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ON THE ROMAN DOMINATION STABLE GRAPHS

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Abstract

A Roman dominating function (or just RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A graph G is Roman domination stable if the Roman domination number of G remains unchanged under removal of any vertex. In this paper we present upper bounds for the Roman domination number in the class of Roman domination stable graphs, improving bounds posed in [V. Samodivkin, *Roman domination in graphs: the class* $\mathcal{R}_{UVR}$, Discrete Math. Algorithms Appl. **8** (2016) 1650049].

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1. Introduction

For notation and graph theory terminology in general we follow [5]. Let $G = (V, E)$ be a simple graph of order n. We denote the open neighborhood of a vertex v of G by $N_G(v)$, or just $N(v)$, and its closed neighborhood by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The *degree* $\deg(x)$ (or $\deg_G(x)$ to refer to G) of a vertex x is the number of neighbors of x in G. The maximum degree and minimum degree among the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A set S of vertices in G is a *dominating set*, if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set S in G is an

efficient dominating set, if $|N[v] \cap S| = 1$ for every vertex $v \in V(G)$. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A graph G is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A subset S of vertices of G is a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair u, v of vertices of S. The 2-*corona* $G \circ K_2$ of a graph G is a graph obtained from G by attaching a path of order two to every vertex of G.

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer to f). A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0, V_1, V_2)$ is called a γ_R-*function* (or $\gamma_R(G)$-*function* when we want to refer f to G), if it is an RDF and $f(V(G)) = \gamma_R(G)$. A graph G is a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. If $f = (V_0, V_1, V_2)$ is an RDF in G then for any vertex $v \in V_2$, we define $pn(v, V_2^f) = \{u \in V_0 : N(u) \cap V_2^f = \{v\}\}$. For references in Roman domination see for example [1, 2, 3, 9].

The affection of vertex removal on Roman domination number in a graph has been studied in [4]. Jafari Rad and Volkmann [6] introduced the concept of Roman domination stable graphs. A graph G is *Roman domination stable* if $\gamma_R(G - v) = \gamma_R(G)$ for all $v \in V(G)$. Let $\mathcal{R}_{UVR}$ be the class of all Roman domination stable graphs. Samodivkin [10] studied properties of Roman domination stable graphs.

Theorem 1 (Samodivkin [10])**.** *Let $G \in \mathcal{R}_{UVR}$ be a connected graph of order n. Then $\gamma_R(G) \le \frac{2n}{3}$. If the equality holds, then for any $\gamma_R(G)$-function f, V_2^f is an efficient dominating set of G and each vertex of V_2^f has degree 2. If G has an efficient dominating set D and each vertex of D has degree 2, then $\gamma_R(G) = \frac{2n}{3}$.*

Problem 2 (Samodivkin [10])**.** Find an attainable constant upper bound for $\frac{\gamma_R(G)}{|V(G)|}$ on all connected graphs $G \in \mathcal{R}_{UVR}$ with $\delta(G) \ge 3$.

In this paper we present upper bounds for the Roman domination number in the class of Roman domination stable graphs. First we characterize Roman domination stable graphs G with $\delta(G) = 2$ that achieve the upper bound of Theorem 1 as the cycles of order divisible by 3. Next, we consider the Roman domination stable graphs G with $\delta(G) \ge 3$. In particular, we improve Theorem 1 for claw-free Roman domination stable graphs. Finally, we present several upper bounds for the Roman domination number in Roman domination stable graphs,

which are expressed in terms of the order, the maximum and the minimum degree of a graph.

2. Known Results

The following proposition of Samodivkin plays an important role in this paper.

Proposition 3 (Samodivkin [10]). *Let a graph G be in $\mathcal{R}_{UVR}$. Then G is a Roman graph. For any $\gamma_R(G)$-function $f = \left(V_0^f, V_1^f, V_2^f\right)$, $V_1^f = \emptyset$, V_2^f is a $\gamma(G)$-set, and $\left|pn\left(v, V_2^f\right)\right| \geq 2$ for any $v \in V_2^f$. If D is a $\gamma(G)$-set, then $h = (V(G) - D, \emptyset, D)$ is a $\gamma_R(G)$-function.*

Let G_1 be a graph obtained from a cycle $C_8 : v_1v_2 \cdots v_8v_1$ by joining v_1 to v_5, and G_2 be a graph obtained from G_1 by joining v_4 to v_8. The following upper bounds for the Roman domination number of a graph are given in [1, 7].

Theorem 4 (Chambers *et al.* [1]). *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_4, C_5, C_8, G_1, G_2\}$, then $\gamma_R(G) \leq \frac{8n}{11}$.*

Theorem 5 (Liu *et al.* [8]). *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_R(G) \leq \frac{2n}{3}$.*

Theorem 6 (Hansberg *et al.* [4]). *Let v be a vertex of a graph G. Then $\gamma_R(G-v) < \gamma_R(G)$ if and only if there is a $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ with $v \in V_1$.*

3. Minimum Degree at Least Two

We characterize graphs with minimum degree at least two that achieve equality for the bound of Theorem 1.

