# PROPER CONNECTION OF DIRECT PRODUCTS 

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#### Abstract

The proper connection number of a graph is the least integer $k$ for which the graph has an edge coloring with $k$ colors, with the property that any two vertices are joined by a properly colored path. We prove that given two connected non-bipartite graphs, one of which is (vertex) 2-connected, the proper connection number of their direct product is 2 .


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An edge coloring of a graph is an assignment of colors to its edges. A proper edge coloring is an edge coloring for which adjacent edges never have the same color. The proper connection number of a graph is the least integer $k$ for which the graph has an edge coloring with $k$ colors, with the property that any two vertices are joined by a properly colored path. The proper connection number of a graph $G$ is denoted $\operatorname{pc}(G)$. This invariant has been studied in $[2,6]$ and is a natural extension of the rainbow connection number of a graph $[3,4,5]$. (The rainbow connection number of $G$ is the minimum number of colors needed to edge-color $G$ in such a way that any two vertices are joined by a path for which no two edges are colored the same.)

The rainbow connection number of graph powers and graph products is investigated in [1]. (See [7] for a survey of graph products.) A recent paper [8] determines the proper connection number of three of the four standard graph
products. For the Cartesian product, the authors show $\operatorname{pc}(G \square H)=2$ for nontrivial connected graphs $G$ and $H$. For the strong product $\mathrm{pc}(G \boxtimes H)$ is either 1 or 2 depending on whether or not $G$ and $H$ are both complete. A similar result holds for the lexicographic product, where $\mathrm{pc}(G \circ H)$ is 1 or 2 , depending on whether or not the product is complete. However, the proper connection number of a direct product $G \times H$ is not known. We prove here that if $G$ and $H$ are connected non-bipartite graphs and one is 2 -connected, then the proper connection number of their direct product is 2 .

Recall that the direct product of $G$ and $H$ is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and edges $\left\{(g, h)\left(g^{\prime}, h^{\prime}\right) \mid g g^{\prime} \in E(G)\right.$ and $\left.h h^{\prime} \in E(H)\right\}$. Figure 1 shows an example. Here neither factor is 2 -connected, and the proper connection number of the product exceeds 2 because in any edge 2 -coloring a pair of the vertices $a, b, c$ is joined only by a monochromatic path. Thus the assumption of 2 -connectivity in our result cannot be relaxed.


Figure 1. A direct product with $\operatorname{pc}(G \times H)>2$.
Our results involve simple undirected finite graphs without loops, though our proofs use orientations. Denote the vertices of an $n$-cycle $C_{n}$ as $0,1,2, \ldots, n-1$; its edges are $i(i+1)$, with addition modulo $n$.

Given two cycles $C_{m}$ and $C_{n}$, we define the standard edge 2 -coloring of the product $C_{m} \times C_{n}$ to be the assignment of two colors, bold and dashed, to the edges of $C_{m} \times C_{n}$ such that any edge of form $(i, j)(i+1, j+1)$ is colored bold, and any edge of form $(i, j)(i+1, j-1)$ is colored dashed (with arithmetic modulo $m$ and $n$ on respective arguments). This is illustrated in Figure 3, for odd cycles $m=2 p+1$ and $n=2 q+1$, where we regard the product as embedded on a torus. The left-most column of vertices is identified with the right-most column, and the top row of vertices is identified with the bottom row.


Figure 2. The four paths when $x$ and $y$ are both even. For clarity the product is shown embedded in a torus.

Lemma 1. The proper connection number of the direct product of two odd cycles is 2. Further, in the standard edge 2 -coloring any two vertices are joined by four types of properly colored paths, namely those that

- begin in bold and end in dashed,
- begin in dashed and end in bold,
- begin in bold and end in bold,
- begin in dashed and end in dashed.

