($q \leq p$) vertices on C_p. Note that $\mathscr{U}_p^{2p} = \left\{U_p^{2p}\right\}$. For $p > 2$ and $0 < q \leqslant p$, we denote by $\bar{U}_p^{2q}$ the graph in $\mathscr{U}_p^{2q}$ whose q vertices of degree 4 are consecutive, whereas we let $\mathscr{V}_p^{2q}$ be the subset of $\mathscr{U}_p^{2q}$ with odd p and its q vertices of degree 4 being non-adjacent to each other. For instance, graphs $U_8^{16}, \bar{U}_8^8$ and $G_1 \in \mathscr{V}_7^6$ are depicted in Figure 1.

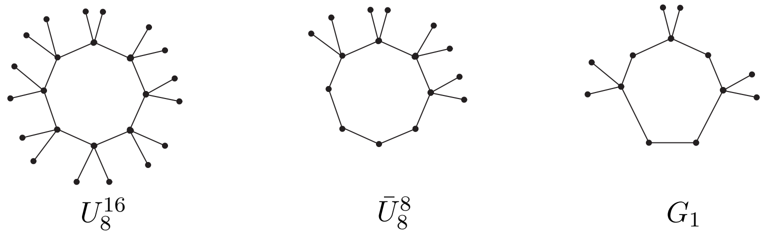

Figure 1. Graphs $U_8^{16}, \bar{U}_8^8$ and $G_1 \in \mathscr{V}_7^6$.

Some structural properties can be deduced from their spectrum, however in general we cannot determine a graph from its adjacency or Laplacian spectrum. Dam and Haemers [10] proposed a natural problem: Which graphs are determined by their spectrum? It is a difficult problem in algebraic graph theory. Spectral characterizations of graphs (with respect to various matrices) did attract much attention in the recent years; see [9, 10]. It has been conjectured by Haemers that almost all graphs are determined by its spectrum. Truth of this conjecture would mean that the spectrum can be used to identify a graph. The paradox is that it is difficult to prove that a given graph is determined by its spectrum. Up to now, many examples of cospectral but non-isomorphic graphs are reported;

see [6]. However, only few of the graphs have been proved to be determined by their spectra [10, 13, 19, 21, 22, 23].

An *odd* (respectively *even*) *sun graph* is obtained by appending a pendant vertex to each vertex of an odd (respectively even) cycle. A *broken sun graph* is a graph obtained by deleting pendant vertices of a sun graph. In 2009, Boulet [3] proved that the sun graph is determined by its Laplacian spectrum and an odd sun graph is determined by its adjacency spectrum. Later in 2010, Mirzakhah and Kiani [18] proved that the sun graph is also determined by its signless Laplacian spectrum. Recently, Bu, Zhou, Li and Wang [5] proved that U_p^{2p} is determined by its signless Laplacian spectrum when $p \neq 32, 64$. They also showed that U_p^{2p} is determined by its Lapacian spectrum.

Motivated from [3, 5, 18], in this paper we show that U_p^{2p} (respectively a graph in $\bar{U}_p^{2q}$ with even p) is determined by its Laplacian spectrum. As well we show that if G is a graph without isolated vertices and A-cospectral with the graph in $\mathscr{V}_p^{p-1} = \{H\}$, then $G \cong H$.

2. Preliminaries

Throughout the text, we shall denote by $\Phi(B) = \det(xI - B)$ the *characteristic polynomial* of the square matrix B. In particular, if $B = A(G)$, we denote $\Phi(A(G))$ by $\phi(G;x)$ and call $\phi(G;x)$ the *characteristic polynomial* of G; if $B = L(G)$, we denote $\Phi(L(G))$ by $\Gamma(G;x)$ and call $\Gamma(G;x)$ the *Laplacian characteristic polynomial* of G. Let $\phi(G;x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ and $SP_A(G)$ (respectively $SP_L(G)$) be the adjacency spectrum (respectively Laplacian spectrum) of the graph G. The *line graph* $l(G)$ of a graph G has the edges of G as its vertices and two vertices of $l(G)$ are adjacent if and only if the corresponding edges in G have a common vertex.

Some fundamental results about the adjacency spectrum and the Laplacian spectrum are the following.

Lemma 2.1 [14]**.** *If H is an induced subgraph of G, then $\lambda_1(H) \leqslant \lambda_1(G)$.*

Lemma 2.2 [12]**.** *Let G be a graph with maximum degree $\Delta(G)$. Then $\lambda_1(G) \geqslant \sqrt{\Delta(G)}$.*

Given a connected graph G, the number of closed walks of length k is denoted by $S_k(G)$.