Theorem 7. *Let $G \in \mathcal{R}_{UVR}$ be a connected graph of order n with $\delta(G) \geq 2$. Then $\gamma_R(G) = \frac{2n}{3}$ if and only if G is a cycle of order $3k$ for some integer k.*

Proof. Let $G \in \mathcal{R}_{UVR}$ be a connected graph of order n with $\delta(G) \geq 2$ and $\gamma_R(G) = \frac{2n}{3}$. Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function. By Theorem 1, V_2^f is an efficient dominating set of G and each vertex of V_2^f has degree 2. By Proposition 3, $V_1^f = \emptyset$. We show that $\Delta(G) = 2$. Suppose that $\Delta(G) \geq 3$. Since each vertex of V_2^f has degree 2, V_0^f has some vertex of degree at least three.

Assume that there are two adjacent vertices $u_1, v_1 \in V_0^f$ with $\deg_G(u_1) \geq 3$ and $\deg_G(v_1) \geq 3$. We remove the edge u_1v_1 to obtain a graph G_1. Clearly,

$\delta(G_1) \geq 2$, $\gamma_R(G_1) = \frac{2n}{3}$. Suppose that $G_1 \notin \mathcal{R}_{UVR}$. Clearly, there is no $\gamma_R(G_1)$-function h with $V_1^h \neq \emptyset$, since any $\gamma_R(G_1)$-function is a $\gamma_R(G)$-function. Thus by Theorem 6, there is a vertex $v \in V(G)$ such that $\gamma_R(G_1 - v) > \gamma_R(G_1)$. Clearly $v \notin V_0^f$. Thus $v \in V_2^f$. By Theorem 1, $\deg_G(v) = 2$. Let $N_G(v) = \{w_1, w_2\}$. Then h defined on $V(G_1 - v)$ by $h(u) = f(u)$ if $u \notin \{w_1, w_2\}$, and $h(w_1) = h(w_2) = 1$, is an RDF for G_1, implying that $\gamma_R(G_1 - v) \leq \gamma_R(G_1)$, a contradiction. Thus $G_1 \in \mathcal{R}_{UVR}$. If there are two adjacent vertices $u_2, v_2 \in V_0^f$ with $\deg_{G_1}(u_2) \geq 3$ and $\deg_{G_1}(v_2) \geq 3$, then we remove the edge u_2v_2 to obtain a graph G_2 with $\delta(G_2) \geq 2$, $\gamma_R(G_2) = \frac{2n}{3}$, and $G_2 \in \mathcal{R}_{UVR}$. Proceeding this process, if necessary, we obtain a graph $H = G_k$ (for some $k \geq 0$) such that $\delta(H) \geq 2$, $\gamma_R(H) = \frac{2n}{3}$, $H \in \mathcal{R}_{UVR}$, and there is no pair of adjacent vertices $u, v \in V_0^f$ with $\deg_H(u) \geq 3$ and $\deg_H(v) \geq 3$. Clearly, f is a $\gamma_R(H)$-function. Since $\delta(H) \geq 2$, H contains a cycle C. Since there is no pair of adjacent vertices $u, v \in V_0^f$ with $\deg_H(u) \geq 3$ and $\deg_H(v) \geq 3$, we find that $V(C) \cap V_2^f \neq \emptyset$. By Theorem 1, $V_2^f \cap V(C)$ is a 2-packing set for C. Thus for each $v \in V_0^f \cap V(C)$, $\left|N_H(v) \cap V(C) \cap V_0^f\right| \geq 1$. By a *special P_k-path* we mean a path of order k in C whose vertices belong to V_0^f (i.e., a path $v_1v_2 \cdots v_k$ with $v_i \in V(C) \cap V_0^f$ for $i = 1, 2, \ldots, k$, such that $v_iv_{i+1} \in E(C)$ for $i = 1, 2, \ldots, k-1$). A *maximal special P_k-path* is a special P_k-path that cannot be extended to a special P_{k+1}-path. Clearly, C has no maximal P_1-path. Assume that C has a maximal special P_k-path with $k \geq 4$. Let v_1, v_2, v_3, v_4 be four consecutive vertices of this maximal special P_k-path. Then $\deg_H(v_2) \geq 3$ and $\deg_H(v_3) \geq 3$, a contradiction. Thus C has no maximal special P_k-path with $k \geq 4$. We consider the following cases.