Proof. Let the cycles be $C_{2 p+1}$ and $C_{2 q+1}$. Give $C_{2 p+1} \times C_{2 q+1}$ the standard edge 2-coloring. We now show that any two vertices in the product are joined by paths of the prescribed types. By symmetry we can assume one vertex is $(0,0)$. Say the other is $(x, y)$. We break into cases, depending on the parity of $x$ and $y$.

First assume $x$ and $y$ are both even. The following paths have the prescribed types. (These paths are illustrated in Figure 2.)
$(0,0)(1,1)(0,2)(1,3) \quad \ldots(1, y-1)(0, y) \mid(1, y+1)(2, y)(3, y+1)(4, y) \ldots(x-1, y+1)(x, y)$
$(0,0)(1,-1)(2,0)(3,-1) \quad \ldots(x-1,-1)(x, 0) \mid(x-1,1)(x, 2)(x-1,3)(x, 4) \ldots(x-1, y-1)(x, y)$
$(0,0)(-1,-1)(-2,0)(-3,-1) \ldots(x+1,0)(x,-1) \mid(x+1,-2)(x,-3)(x+1,-4) \ldots(x+1, y+1)(x, y)$
$(0,0)(1,-1)(0,-2)(1,-3) \quad \ldots(0, y+1)(1, y) \mid(0, y-1)(-1, y)(-2, y-1) \ldots(x+1, y-1)(x, y)$
For clarity an artificial separating bar | shows where the pattern switches from alternating back and forth along an edge in one factor to alternating in the other.

The case in which $x$ and $y$ are both odd is similar, though we will not write the four paths explicitly. The construction is illustrated in Figure 3. The case where $x$ and $y$ have opposite parity is shown in Figure 4.


Figure 3. The four paths when $x$ and $y$ are both odd.


Figure 4. The four paths when $x$ and $y$ have opposite parity.

The proof of our main result will use ear decompositions. Recall that an ear decomposition of a graph is an edge-disjoint sequence $C, P_{1}, P_{2}, P_{3}, \ldots, P_{k}$, where $C$ is a cycle in the graph, each $P_{i}$ is a path whose internal vertices have degree 2 in $C \cup P_{1} \cup P_{2} \cup \cdots \cup P_{i}$, and any edge of the graph belongs to a unique member of the sequence. A theorem of Whitney $[9,10]$ holds that a graph is 2 -connected if and only if it has an ear decomposition, and, moreover, an ear decomposition may begin with any cycle of the graph.

Theorem 2. If $G$ and $H$ are connected non-bipartite graphs, and one of them is (vertex) 2-connected, then $\operatorname{pc}(G \times H)=2$.

Proof. Let $G$ and $H$ be as stated, with $H$ 2-connected.
First we argue that it suffices to assume that $G$ has a particularly simple structure. Let $K$ be a connected spanning subgraph of $G$ that has only one cycle, $B$, which is an odd cycle (as in Figure 5). Then $K \times H$ is a connected spanning non-complete subgraph of $G \times H$, so $1<\mathrm{pc}(G \times H) \leq \mathrm{pc}(K \times H)$. Thus it suffices to prove the proposition for $K \times H$ instead of $G \times H$. Equivalently, there is no loss of generality in assuming that $G$ has only one cycle $B$, which is odd. We assume this henceforward.

Next we define an edge 2-coloring of $G \times H$. (We will eventually show that under this coloring, any two vertices of $G \times H$ are joined by a properly colored path.) Our coloring will be defined in terms of certain orientations of $G$ and $H$.

Give $G$ an orientation for which $B$ is a directed cycle and all other edges are oriented toward it, as shown in Figure 5.


Figure 5. Orientation of the graph $G$.
We next construct an orientation of $H$ that has neither sources nor sinks. Give $H$ an ear decomposition $C, P_{1}, P_{2}, P_{3}, \ldots, P_{k}$ for which $C$ is an odd cycle. Orient the edges of $C$ so that it is a directed cycle, and orient the edges of each $P_{i}$ so that it is a directed path, as in Figure 6. (Each $P_{i}$ has two such orientations; choose one arbitrarily.) By construction this orientation has neither sources nor sinks.