Lemma 2.3 [7]**.** *For a connected graph G, $S_0(G) = n, S_1(G) = l, S_2(G) = 2m$, $S_3(G) = 6c_3$, where n, l, m, c_3 denote the number of vertices, the number of loops, the number of edges and the number of triangles contained in G, respectively.*

Lemma 2.4 [8]**.** *Let G be a graph on n vertices, c_4 4-cycles and let n_i be the number of vertices of degree i. Then*

$$S_4(G) = 8c_4 + \sum_i in_i + 4\sum_i \frac{i(i-1)}{2}n_i.$$

The following corollary is a direct consequence of Lemma 2.4.

Corollary 2.5. *Let G be a graph on n vertices, m edges, c_4 4-cycles and vertex degrees $d_1, d_2, \ldots, d_n$. Then*

$$S_4(G) = 2\sum_{i=1}^{n} d_i^2 + 8c_4 - 2m.$$

A graph is called an *elementary figure* if it is either a K_2 or a cycle C_q, $q \geqslant 3$. We call U a *basic figure* if all of its connected components are elementary figures.

Lemma 2.6 [7]**.** *Let $p(U)$ be the number of connected components of U and $c(U)$ the number of cycles in U. Then the coefficient a_i of $\phi(G;x)$ is*

$$a_i = \sum_{U \in \mathscr{U}_i} (-1)^{p(U)} \cdot 2^{c(U)},\ i = 1, 2, \ldots, n,$$

where $\mathscr{U}_i$ is the set of basic figures with i vertices of G.

The following lemma is a consequence of Lemma 2.6.

Lemma 2.7 [7]**.** *The length of the shortest odd cycle in G and the number of such cycles are given by the smallest odd index p such that $a_p \neq 0$.*

Lemma 2.7 ensures that a graph is bipartite if and only if its adjacency spectrum is symmetric.

Lemma 2.8 [11]**.** *Let G be a bipartite graph with n vertices, let $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)$ be the Laplacian eigenvalues of G and $\lambda_1(l(G)) \geqslant \lambda_2(l(G)) \geqslant \cdots \geqslant \lambda_n(l(G))$ be the adjacency eigenvalues of the line graph of G. Then $\mu_i(G) = \lambda_i(l(G)) + 2$ for $1 \leqslant i \leqslant n$.*

Lemma 2.9 [15, 16]**.** *Let $G = (V_G, E_G)$ be a graph with maximum degree $\Delta(G)$. Then we have*

$$\Delta(G) + 1 \leqslant \mu_1(G) \leqslant \max\left\{\frac{d_{v_i}(d_{v_i} + m_{v_i}) + d_{v_j}(d_{v_j} + m_{v_j})}{d_{v_i} + d_{v_j}}, v_iv_j \in E_G\right\},$$

where m_{v_i} is the average of the degrees of the vertices adjacent to vertex v_i.

Lemma 2.10 [20]**.** *Let G be an n-vertex graph of size m with $V_G=\{v_1,v_2,\dots,v_n\}$. Put $X_k := \sum_{i=1}^{n}[\mu_i(G)]^k$. If G contains c_3 triangles, c_4 4-cycles and t_i triangles containing vertex v_i, $i=1,2,\dots,n$, then*

$$\begin{aligned}(1)\qquad & X_0 = n,\quad X_1 = 2m,\quad X_2 = 2m+\sum_{i=1}^{n} d_i^2,\quad X_3=\sum_{i=1}^{n} d_i^3+3\sum_{i=1}^{n} d_i^2-6c_3,\\ & X_4=\sum_{i=1}^{n} d_i^4+6\sum_{i=1}^{n} d_i^3+2\sum_{i=1}^{n} d_i^2-8\sum_{i=1}^{n} d_it_i+8c_4-2m.\end{aligned}$$

The *join* of two disjoint graphs G and H, denoted by $G\vee H$, is the graph obtained from $G\cup H$ by joining each vertex of G to each vertex of H by an edge. A graph $G=(V_G,E_G)$ is *unicyclic* if G is connected and $|V_G|=|E_G|$.