Case 1. C has no maximal special P_3-path. Since V_2^f is an efficient dominating set of G, and thus an efficient dominating set of H, we conclude that $V_2^f \cap V(C)$ is an efficient dominating set of C. Thus $|V(C)| = 3t$, for some integer $t \geq 1$. Without loss of generality, assume that $V(C) = \{v_0, v_1, \ldots, v_{3t-1}\}$, where v_i is adjacent to v_{i+1} for $i = 0, 1, 2, \ldots, 3t-2$, and v_0 is adjacent to v_{3t-1}. We may assume that $V_2^f \cap C = \{v_{3i} : i = 0, 1, \ldots, t-1\}$. Since $\Delta(G) \geq 3$ and G is connected, there is a vertex $v_s \in V(C)$ with $\deg_G(v_s) \geq 3$. Clearly, $s \equiv 1$ or $2 \pmod 3$. Assume that $s \equiv 1 \pmod 3$. Let g be defined on $V(H)$ by $g(u) = f(u)$ if $u \in V(H) - V(C)$, and $g(v_i) = f(v_{i-1}) \pmod{3t}$ for $i = 0, 1, 2, \ldots, 3t-1$. Then g is a $\gamma_R(G)$-function, contradicting Theorem 1. If $j \equiv 2 \pmod 3$, then h defined on $V(H)$ by $g(u) = f(u)$ if $u \in V(H) - V(C)$, and $g(v_i) = f(v_{i-2}) \pmod{3t}$ for $i = 0, 1, 2, \ldots, 3t-1$ is a $\gamma_R(G)$-function, contradicting Theorem 1.

Case 2. C has some maximal special P_3-path. Let $j \geq 1$ be the number of special P_3-paths in C, and $x_iy_iz_i$ $(i = 1, 2, \ldots, j)$ be the maximal special P_3-paths in C, where there is no maximal special P_3-path on C between z_i and x_{i+1}, $i = 1, 2, \ldots, j \pmod j$. Observe that $(N(x_i) \cap V(C)) - \{y_i\} \subseteq V_2^f$, and

$(N(z_i) \cap V(C)) - \{y_i\} \subseteq V_2^f$, for $i = 1, 2, \ldots, j$. Let $(N(z_i) \cap V(C)) - \{y_i\} = \{z_i'\}$ and $(N(x_i) \cap V(C)) - \{y_i\} = \{x_i'\}$, for $i = 1, 2, \ldots, j$. On the other hand, any vertex on $V(C) \cap V_0^f$ lying between z_i and x_{i+1}, if any, belongs to some maximal special P_2-path. Note that it is possible that there are no vertices on $V(C) \cap V_0^f$ lying between z_i and x_{i+1} when z_i and x_{i+1} have a common neighbor in V_2^f. Since $V_2^f \cap V(C)$ is independent, for each i, the path on C starting at z_i' and ending at x_{i+1}' has $3k_i + 1$ vertices for some integer $k_i \geq 0$. Let $v_0^i v_1^i \cdots v_{3k_i}^i$ be the path on C starting at z_i' and ending at x_{i+1}', where $z_i' = v_0^i$ and $x_{i+1}' = v_{3k_i}^i$. Thus C is the cycle $x_1 y_1 z_1 v_0^1 v_1^1 \cdots v_{3k_1}^1 x_2 y_2 z_2 \cdots x_j y_j z_j v_0^j v_1^j \cdots v_{3k_j}^j x_1$. For $i = 1, 2, \ldots, j$, $\{v_{3t}^i : 0 \leq t \leq k_i\} \subseteq V_2^f$ and $\{v_{3t+1}^i, v_{3t+2}^i : 0 \leq t \leq k_i - 1\} \subseteq V_0^f$. For $i = 1, 2, \ldots, j$, let $N(y_i) \cap V_2^f = \{y_i'\}$. Clearly, for $i = 1, 2, \ldots, j$, $y_i' \notin C$, and $\deg_G(y_i') = 2$ by Theorem 1. Let $N(y_i') - \{y_i\} = \{y_i''\}$ for $i = 1, 2, \ldots, j$. If $y_i'' \in V(C)$ then $y_i'' \in \{y_1, \ldots, y_j\} - \{y_i\}$, since V_2^f is an efficient dominating set for G. Let $D = \{y_i'' : i = 1, 2, \ldots, j\} \cap V(C)$. Let g be defined on $V(G)$ by $g(u) = f(u)$ if $u \notin V(C) \cup \{y_i', y_i'' : i = 1, 2, \ldots, j\}$, $g(u) = 2$ if $u \in \bigcup_{i=1}^j \{y_i, v_{3t+1}^i : 0 \leq t \leq k_i - 1\}$, $g(u) = 0$ if $u \in \bigcup_{i=1}^j \{x_i, z_i, y_i', v_{3t}^i, v_{3t+2}^i : 0 \leq t \leq k_i - 1\}$, and $g(u) = 1$ if $u \in \bigcup_{i=1}^j \{v_{3k_i}, y_i''\} - D$. It is straightforward to see that if $D \neq \emptyset$ then g is an RDF for G of weight less than $\gamma_R(G)$, and if $D = \emptyset$ then g is a $\gamma_R(G)$-function with $V_1^f \neq \emptyset$, a contradiction.