Now we define our edge 2-coloring of $G \times H$. Color an edge $(g, h)\left(g^{\prime}, h^{\prime}\right)$ bold if $g g^{\prime}$ is directed from $g$ to $g^{\prime}$ in the orientation of $G$ and $h h^{\prime}$ is directed from $h$ to $h^{\prime}$ in the orientation of $H$. (Or, symmetrically, if $g g^{\prime}$ is directed from $g^{\prime}$ to $g$ and $h h^{\prime}$ from $h^{\prime}$ to $h$.) Color $(g, h)\left(g^{\prime}, h^{\prime}\right)$ dashed if $g g^{\prime}$ is directed from $g$ to $g^{\prime}$ but $h h^{\prime}$ is directed from $h^{\prime}$ to $h$.

In essence, $(g, h)\left(g^{\prime}, h^{\prime}\right)$ is colored bold if $g g^{\prime}$ and $h h^{\prime}$ are oriented the same (both left to right, or both right to left), and $(g, h)\left(g^{\prime}, h^{\prime}\right)$ is colored dashed if $g g^{\prime}$ and $h h^{\prime}$ are oriented oppositely.


Figure 6. Orientation of the ear decomposition of $H$.

Notice that under this coloring the subgraph $B \times C$ has the standard edge 2 -coloring for the product of two cycles. Lemma 1 says that any two vertices of $B \times C$ are joined by properly colored paths that begin and end with edges of any color we desire. We claim that this same property holds for $B \times H$.

Claim. Consider the subgraph $B \times H \subseteq G \times H$. With the 2 -coloring inherited from $G \times H$, the graph $B \times H$ has the property that any two of its vertices can be joined by paths that begin and end with all possible combinations of the two colors (as in Lemma 1).

To prove the claim, take two vertices $\left(b_{0}, h_{0}\right)$ and $\left(b_{0}^{\prime}, h_{0}^{\prime}\right)$ of $B \times H$. We now produce properly colored paths that join them and meet the requirements of the proposition. If it happens that both $\left(b_{0}, h_{0}\right)$ and $\left(b_{0}^{\prime}, h_{0}^{\prime}\right)$ belong to $B \times C$, then the claim follows from Lemma 1 because the 2-coloring of $G \times H$ restricts to the standard edge 2 -coloring of the product of cycles $B \times C$. Otherwise, at least one of $h_{0}$ and $h_{0}^{\prime}$ is not a vertex of $C$ (though possibly $h_{0}=h_{0}^{\prime}$ ). Because $H$ is 2-connected, $H-E(C)$ has two paths $P: h_{0} h_{1}, h_{2} \cdots h_{k}$ and $P^{\prime}: h_{0}^{\prime} h_{1}^{\prime}, h_{2}^{\prime} \cdots h_{\ell}^{\prime}$ that are vertex-disjoint (except possibly $h_{0}=h_{0}^{\prime}$ ), and whose terminal vertices $h_{k}$ and $h_{\ell}^{\prime}$ belong to $C$, but for which no internal vertices belong to $C$. (Possibly one of these paths is trivial if $h_{0}$ or $h_{0}^{\prime}$ already belongs to $C$.)