Lemma 2.11 [4]**.** *Let G be a unicyclic graph with n ($n\geqslant 6$) vertices. If G is determined by its Laplacian spectrum and $G\ncong C_6$, then $G\vee K_r$ is determined by its Laplacian spectrum for any positive integer r.*

It is known [1] that the number of closed walks of length $k\geqslant 2$ in G is $\sum_{\lambda\in SP_A(G)}\lambda^k$. Let M be a graph and $k>1$ be an integer. Then a *k-covering closed walk* in M is a closed walk of length k in M running through all the edges of M at least once. For a graph G, let $\zeta_M(G)$ (or ζ_M for short) denote the number of all distinct subgraphs (not necessarily induced) of G isomorphic to M and let $\zeta_G(i)$ be the number of closed walks of length i in G. The number of k-covering closed walks in M is denoted by $w_k(M)$ and we define the set $\mathcal{M}_k=\{M, w_k(M)>0\}$. As a consequence (see also in [4]), the number of closed walks of length k in G is

$$(2)\qquad \sum_{\lambda\in SP_A(G)}\lambda^k=\sum_{M\in\mathcal{M}_k} w_k(M)\zeta_M(G).$$

Lemma 2.12 [17]**.** *The number of closed walks of lengths $6,7$ for a graph G are respectively determined as follows, where m is number of edges of G and the graphs used are depicted in Figure 2.*

$$\begin{aligned}\zeta_G(6) &= 2m+12\zeta_G(P_3)+6\zeta_G(P_4)+48\zeta_G(C_4)+12\zeta_G(C_6)+24\zeta_G(K_3)\\ (3)\qquad &\quad + 12\zeta_G(K_{1,3})+36\zeta_G(G_h)+12\zeta_G(G_j)+24\zeta_G(G_o).\\ \zeta_G(7) &= 126\zeta_G(C_3)+70\zeta_G(C_5)+14\zeta_G(C_7)+84\zeta_G(G_a)+14\zeta_G(G_b)\\ &\quad + 14\zeta_G(G_c)+14\zeta_G(G_d)+28\zeta_G(G_e)+42\zeta_G(G_f)+28\zeta_G(G_g)\\ (4)\qquad &\quad + 112\zeta_G(G_h)+84\zeta_G(H_b).\end{aligned}$$

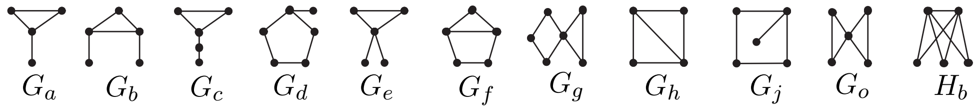

Figure 2. Graphs related to $\zeta_G(6)$ and $\zeta_G(7)$.

3. Laplacian Spectral Characterizations of U_p^{2p} and Graphs in $\mathscr{U}_p^{2q}$

It is known that two L-cospectral (respectively A-cospectral) graphs have the same order and size. We know from [14], that a graph is connected if and only if its Laplacian spectrum contains just one zero eigenvalue, whereas the number of spanning trees in a connected graph G is $\frac{1}{n}\prod_{i=1}^{n-1}\mu_i$, where $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} > \mu_n = 0$ are the Laplacian eigenvalues of G.

Lemma 3.1. *Let G be a graph L-cospectral with a graph H in $\mathscr{U}_p^{2q}$. Then G is a unicyclic graph of girth p.*

Proof. Since H is connected, G is connected. Then Lemma 3.1 follows from the facts: A unicyclic graph is connected and its size equals its order. The number of spanning trees of a unicyclic graph equals the length of the cycle contained in it. ■

The following Lemma follows directly from Lemma 2.9.

Lemma 3.2. *Let G be a graph L-cospectral with a graph in $\mathscr{U}_p^{2q}$, $\Delta(G)$ be the maximum degree of G. Then $\Delta(G) \leqslant 5$.*

Theorem 3.3. *Let G be a graph L-cospectral with a graph H in $\mathscr{U}_p^{2q}$ with $p > 2$ and $1 \leqslant q \leqslant p$. Then G is also in $\mathscr{U}_p^{2q}$.*

Proof. We first show that G has the same degree sequence with that of H. By Lemmas 3.1 and 3.2, G is a unicyclic graph of girth p with $\Delta(G) \leqslant 5$. Let n_i be the number of vertices of degree i, $i = 1, 2, 3, 4, 5$, of G. On the one hand, by Lemma 2.10, $\sum d_i$, $\sum d_i^2$, $\sum d_i^3$ can be determined by the $SP_L(G)$. Hence, in view of (1), if $p \geqslant 4$, $\sum d_i^4$ can be also determined by the $SP_L(G)$. On the other hand, for the graph H, let d_i' denotes the degree of v_i in H, $i = 1, 2, \ldots, n$. Then we have

$$\begin{aligned}
\sum d_i' &= 2q + 2(p-q) + 4q = 2p + 4q,\\
\sum d_i'^2 &= 2q + 4(p-q) + 16q = 4p + 14q,\\
\sum d_i'^3 &= 2q + 8(p-q) + 64q = 8p + 58q,\\
\sum d_i'^4 &= 2q + 16(p-q) + 256q = 16p + 242q.
\end{aligned}$$