We conclude that $\Delta(G) = 2$, and thus G is a cycle. Consequently, G is a cycle of order $3k$ for some integer k. The converse is obvious. ■

Proposition 2 demonstrates that a graph $G \in \mathcal{R}_{UVR}$ with $\Delta(G) > 2$ should have the Roman domination number less than $\frac{2n}{3}$.

Corollary 8. *If $G \in \mathcal{R}_{UVR}$ is a connected graph of order n with $2 \leq \delta(G) < \Delta(G)$, then $\gamma_R(G) \leq \frac{2n-2}{3}$. This bound is sharp.*

Proof. Let $G \in \mathcal{R}_{UVR}$ be a connected graph of order n with $\delta(G) \geq 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. Clearly, each vertex of V_2 has at least two private neighbors in V_0. If each vertex of V_2 has degree two, then by Theorem 1, $\gamma_R(G) = \frac{2n}{3}$, a contradiction. Thus there is a vertex $x \in V_2$ with $\deg(x) \geq 3$. Then

$$n \geq \deg(x) + 2(|V_2| - 1) + |V_2| = 3|V_2| + \deg(x) - 2 = \frac{3\gamma_R(G)}{2} + \deg(x) - 2$$

implying that $\gamma_R(G) \leq \frac{2n - 2\deg(x) + 4}{3} \leq \frac{2n-2}{3}$. To see the sharpness consider the graph $K_4 - e$, where $e \in E(K_4)$. ■

4. Minimum Degree at Least Three

We begin with the following lemma.

Lemma 9. *Let $G \in \mathcal{R}_{UVR}$ and $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function. If $v \in V_2^f$ is a vertex such that $\left|pn\left(v, V_2^f\right)\right| = 2$, then $\deg_{G[V_2^f]}(v) = 0$.*

Proof. Let $v \in V_2^f$ be a vertex such that $\left|pn\left(v, V_2^f\right)\right| = 2$. Suppose that v is adjacent to a vertex $x \in V_2^f$. Let $pn\left(v, V_2^f\right) = \{v_1, v_2\}$. Then $\left(V_0^f \cup \{v\}, \{v_1, v_2\}, V_2^f - \{v\}\right)$ is a $\gamma_R(G)$-function, contradicting Proposition 3. ■

In the following we present a sharp upper bound for the Roman domination number of a claw-free graph $G \in \mathcal{R}_{UVR}$.

Theorem 10. *If $G \in \mathcal{R}_{UVR}$ is a claw-free graph of order n with $\delta(G) \geq 3$, then $\gamma_R(G) \leq \frac{4n-2}{7}$. This bound is sharp.*

Proof. Let $G \in \mathcal{R}_{UVR}$ be a claw-free graph of order n with $\delta(G) \geq 3$. Clearly, any $\gamma_R(G)$-function satisfies Proposition 3.

Claim 1. *There is a $\gamma_R(G)$-function $f = \left(V_0^f, V_1^f, V_2^f\right)$ such that one of the following holds:*
(1) $\left|pn\left(v, V_2^f\right)\right| \geq 3$ *for some* $v \in V_2^f$.
(2) $|N(x) \cap N(y)| \geq 2$ *for some vertices* $x, y \in V_2^f$.

Proof. Suppose that for every $\gamma_R(G)$-function $g = (V_0^g, V_1^g, V_2^g)$, $|pn(v, V_2^g)| = 2$ for all $v \in V_2^g$, and $|N(x) \cap N(y)| \leq 1$ for any pair of vertices $x, y \in V_2^g$. Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function. Clearly, $\left|V_2^f\right| \geq 2$. By Proposition 3, $V_1^f = \emptyset$. By our assumption, $\left|pn\left(v, V_2^f\right)\right| = 2$ for all $v \in V_2^f$, and $|N(x) \cap N(y)| \leq 1$ for any pair of vertices $x, y \in V_2^f$. Let $u \in V_2^f$, $pn\left(u, V_2^f\right) = \{u_1, u_2\}$. Let $v \in N(u) - \{u_1, u_2\}$. By Lemma 9, $v \in V_0^f$. Let $w \in N(v) \cap \left(V_2^f - \{u\}\right)$. Let $pn\left(w, V_2^f\right) = \{w_1, w_2\}$. Since G is claw-free, $\{vw_1, vw_2, w_1w_2\} \cap E(G) \neq \emptyset$ and $\{vu_1, vu_2, u_1u_2\} \cap E(G) \neq \emptyset$. Assume that $u_1u_2 \in E(G)$. Then $g = \left(V_0^f \cup \{u\}, \emptyset, \left(V_2^f - \{u\}\right) \cup \{u_1\}\right)$ is a $\gamma_R(G)$-function with $|pn(w, V_2^g)| = 3$, a contradiction. Thus $u_1u_2 \notin E(G)$ and similarly $w_1w_2 \notin E(G)$. Without loss of generality assume that $vw_1, vu_1 \in E(G)$. Observe that $N(u) \cap N(w) = \{v\}$. Now $\left(\left(V_0^f - \{u_2, w_2, v\}\right) \cup \{u, w\}, \{u_2, w_2\}, \left(V_2^f - \{u, w\}\right) \cup \{v\}\right)$ is a $\gamma_R(G)$ function contradicting Proposition 3. This completes the proof of Claim 1. □

Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function satisfying Claim 1. Let $A = \left\{v \in V_2^f : \left|pn\left(v, V_2^f\right)\right| = 2\right\}$ and $B = \left\{v \in V_2^f : \left|pn\left(v, V_2^f\right)\right| \geq 3\right\}$. Clearly, $V_2^f = A \cup B$. By Lemma 9, A is independent. Now we count $\left|V_0^f\right|$. Let $Z_0 = \left\{v \in V_0^f : v \in pn\left(x, V_2^f\right) \text{ for some } x \in V_2^f\right\}$ and $Z_1 = V_0^f - Z_0$. Clearly, $\left|V_0^f\right| = |Z_0| + |Z_1|$. For any vertex $x \in A$, $\left|pn\left(x, V_2^f\right)\right| = 2$, and for any vertex $x \in B$, $\left|pn\left(x, V_2^f\right)\right| \geq 3$. Thus $|Z_0| \geq 2|A| + 3|B|$. For any vertex $x \in A$, by Lemma 9, $N(x) \cap Z_1 \neq \emptyset$, since $\delta(G) \geq 3$. On the other hand, for any $y \in Z_1$, $|N(y) \cap A| \leq 2$, since G is claw-free and A is independent. Assume that $\left|pn\left(v, V_2^f\right)\right| \geq 3$ for some $v \in V_2^f$. Then $|Z_1| \geq \frac{|A|}{2}$. Hence

$$\left|V_0^f\right| = |Z_0| + |Z_1| \geq 2|A| + 3|B| + \frac{|A|}{2} \geq 2\left|V_2^f\right| + |B| + \frac{|A|}{2} = \frac{5\left|V_2^f\right|}{2} + \frac{|B|}{2}.$$

Now $n = \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{5\left|V_2^f\right|}{2} + \frac{|B|}{2} + \left|V_2^f\right| = \frac{7\left|V_2^f\right|}{2} + \frac{|B|}{2} \geq \frac{7\left|V_2^f\right|}{2} + \frac{1}{2}$. Consequently, $\gamma_R(G) \leq \frac{4n-2}{7}$. Next assume that $|N(x) \cap N(y)| \geq 2$ for some vertices $x, y \in V_2^f$. Then $|Z_1| \geq \frac{|A|+1}{2}$. Hence

$$\begin{aligned}\left|V_0^f\right| &= |Z_0| + |Z_1| \geq 2|A| + 3|B| + \frac{|A|+1}{2} \\ &\geq 2\left|V_2^f\right| + |B| + \frac{|A|+1}{2} = \frac{5\left|V_2^f\right|}{2} + \frac{|B|}{2} + \frac{1}{2}.\end{aligned}$$

Now $n = \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{5\left|V_2^f\right|}{2} + \frac{|B|}{2} + \frac{1}{2} + \left|V_2^f\right| = \frac{7\left|V_2^f\right|}{2} + \frac{|B|}{2} + \frac{1}{2} \geq \frac{7\left|V_2^f\right|}{2} + \frac{1}{2}$. Consequently, $\gamma_R(G) \leq \frac{4n-2}{7}$. To see the sharpness consider the complete graph K_4. ■

We next present a sharp upper bound for the Roman domination number of a graph $G \in \mathcal{R}_{UVR}$ in terms of maximum degree.

Theorem 11. *If $G \in \mathcal{R}_{UVR}$ is a graph of order n with $\delta(G) \geq 3$, then*

$$\gamma_R(G) \leq \frac{2n}{3}\left(\frac{1}{1 + \frac{\delta(G)-2}{3\Delta(G)}}\right).$$

This bound is sharp.