Note that neither $P$ nor $P^{\prime}$ is necessarily a directed path in the orientation of $H$. In traversing them we may go with the orientation and also against it. But we can find a walk $W: b_{0} b_{1} b_{2} \cdots b_{k}$ in $B$ for which the path $\left(b_{0}, h_{0}\right)\left(b_{1}, h_{1}\right)\left(b_{2}, h_{2}\right) \cdots$ $\left(b_{k}, h_{k}\right)$ in $B \times H$ is properly colored, and begins with an edge that is either solid or bold. If we want $\left(b_{0}, h_{0}\right)\left(b_{1}, h_{1}\right)$ to be solid, we select $b_{1}$ so that $b_{0} b_{1}$ has the same orientation as $h_{0} h_{1}$, and if we want it dashed we go the other way on $B$, selecting $b_{1}$ so $b_{0} b_{1}$ is oriented opposite to $h_{0} h_{1}$. Moving on to $\left(b_{1}, h_{1}\right)\left(b_{2}, h_{2}\right)$ we can make this edge either solid or dashed with a judicious choice of $b_{2}$. Continuing this process, we get a path $Q:\left(b_{0}, h_{0}\right)\left(b_{1}, h_{1}\right)\left(b_{2}, h_{2}\right) \cdots\left(b_{k}, h_{k}\right)$ in $B \times H$ that is properly colored, and we are free to choose the color of the first edge.

Likewise there is a path $Q^{\prime}:\left(b_{0}^{\prime}, h_{0}^{\prime}\right)\left(b_{1}^{\prime}, h_{1}^{\prime}\right)\left(b_{2}^{\prime}, h_{2}^{\prime}\right) \cdots\left(b_{\ell}^{\prime}, h_{\ell}^{\prime}\right)$ in $B \times H$ that is properly colored, and again we are free to choose the color of the first edge. By construction $Q$ and $Q^{\prime}$ are vertex-disjoint, and they terminate in $B \times C$. With the exception of their terminal vertices $\left(b_{k}, h_{k}\right)$ and $\left(b_{\ell}^{\prime}, h_{\ell}^{\prime}\right)$, no other vertex belongs to $B \times C$. Lemma 1 guarantees a path $R$ in $B \times C$ from $\left(b_{k}, h_{k}\right)$ to ( $\left.b_{\ell}^{\prime}, h_{\ell}^{\prime}\right)$ for which $Q \cup R \cup Q^{\prime}$ is properly colored. The combinations of these paths yield the desired set of four paths. This completes the claim.

To finish the proof we take two arbitrary vertices $\left(g_{0}, h_{0}\right)$ and $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$ of $G \times H$, and produce a properly colored path joining them.

Now, $G-E(B)$ has directed (possibly trivial) paths $P: g_{0} g_{1} g_{2} \cdots g_{k}$ and $P^{\prime}: g_{0}^{\prime} g_{1}^{\prime} g_{2}^{\prime} \cdots g_{\ell}^{\prime}$ that terminate at vertices of $B$. Our plan is to use them to construct two disjoint properly colored paths in $(G-E(B)) \times H$, joining ( $g_{0}, h_{0}$ ) and $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$ to distinct vertices of $B \times H$, and then use the above claim to join these endpoints with an appropriate properly colored path in $B \times H$.

Case 1. Suppose $g_{0}$ and $g_{0}^{\prime}$ are in different components of $G-E(B)$, so $P$ and $P^{\prime}$ do not meet. Choose arbitrary edges $h_{0} h_{1}$ and $h_{0}^{\prime} h_{1}^{\prime}$ of $H$. In $(G-E(B)) \times H$ we have vertex-disjoint properly colored paths

$$
\begin{aligned}
Q & :\left(g_{0}, h_{0}\right)\left(g_{1}, h_{1}\right)\left(g_{2}, h_{0}\right)\left(g_{3}, h_{1}\right)\left(g_{4}, h_{0}\right) \cdots\left(g_{k}, h_{*}\right), \\
Q^{\prime} & :\left(g_{0}^{\prime}, h_{0}^{\prime}\right)\left(g_{1}^{\prime}, h_{1}^{\prime}\right)\left(g_{2}^{\prime}, h_{0}^{\prime}\right)\left(g_{3}^{\prime}, h_{1}^{\prime}\right)\left(g_{4}^{\prime}, h_{0}^{\prime}\right) \cdots\left(g_{\ell}^{\prime}, h_{*}^{\prime}\right),
\end{aligned}
$$

where $h_{*}=h_{0}$ or $h_{*}=h_{1}$ (depending on the parity of $k$ ), and $h_{*}^{\prime}=h_{0}^{\prime}$ or $h_{*}^{\prime}=h_{1}^{\prime}$. By the above claim, $B \times H$ has a path $R$ joining $\left(g_{k}, h_{*}\right)$ to $\left(g_{\ell}^{\prime}, h_{*}^{\prime}\right)$, for which the path $Q \cup R \cup Q^{\prime}$ is properly colored.