Note that G, H are L-cospectral. Hence, we have

$$(5)\qquad \begin{cases} n_1+n_2+n_3+n_4+n_5 = p+2q, \\ n_1+2n_2+3n_3+4n_4+5n_5 = 2p+4q, \\ n_1+4n_2+9n_3+16n_4+25n_5 = 4p+14q, \\ n_1+8n_2+27n_3+64n_4+125n_5 = 8p+58q. \end{cases}$$

Solving (5), we get

$$(6)\qquad \begin{cases} n_1 = 2q+n_5, \\ n_2 = p-q-4n_5, \\ n_3 = 6n_5, \\ n_4 = q-4n_5. \end{cases}$$

If $p=3$, then $q \leqslant 3$. Note that $n_4 \geqslant 0$; hence we have $4n_5 \leqslant q \leqslant 3$, $n_5 = 0$. So we get $n_1 = 2q, n_2 = p-q, n_4 = q, n_3 = n_5 = 0$.

If $p \geqslant 4$, then we have

$$(7)\qquad n_1+16n_2+81n_3+256n_4+625n_5 = 16p+242q.$$

Substituting (6) into (7) yields $n_5 = 0$. So we get $n_1 = 2q, n_2 = p-q, n_4 = q, n_3 = n_5 = 0$.

Thus G and H have the same degree sequence. If there are $q' < q$ vertices of degree 4 belonging to the p-cycle in G, then $n_2 \geqslant p-q' > p-q$, a contradiction. Therefore, there are just q vertices of degree 4 on the cycle C_p, whence G is in $\mathscr{U}_p^{2q}$.

This completes the proof. ∎

Note that $\mathscr{U}_p^{2p} = \left\{U_p^{2p}\right\}$; the following result is a direct consequence of Theorem 3.3, which is one of the main results obtained by Bu *et al.* in [5].

Corollary 3.4 [5]**.** *U_p^{2p} is determined by its Laplacian spectrum.*

Corollary 3.5. *$U_p^{2p} \vee K_r$ is determined by its Laplacian spectrum for any positive r.*

Proof. It follows immediately from Lemma 2.11 and Corollary 3.4. ∎

Lemma 3.6. *Given two integers p, q with $p \in \mathbb{N}\backslash\{0,1,2,3,4,5,7\}$ and $1 < q < p-1$, let G be a graph in $\mathscr{U}_p^{2q} \setminus \left\{\bar{U}_p^{2q}\right\}$. Then we have*

$$\sum_{\lambda \in SP_A(l(G))} \lambda^7 < \sum_{\lambda \in SP_A\left(l\left(\bar{U}_p^{2q}\right)\right)} \lambda^7.$$

Proof. In view of (2), we have

$$\sum_{\lambda \in SP_A(l(G))} \lambda^7 = \sum_{M \in \mathcal{M}_7} w_7(M)\zeta_M(l(G)), \tag{8}$$

where $\mathcal{M}_7 = \{C_3, G_a, G_b, G_c, G_e, G_g, G_h, C_7\}$ if $p = 6$ and $\mathcal{M}_7 = \{C_3, G_a, G_b, G_c, G_e, G_g, G_h\}$ otherwise. Graphs $G_a, G_b, G_c, G_e, G_g, G_h$ are depicted in Figure 2.

As an odd closed walk necessarily runs through an odd cycle, it is routine to check that if $M \in \mathcal{M}_7$ is a subgraph of $l(G)$ or $l\left(\bar{U}_p^{2q}\right)$, then M contains at most 2 triangles or M contains one and only one 7-cycle. Only the graphs C_3 and $G_a, G_b, G_c, G_e, G_g, G_h \in \mathcal{M}_7$ depicted in Figure 2 and the 7-cycle C_7 can arise as subgraphs of $l(G)$ and $l\left(\bar{U}_p^{2q}\right)$, and the 7-cycle C_7 can arise if and only if $p = 6$.