Proof. Let $G \in \mathcal{R}_{UVR}$ be a graph of order n with $\delta(G) \geq 3$. Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function. By Proposition 3, $V_1^f = \emptyset$. Let A,B, Z_0 and Z_1 be defined as in the proof of Theorem 10. Thus $|Z_0| \geq 2|A| + 3|B|$. Any vertex of A has at least $\delta(G) - 2$ neighbors in Z_1. Consequently, there are at least $(\delta(G) - 2)|A|$ edges between V_2^f and Z_1. But any vertex of Z_1 is adjacent to at most $\Delta(G)$ vertices of V_2^f. We conclude that $|Z_1| \geq \frac{(\delta(G)-2)|A|}{\Delta(G)}$. Hence

$$\left|V_0^f\right| = |Z_0| + |Z_1| \geq 2|A| + 3|B| + \frac{(\delta(G) - 2)|A|}{\Delta(G)} \tag{1}$$

$$= 2\left|V_2^f\right| + |B| + \frac{(\delta(G) - 2)|A|}{\Delta(G)} \tag{2}$$

Now

$$n = \left|V_0^f\right| + \left|V_2^f\right| \geq 2\left|V_2^f\right| + |B| + \frac{(\delta(G) - 2)|A|}{\Delta(G)} + \left|V_2^f\right| \tag{3}$$

$$= \frac{3\Delta(G)\left|V_2^f\right| + \Delta(G)|B| + (\delta(G) - 2)|A|}{\Delta(G)} \tag{4}$$

$$= \frac{(3\Delta(G) + \delta(G) - 2)\left|V_2^f\right| + (\Delta(G) - (\delta(G) - 2))|B|}{\Delta(G)} \tag{5}$$

$$\geq \frac{(3\Delta(G) + \delta(G) - 2)\left|V_2^f\right|}{\Delta(G)} \tag{6}$$

and thus $\gamma_R(G) \leq \frac{2n\Delta(G)}{3\Delta(G)+\delta(G)-2} = \frac{2n}{3}\left(\frac{1}{1+(\delta(G)-2)/3\Delta(G)}\right)$. To see the sharpness consider the graph G shown in Figure 1. Note that $\gamma_R(G) = 6$, and $G \in \mathcal{R}_{UVR}$. ■

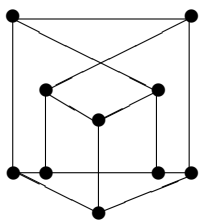

Figure 1. The graph $G \in \mathcal{R}_{UVR}$ with $\gamma_R(G) = 6$ and $n = 10$.

Corollary 12. *If $G \in \mathcal{R}_{UVR}$ is a cubic graph of order n, then $\gamma_R(G) \leq \frac{3n}{5}$, and this bound is sharp.*

We next improve Theorem 11 for C_5-free graphs G with $\delta(G) = 3$ and $\Delta(G) \geq 4$.

Theorem 13. *If $G \in \mathcal{R}_{UVR}$ is a C_5-free graph of order n with $\delta(G) = 3$ and $\Delta(G) \geq 4$, then*

$$\gamma_R(G) \leq \frac{2n}{3}\left(\frac{\Delta(G) - 1/n}{\Delta(G) + 1/3}\right).$$

Proof. Let $G \in \mathcal{R}_{UVR}$ be a C_5-free of order n with $\delta(G) = 3$ and $\Delta(G) \geq 4$. Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function satisfying Proposition 3. Let A, B, Z_0 and Z_1 be defined as in the proof of Theorem 10. By Theorem 11, $\gamma_R(G) \leq \frac{2n\Delta(G)}{3\Delta(G)+1}$. We show that $\gamma_R(G) < \frac{2n\Delta(G)}{3\Delta(G)+1}$. Suppose that $\gamma_R(G) = \frac{2n\Delta(G)}{3\Delta(G)+1}$. Then each of the inequalities in the proof of Theorem 11 will be equality. From (5) and (6) we find that $B = \emptyset$, and from (1) we obtain $|Z_1| = \frac{|A|}{\Delta(G)}$. Consequently, each vertex of Z_1 is adjacent to precisely $\Delta(G)$ vertices of A, and each vertex of A has precisely one neighbor in Z_1 and two neighbors in Z_0. Observe that $\gamma_R(G) = 2|A|$ and $|Z_1| = \frac{|A|}{\Delta(G)}$ and $|Z_0| = 2|A|$. Let H be the graph obtained from G by removal of the vertices of $N[x]$ for all $x \in Z_1$. Clearly, $V(H) = Z_0$. If H has no component isomorphic to C_4 or C_8, then by Theorem 4, $\gamma_R(H) \leq \frac{8|V(H)|}{11} = \frac{16|A|}{11}$. Let g be a $\gamma_R(H)$-function. Then g_1 defined on $V(G)$ by $g_1(x) = g(x)$ if $x \in V(H)$, $g_1(x) = 2$ if $x \in Z_1$, and $g_1(x) = 0$ if $x \in A$, is an RDF for G. Thus

$$\gamma_R(G) \leq \frac{16|A|}{11} + \frac{2|A|}{\Delta(G)} = \frac{16\Delta(G)|A| + 22|A|}{11\Delta(G)} < 2|A|,$$