Case 2. Suppose $g_{0}$ and $g_{0}^{\prime}$ are in the same component of $G-E(B)$. Now, $P$ and $P^{\prime}$ terminate at the same vertex $g_{k}=g_{\ell}^{\prime}$ of $B$, and they merge at some vertex $g_{k-a}=g_{\ell-a}^{\prime}$. That is, $a$ is the largest non-negative integer for which $g_{k-i}=g_{\ell-i}^{\prime}$ for $a \geq i \geq 0$. (Possibly $a=0$, in which case $P$ and $P^{\prime}$ meet only at $g_{k}=g_{\ell}^{\prime}$. At the other extreme, $P \subseteq P^{\prime}$ if $a=k$, and $P^{\prime} \subseteq P$ if $a=\ell$.)

First suppose $k-a$ and $\ell-a$ have opposite parity (and w.l.o.g., suppose it is $k-a$ that is even). Choose $h_{0} h_{1} \in E(H)$ and $h_{0}^{\prime} h_{1}^{\prime} \in E(H)$ with $h_{0} \neq h_{1}^{\prime}$ and $h_{1} \neq h_{0}^{\prime}$. Form the following properly colored paths in $(G-E(B)) \times H$ :
$Q:\left(g_{0}, h_{0}\right)\left(g_{1}, h_{1}\right)\left(g_{2}, h_{0}\right)\left(g_{3}, h_{1}\right)\left(g_{4}, h_{0}\right) \cdots\left(g_{k-a}, h_{0}\right)\left(g_{k-a+1}, h_{1}\right)\left(g_{k-a+2}, h_{0}\right) \cdots\left(g_{k}, h_{*}\right)$,
$Q^{\prime}:\left(g_{0}^{\prime}, h_{0}^{\prime}\right)\left(g_{1}^{\prime}, h_{1}^{\prime}\right)\left(g_{2}^{\prime}, h_{0}^{\prime}\right)\left(g_{3}^{\prime}, h_{1}^{\prime}\right)\left(g_{4}^{\prime}, h_{0}^{\prime}\right) \cdots\left(g_{\ell-a}^{\prime}, h_{1}^{\prime}\right)\left(g_{\ell-a+1}^{\prime}, h_{0}^{\prime}\right)\left(g_{\ell-a+2}^{\prime}, h_{1}^{\prime}\right) \cdots\left(g_{\ell}^{\prime}, h_{*}^{\prime}\right)$.
Notice $h_{*} \neq h_{*}^{\prime}$, and these paths are disjoint and end in $B \times H$. By our claim, $B \times H$ has a path $R$ joining ( $g_{k}, h_{*}$ ) to ( $g_{\ell}^{\prime}, h_{*}^{\prime}$ ), for which the path $Q \cup R \cup Q^{\prime}$ is properly colored.