In view of (4), we have

$$\begin{aligned}
&w_7(C_3) = 126, \quad w_7(G_a) = 84, \quad w_7(G_e) = 28, \quad w_7(G_b) = 14,\\
&w_7(G_c) = 14, \quad w_7(G_h) = 112, \quad w_7(G_g) = 28.
\end{aligned}$$

For the graph $l\left(\bar{U}_p^{2q}\right)$ with $q < p - 1$, we count the number of its subgraphs isomorphic to C_3 (respectively G_a, G_b, G_c, G_e, G_g, G_h) as follows.

$$\begin{aligned}
&\zeta_{C_3} = 4q, \quad \zeta_{G_a} = 2 \cdot 9(q-2) + 2 \cdot 6(q-2) + 2 \cdot (2 \cdot 7 + 4 + 6) = 30q - 12,\\
&\zeta_{G_b} = 2 \cdot 21(q-2) + 2 \cdot 6(q-2) + 2 \cdot (2 \cdot 11 + 2 + 6) = 54q - 48,\\
&\zeta_{G_c} = 2 \cdot 18(q-4) + 2 \cdot 18(q-4) + 2 \cdot (2 \cdot 16 + 18 + 16 + 2 \cdot 10 + 10 + 12)\\
&\quad = 72q - 72,\\
&\zeta_{G_e} = 2 \cdot 12(q-2) + 2 \cdot 6(q-2) + 2 \cdot (2 \cdot 7 + 1 + 6) = 36q - 30,\\
&\zeta_{G_g} = 3 \cdot 6(q-2) + 2 \cdot 3 \cdot 3 = 18q - 18, \quad \zeta_{G_h} = 6q.
\end{aligned}$$

Moreover, $\zeta_{C_7} = 2q\delta_p^6$ and $w_7(C_7) = 14$, where $\delta_p^6 = 1$ if $p = 6$ and 0 otherwise. Thus we have

$$\sum_{\lambda \in SP_A\left(l\left(\bar{U}_p^{2q}\right)\right)} \lambda^7 = \sum_{M \in \mathcal{M}_7} w_7(M)\zeta_M\left(l\left(\bar{U}_p^{2q}\right)\right) = 6972q - 4032 + 28q\delta_p^6. \tag{9}$$

We call H the *chain of* K_4 if H is the line graph of a tree obtained by appending two pendant vertices to each vertex of degree 2 of a path (see Figure 3 for an example) and we define the *length* of a chain of K_4 as the number of K_4 contained in it.

Let $l_1, l_2, \ldots, l_r$ ($r \geqslant 2$) be the maximal lengths of the chains of K_4 contained in $l(G)$. Note that $G \in \mathscr{U}_p^{2q} \setminus \left\{\bar{U}_p^{2q}\right\}$; hence we have $\sum_{i=1}^{r} l_i = q$. For

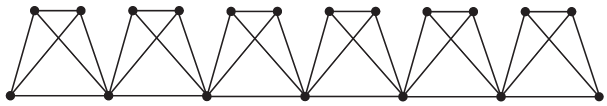

Figure 3. A chain of K_4 of length 6.

the graph $l(G)$, we count the number of its subgraphs isomorphic to G_a (resp. $G_b, G_c, G_e, G_g, C_3, C_7, G_h$) as follows.

$$\begin{aligned}
\zeta_{G_a}(l(G)) &= \sum_{i=1}^{r}[2\cdot 9(l_i-2)+2\cdot 6(l_i-2)+2\cdot(2\cdot 7+4+6)] = 30q-12r,\\
\zeta_{G_b}(l(G)) &< \sum_{i=1}^{r}(2\cdot 21l_i+2\cdot 6l_i) = 54q,\\
\zeta_{G_c}(l(G)) &< \sum_{i=1}^{r}(2\cdot 18l_i+2\cdot 18l_i) = 72q,\\
\zeta_{G_e}(l(G)) &= \sum_{i=1}^{r}[2\cdot 12(l_i-2)+2\cdot 6(l_i-2)+2\cdot(2\cdot 7+1+6)] = 36q-30r,\\
\zeta_{G_g}(l(G)) &= \sum_{i=1}^{r}[3\cdot 6(l_i-2)+2\cdot 3\cdot 3] = 18q-18r,\\
\zeta_{C_3}(l(G)) &= \sum_{i=1}^{r}4l_i = 4q, \quad \zeta_{C_7}(l(G)) = 2q\delta_p^6, \quad \zeta_{G_h}(l(G)) = \sum_{i=1}^{r}6l_i = 6q.
\end{aligned}$$

Together with (8) we have

$$\begin{aligned}
\sum_{\lambda\in SP_A(l(G))}\lambda^7 &= \sum_{M\in\mathcal{M}_7} w_7(M)\zeta_M(l(G))\\
&< 6972q-2352r+28q\delta_p^6 \leqslant 6972q-4704+28q\delta_p^6,
\end{aligned}$$

where the last inequality follows by $r \geqslant 2$. Thus comparing with (9) it yields

$$\sum_{\lambda\in SP_A(l(G))}\lambda^7 < \sum_{\lambda\in SP_A\left(l\left(\bar{U}_p^{2q}\right)\right)}\lambda^7,$$

as desired. ■

Theorem 3.7. *The graph $\bar{U}_p^{2q}$ with even $p > 2$, $0 < q < p$ is determined by its Laplacian spectrum.*