since $\Delta(G) \geq 4$. This is a contradiction. Thus assume that H has some component isomorphic to C_4 or C_8. Let H has r_1 components isomorphic to C_4 and r_2 components isomorphic to C_8. For any component C of H with $C \notin \{C_4, C_8\}$, by Theorem 4, $\gamma_R(C) \leq \frac{8|V(C)|}{11}$. Let H'' be the union of C_4-components and C_8-components of H, and $H' = H - H''$. (Thus H' is obtained from H by removing each C_4-component and also each C_8-component of H.) Thus $\gamma_R(H') \leq \frac{8|V(H')|}{11}$. Let g be a $\gamma_R(H')$-function, and g_1 be a $\gamma_R(H'')$-function with $V_1^{g_1} \neq \emptyset$. Now define h on $V(G)$ by $h(x) = g(x)$ if $x \in V(H')$, $h(x) = g_1(x)$ if $x \in V(H'')$, $h(x) = 2$ if $x \in Z_1$, and $h(x) = 0$ if $x \in A$. Then h is an RDF for G. Then by Theorem 4,

$$\begin{aligned}
\gamma_R(G) &\leq \frac{8|V(H')|}{11} + 3r_1 + 6r_2 + 2|Z_1| \\
&= \frac{8(2|A| - 4r_1 - 8r_2)}{11} + 3r_1 + 6r_2 + 2\frac{|A|}{\Delta(G)} \\
&= \frac{16\Delta(G)|A| + r_1\Delta(G) + 2r_2\Delta(G) + 22|A|}{11\Delta(G)}.
\end{aligned}$$

But $4r_1 + 8r_2 \leq 2|A|$, and thus $\Delta(G)(r_1 + 2r_2) \leq \frac{\Delta(G)|A|}{2}$. Thus $\gamma_R(G) \leq$

$\frac{(33\Delta(G)+44)|A|}{22\Delta(G)} \leq 2|A|$, since $\Delta(G) \geq 4$. This produces a contradiction, since $G \in \mathcal{R}_{UVR}$. We deduce that $\gamma_R(G) < \frac{2n\Delta(G)}{3\Delta(G)+1}$.

We conclude that $B \neq \emptyset$ or $|Z_1| > \frac{|A|}{\Delta(G)}$. Assume that $B \neq \emptyset$. Thus $|B| \geq 1$. Now

$$\begin{aligned} n &= \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{(2\Delta(G)+1)\left|V_2^f\right| + (\Delta(G)-1)|B|}{\Delta(G)} + \left|V_2^f\right| \\ &\geq \frac{(3\Delta(G)+1)\left|V_2^f\right| + (\Delta(G)-1)}{\Delta(G)} \end{aligned}$$

and thus $\gamma_R(G) \leq \frac{2n\Delta(G)-2\Delta(G)+2}{3\Delta(G)+1}$. Next assume that $B = \emptyset$. Then $|Z_1| > \frac{|A|}{\Delta(G)}$. We have the following possibilities:

Possibility 1. There is a vertex $a \in A$ such that $|N(a) \cap Z_1| \geq 2$.

Possibility 2. There is a vertex $x \in Z_1$ such that $|N(x) \cap A| \leq \Delta(G) - 1$.

In Possibility 1, it is obvious that $|Z_1| \geq \frac{|A|+1}{\Delta(G)}$, and in Possibility 2,

$$|Z_1| \geq \frac{|A| - (\Delta(G)-1)}{\Delta(G)} + 1 = \frac{|A|+1}{\Delta(G)}.$$

Now

$$n = \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{(2\Delta(G)+1)\left|V_2^f\right| + 1}{\Delta(G)} + \left|V_2^f\right| = \frac{(3\Delta(G)+1)\left|V_2^f\right| + 1}{\Delta(G)}$$

and thus $\gamma_R(G) \leq \frac{2n\Delta(G)-2}{3\Delta(G)+1} = \frac{2n}{3}\left(\frac{\Delta(G)-1/n}{\Delta(G)+1/3}\right)$. ■

We next improve Theorem 11 for graphs G with $\delta(G) \geq 4$.