Next suppose $k-a$ and $\ell-a$ are both even. Choose $h_{0} h_{1} \in E(H)$ and $h_{0}^{\prime} h_{1}^{\prime} \in E(H)$ with $h_{1} \neq h_{1}^{\prime}$, and also so that their orientations are opposite (i.e.,
$h_{0} h_{1}$ is directed from $h_{0}$ to $h_{1}$, and $h_{0}^{\prime} h_{1}^{\prime}$ is directed from $h_{1}^{\prime}$ to $h_{0}^{\prime}$, or vice versa). This is possible because $H$ is 2 -connected and its orientation has neither sources nor sinks. We have paths

$$
\begin{aligned}
Q & :\left(g_{0}, h_{0}\right)\left(g_{1}, h_{1}\right)\left(g_{2}, h_{0}\right)\left(g_{3}, h_{1}\right)\left(g_{4}, h_{0}\right)\left(g_{5}, h_{1}\right) \cdots\left(g_{k-a+1}, h_{1}\right)\left(g_{k-a}, h_{0}\right), \\
Q^{\prime} & :\left(g_{0}^{\prime}, h_{0}^{\prime}\right)\left(g_{1}^{\prime}, h_{1}^{\prime}\right)\left(g_{2}^{\prime}, h_{0}^{\prime}\right)\left(g_{3}^{\prime}, h_{1}^{\prime}\right)\left(g_{4}^{\prime}, h_{0}^{\prime}\right)\left(g_{5}, h_{1}^{\prime}\right) \cdots\left(g_{\ell-a+1}, h_{1}^{\prime}\right)\left(g_{\ell-a}^{\prime}, h_{0}^{\prime}\right) .
\end{aligned}
$$

The first begins with a bold edge and ends with a dashed edge. The second begins dashed and ends bold. If it happens that $h_{0}=h_{0}^{\prime}$, then $Q$ and $Q^{\prime}$ intersect only at their last vertex, so $Q \cup Q^{\prime}$ is a properly colored path from $\left(g_{0}, h_{0}\right)$ to $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$. If $h_{0} \neq h_{0}^{\prime}$ then the paths may be continued as indicated until reaching $B \times H$. Then, by our claim, $B \times H$ has a path $R$ for which $Q \cup R \cup Q^{\prime}$ is a properly colored path joining $\left(g_{0}, h_{0}\right)$ to $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$.

Finally suppose $k-a$ and $\ell-a$ are both odd. Let $h_{0} h_{1}$ and $h_{0}^{\prime} h_{1}^{\prime}$ be as in the previous paragraph. We have properly colored paths

$$
\begin{aligned}
Q & :\left(g_{0}, h_{0}\right)\left(g_{1}, h_{1}\right)\left(g_{2}, h_{0}\right)\left(g_{3}, h_{1}\right)\left(g_{4}, h_{0}\right)\left(g_{5}, h_{1}\right) \cdots\left(g_{k-a}, h_{1}\right)\left(g_{k-a+1}, h_{0}\right), \\
Q^{\prime} & :\left(g_{0}^{\prime}, h_{0}^{\prime}\right)\left(g_{1}^{\prime}, h_{1}^{\prime}\right)\left(g_{2}^{\prime}, h_{0}^{\prime}\right)\left(g_{3}^{\prime}, h_{1}^{\prime}\right)\left(g_{4}^{\prime}, h_{0}^{\prime}\right)\left(g_{5}^{\prime}, h_{1}^{\prime}\right) \cdots\left(g_{\ell-a}^{\prime}, h_{1}^{\prime}\right)\left(g_{\ell-a+1}^{\prime}, h_{0}^{\prime}\right) .
\end{aligned}
$$

Now, $Q$ begins bold and ends dashed, and $Q^{\prime}$ begins dashed and ends bold. If $h_{0}=h_{0}^{\prime}$, then $Q$ and $Q^{\prime}$ meet only at their last vertex, so $Q \cup Q^{\prime}$ is a properly colored path from $\left(g_{0}, h_{0}\right)$ to $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$. If $h_{0} \neq h_{0}^{\prime}$ then the paths may be continued as indicated until reaching $B \times H$. Then $B \times H$ has a path $R$ for which $Q \cup R \cup Q^{\prime}$ is a properly colored path joining $\left(g_{0}, h_{0}\right)$ to $\left(g_{0}^{\prime}, h_{0}^{\prime}\right)$.

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