Proof. Let G be a graph L-cospectral with $\bar{U}_p^{2q}$ for some fixed p, q, where $p \geqslant 4$ is even and $0 < q < p$. By Theorem 3.3, G is in $\mathscr{U}_p^{2q}$. If $q = 1$ or $q = p-1$, then

$\left|\mathscr{U}_p^{2q}\right| = 1$ and hence G is isomorphic to $\bar{U}_p^{2q}$ in this case. So in what follows, we consider $1 < q < p-1$.

Figure 4. Graphs $\bar{U}_4^4$ and H belonging to $\mathscr{U}_4^4$.

Case 1. $p = 4$. In this case, we obtain $q = 2$ and $\mathscr{U}_4^4 = \left\{\bar{U}_4^4, H\right\}$, where $\bar{U}_4^4, H$ are depicted in Figure 4. By direct calculation, we have

$$\begin{aligned} \Gamma\left(\bar{U}_4^4; x\right) &= x^8 - 16x^7 + 98x^6 - 296x^5 + 477x^4 - 416x^3 + 184x^2 - 32x, \\ \Gamma(H; x) &= x^8 - 16x^7 + 98x^6 - 296x^5 + 481x^4 - 424x^3 + 188x^2 - 32x. \end{aligned}$$

which implies that $\bar{U}_4^4$ and H are not L-cospectral. Hence, $G \cong \bar{U}_4^4$.

Case 2. $p \geqslant 6$. Note that G and $\bar{U}_p^{2q}$ are bipartite; hence by Lemma 2.8 we have $l(G)$ and $l\left(\bar{U}_p^{2q}\right)$ have the same adjacency spectrum. If $G \ncong \bar{U}_p^{2q}$, then by Lemma 3.6 we have

$$\sum_{\lambda \in SP_A(l(G))} \lambda^7 < \sum_{\lambda \in SP_A\left(l\left(\bar{U}_p^{2q}\right)\right)} \lambda^7,$$

a contradiction.

This completes the proof. ∎

By Lemma 2.11 and Theorem 3.7 the next result follows immediately.

Corollary 3.8. *The graph $\bar{U}_p^{2q} \vee K_r$ is determined by its Laplacian spectrum, where $p > 2$ is even, $0 < q < p$ and r is a positive integer.*

4. ADJACENCY SPECTRAL CHARACTERIZATIONS OF GRAPHS IN $\mathscr{V}_p^{2q}$

In this section we study the adjacency spectral characterizations of graphs in $\mathscr{V}_p^{2q}$. For convenience, let $\lambda^{(k)}(G)$ denote an adjacency eigenvalue λ of graph G with multiplicity k.

The *corona* $G_1 \circ G_2$ is obtained by taking one copy of G_1 and $|V_{G_1}|$ copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1 by an edge, $i = 1, 2, \ldots, |V_{G_1}|$.

Lemma 4.1 [1]**.** *Let G_1 be a graph with n_1 vertices, G_2 be an r-regular graph with n_2 vertices. Then the adjacency spectrum of $G_1 \circ G_2$ is*

$$\left\{ \frac{\lambda_i(G_1) + r \pm \sqrt{(r - \lambda_i(G_1))^2 + 4n_2}}{2},\ i = 1, 2, \dots, n_1 \right\} \cup \left\{ \lambda_j^{(n_1)}(G_2),\ j = 2, 3, \dots, n_2 \right\},$$

where the exponent (n_1) denotes the multiplicity of eigenvalues λ_j, $j = 2, 3, \dots, n_2$.

Lemma 4.2. *If G is A-cospectral with a graph in $\mathscr{U}_p^{2q}$, then $\lambda_1(G) \leqslant 1 + \sqrt{3}$ and $\Delta(G) \leqslant 7$.*

Proof. Assume that G, H are A-cospectral with $H \in \mathscr{U}_p^{2q}$. The adjacency spectra of C_p and $2K_1$ are $2\cos\frac{2\pi i}{p}$ $(i = 1, 2, \dots, p)$ and $0^{(2)}$, respectively. Note that $U_p^{2p} = C_p \circ 2K_1$. Hence, by Lemma 4.1 we have $\lambda_1\left(U_p^{2p}\right) = 1 + \sqrt{3}$. It is clear that H is a subgraph of U_p^{2p}. Then Lemma 4.2 follows directly from Lemmas 2.1 and 2.2. ■

Lemma 4.3. *Let G be a graph A-cospectral with $H \in \mathscr{U}_p^{2q}$, p odd and $q \leqslant p$. Then*

(i) *The coefficient a_t, t odd, $t \neq p$, of $\phi(G; x)$ is zero;*
(ii) *The coefficient a_p of $\phi(G; x)$ is non-zero;*
(iii) *The length of the shortest odd cycle of G is p and G has one and only one p-cycle.*

Proof. (i) As t is odd, by Lemma 2.6, a basic figure with t $(\neq p)$ vertices necessarily contains an odd cycle, and it becomes clear that there are no such basic figures in H.

(ii) The length of the shortest odd cycle of H is p and we apply Lemma 2.7.
(iii) It is a direct consequence of Lemma 2.7. ■

Lemma 4.4. *Let G be a graph A-cospectral with $H \in \mathscr{U}_p^{2q}, p$ odd and $q \leqslant \frac{p}{2}$.*

(i) *A connected component of G different from an isolated vertex cannot be bipartite.*
(ii) *If G has no isolated vertices, then G is unicyclic.*

Proof. (i) Let λ be a non-zero eigenvalue of the adjacency matrix of G. Note that $\phi(G; -\lambda) = (-1)^n(\phi(G; \lambda) - 2a_p\lambda^{n-p}) = (-1)^{n+1} \cdot 2a_p\lambda^{n-p}$; hence by Lemma 4.3 we have $\phi(G; -\lambda) \neq 0$. Thus we obtain that if $\lambda \neq 0$ is an eigenvalue of G, then $-\lambda$ is not an eigenvalue of G. A connected component of G different from an isolated vertex cannot be bipartite, since otherwise there would exist an eigenvalue λ such that $-\lambda$ is also an eigenvalue (based on the fact that the spectrum of a bipartite graph is symmetric).

(ii) According to (i), each connected component of G is non-bipartite, which implies each connected component must contain an odd cycle. By Lemma 4.3, the length of the shortest odd cycle of G is p and G has one and only one p-cycle. Note that $|V_G| = p + 2q \leqslant 2p$; it ensures that G cannot have more than one connected component. Thus G is unicyclic. ■

Lemma 4.5. *Let G be a graph without isolated vertices and A-cospectral with $H \in \mathscr{U}_p^{2q}, p$ odd and $q \leqslant \frac{p}{2}$. Then there are no vertices at distance $d > 1$ from the p-cycle and a vertex of G is at distance 1 from the p-cycle if and only if it is a pendant vertex.*

Proof. In view of Lemmas 4.3 and 4.4, we obtain that G is connected and unicyclic, C_p is the unique cycle contained in G. It is routine to check that $a_{p+2} = 0$ (based on Lemma 4.3). Combining with Lemma 2.6, we get that $C_p \cup K_2$ is not a subgraph of G. Therefore, there do not exist vertices at distance $d > 1$ from the p-cycle in G. It involves that a pendant vertex is at distance 1 from the p-cycle and that a vertex at distance 1 from the p-cycle is necessarily a pendant vertex. ■

Lemma 4.6. *Let G be a graph without isolated vertices and A-cospectral with $H \in \mathscr{V}_p^{2q}$ and $q \leqslant \frac{p-1}{2}$. Then $G \in \mathscr{V}_p^{2q}$.*

Proof. By Lemmas 4.3–4.5, G is a unicyclic graph containing a p-cycle and each pendant vertex of G is at distance 1 from the p-cycle. Let d_i (respectively d_i') denote the vertex degree of v_i in G (respectively H).

By Lemma 4.2, we have $\Delta(G) \leqslant 7$. Let n_i be the number of vertices of degree $i, i = 1, 2, \ldots, 7$, of G. On the one hand, G, H are A-cospectral, based on $\sum \lambda_i^2 = S_2(G) = 2|E_G| = \sum d_i$, and we obtain that $SP_A(G)$ determines $\sum d_i$ (respectively $\sum d_i'$). Note that G, H contain no 4-cycle, by Corollary 2.5, $SP_A(G)$ also determines $\sum d_i^2$ (respectively $\sum d_i'^2$). On the other hand, for the graph H we have $\sum d_i' = 2q+2(p-q)+4q = 2p+4q$ and $\sum d_i'^2 = 2q+4(p-q)+16q = 4p+14q$. So we get

$$(10) \qquad \begin{cases} n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = p + 2q, \\ n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7 = 2p + 4q, \\ n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 + 36n_6 + 49n_7 = 4p + 14q. \end{cases}$$

By Lemma 4.5, the vertices of G of degree strictly greater than 1 are exactly the vertices on the p-cycle. Hence, we have

$$(11) \qquad n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = p.$$

From (10) and (11), we have

$$
\begin{cases}
n_1 = 2q, \\
n_2 = p - q - n_5 - 3n_6 - 6n_7, \\
n_3 = 3n_5 + 8n_6 + 15n_7, \\
n_4 = q - 3n_5 - 6n_6 - 10n_7.
\end{cases} \tag{12}
$$

If $p = 3$, then $q = 1$. As $n_4 \geqslant 0$, we have $3n_5 + 6n_6 + 10n_7 \leqslant q = 1$, $n_5 = n_6 = n_7 = 0$. So we get $n_1 = 2, n_2 = 2, n_3 = 0, n_4 = 1, n_5 = n_6 = n_7 = 0$. Thus $G \cong \bar{U}_3^2$, which is in $\mathscr{V}_3^2$, as desired. So, in what follows we consider $p \geqslant 5$.

According to the structure of G (respectively H), it is routine to check that only the graphs $K_2, P_3, P_4, K_{1,3} \in \mathcal{M}_6$ can arise as subgraphs of G (respectively H). In view of (3), we have

$$w_6(K_2) = 2, \quad w_6(P_3) = 12, \quad w_6(P_4) = 6, \quad w_6(K_{1,3}) = 12.$$

As graph H is in $\mathscr{V}_p^{2q}$, $q \leqslant \frac{p}{2}$, we get

$$
\begin{aligned}
&\zeta_{K_2}(H) = |E_H| = p + 2q, && \zeta_{P_3}(H) = \textstyle\sum \binom{d_i}{2} = p + 5q, \\
&\zeta_{P_4}(H) = p + 4q, && \zeta_{K_{1,3}}(H) = \textstyle\sum \binom{d_i}{3} = 4q.
\end{aligned}
$$

In view of (2), we have

$$\sum_{\lambda \in SP_A(H)} \lambda^6 = \sum_{M \in \mathcal{M}_6} w_6(M)\zeta_M(H) = 20p + 136q.$$

Similarly, for the graph G, we have

$$
\begin{aligned}
\zeta_{K_2}(G) &= |E_G| = \frac{n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 + 7n_7}{2}, \\
\zeta_{P_3}(G) &= \sum_{i=1}^{7} \binom{i}{2} n_i = n_2 + 3n_3 + 6n_4 + 10n_5 + 15n_6 + 21n_7, \\
\zeta_{K_{1,3}}(G) &= \sum_{i=1}^{7} \binom{i}{3} n_i = n_3 + 4n_4 + 10n_5 + 20n_6 + 35n_7, \\
\zeta_{P_4}(G) &\geqslant p + 2n_1.
\end{aligned} \tag{13}
$$

The equality in (13) holds if and only if the vertices of G of degree strictly greater than 2 are not adjacent to each other.

Once again, by (2) we have

$$
\begin{aligned}
\sum_{\lambda \in SP_A(G)} \lambda^6 = \sum_{M \in \mathcal{M}_6} & w_6(M)\zeta_M(G) \\
& \geqslant 6p + 13n_1 + 14n_2 + 51n_3 + 124n_4 + 245n_5 + 426n_6 + 679n_7
\end{aligned}
$$

with equality if and only if the vertices of G of degree strictly greater than 2 are not adjacent to each other.

Note that $\sum_{\lambda\in SP_A(H)} \lambda^6 = \sum_{\lambda\in SP_A(G)} \lambda^6$ (based on G, H being A-cospectral). Hence,

$$13n_1 + 14n_2 + 51n_3 + 124n_4 + 245n_5 + 426n_6 + 679n_7 \leqslant 14p + 136q, \tag{14}$$

with equality if and only if the vertices of G of degree strictly greater than 2 are not adjacent to each other.

Substituting (12) into (14), we get

$$n_5 + 4n_6 + 10n_7 \leqslant 0, \tag{15}$$

with equality if and only if the vertices of G of degree strictly greater than 2 are not adjacent to each other, whereas (15) implies that $n_5 = n_6 = n_7 = 0$, hence $n_3 = 0$ and $n_1 = 2q, n_2 = p - q, n_4 = q$ (based on (12)). Therefore, G is in $\mathscr{V}_p^{2q}$, as desired. ∎

Based on Lemma 4.6, one can deduce the main result in this section directly.

Theorem 4.7. *Let G be a graph without isolated vertices and A-cospectral with $H \in \mathscr{V}_p^{p-1}$. Then $G \cong H$.*

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