Theorem 14. *If $G \in \mathcal{R}_{UVR}$ is graph of order n with $\Delta(G) > 3\delta(G) - 6$ and $\delta(G) \geq 4$, then*

$$\gamma_R(G) \leq \frac{2n}{3}\left(\frac{\Delta(G) - 1/n}{\Delta(G) + (\delta(G)-2)/3}\right).$$

Proof. Let $G \in \mathcal{R}_{UVR}$ be a graph of order n with $\delta(G) \geq 4$. Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_R(G)$-function satisfying Proposition 3. Let A, B, Z_0 and Z_1 be defined as in the proof of Theorem 10. By Theorem 11, $\gamma_R(G) \leq \frac{2n\Delta(G)}{3\Delta(G)+\delta(G)-2}$. We show that $\gamma_R(G) < \frac{2n\Delta(G)}{3\Delta(G)+\delta(G)-2}$. Suppose that $\gamma_R(G) = \frac{2n\Delta(G)}{3\Delta(G)+\delta(G)-2}$. Then each of the inequalities in the proof of Theorem 11 will be equality. From (5) and (6) we find that $B = \emptyset$, and from (1) we obtain $|Z_1| = \frac{(\delta(G)-2)|A|}{\Delta(G)}$. Consequently, each

vertex of Z_1 is adjacent to precisely $\Delta(G)$ vertices of A, and each vertex of A is of degree $\delta(G)$ and has precisely $\delta(G)-2$ neighbors in Z_1 and two neighbors in Z_0.

Observe that $\gamma_R(G) = 2|A|$, $|Z_0| = 2|A|$, and $|Z_1| = \frac{(\delta(G)-2)|A|}{\Delta(G)}$. Let H be the graph obtained from G by removal of the vertices of $N[x]$ for all $x \in Z_1$. Clearly, $V(H) = Z_0$ and $|V(H)| = 2|A| = \gamma_R(G)$. Since $\delta(H) \geq 3$, by Theorem 5, $\gamma_R(H) \leq \frac{2|V(H)|}{3} = \frac{4|A|}{3}$. Let g be a $\gamma_R(H)$-function. Then g_1 defined on $V(G)$ by $g_1(x) = g(x)$ if $x \in V(H)$, $g_1(x) = 2$ if $x \in Z_1$, and $g_1(x) = 0$ if x is adjacent to some vertex of Z_1, is an RDF for G. Thus

$$\gamma_R(G) \leq \frac{4|A|}{3} + 2|Z_1| = \frac{(4\Delta(G) + 6\delta(G) - 12)|A|}{3\Delta(G)} < 2|A|,$$

since $\Delta(G) > 3\delta(G) - 6$. This is a contradiction. Thus $\gamma_R(G) < \frac{2n\Delta(G)}{3\Delta(G)+\delta(G)-2}$.

We conclude that $B \neq \emptyset$ or $|Z_1| > \frac{(\delta(G)-2)|A|}{\Delta(G)}$. Assume that $B \neq \emptyset$. Then

$$\begin{aligned} n &= \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{(2\Delta(G) + \delta(G) - 2)\left|V_2^f\right| + (\Delta(G) - (\delta(G) - 2))|B|}{\Delta(G)} + \left|V_2^f\right| \\ &\geq \frac{(2\Delta(G) + \delta(G) - 2)\left|V_2^f\right| + (\Delta(G) - (\delta(G) - 2))}{\Delta(G)} + \left|V_2^f\right| \\ &= \frac{(3\Delta(G) + \delta(G) - 2)\left|V_2^f\right| + (\Delta(G) - (\delta(G) - 2))}{\Delta(G)} \end{aligned}$$

and thus $\gamma_R(G) \leq \frac{2n\Delta(G)-2(\Delta(G)-(\delta(G)-2))}{3\Delta(G)+\delta(G)-2} \leq \frac{2n\Delta(G)-2}{3\Delta(G)+\delta(G)-2}$. Thus assume that $B = \emptyset$. Then $|Z_1| > \frac{(\delta(G)-2)|A|}{\Delta(G)}$. We have the following possibilities:

Possibility 1. There is a vertex $a \in A$ such that $|N(a) \cap Z_1| \geq (\delta(G) - 2)$.

Possibility 2. There is a vertex $x \in Z_1$ such that $|N(x) \cap A| \leq \Delta(G) - 1$.

In Possibility 1, it is obvious that $|Z_1| \geq \frac{(\delta(G)-2)|A|+1}{\Delta(G)}$, and in Possibility 2,

$$|Z_1| \geq \frac{(\delta(G) - 2)|A| - (\Delta(G) - 1)}{\Delta(G)} + 1 = \frac{(\delta(G) - 2)|A| + 1}{\Delta(G)}.$$

Now

$$\begin{aligned} n &= \left|V_0^f\right| + \left|V_2^f\right| \geq \frac{(2\Delta(G) + (\delta(G) - 2))\left|V_2^f\right| + 1}{\Delta(G)} + \left|V_2^f\right| \\ &= \frac{(3\Delta(G) + (\delta(G) - 2))\left|V_2^f\right| + 1}{\Delta(G)}. \end{aligned}$$

Thus $\gamma_R(G) \leq \frac{2n\Delta(G)-2}{3\Delta(G)+\delta(G)-2} = \frac{2n}{3}\left(\frac{\Delta(G)-1/n}{\Delta(G)+(\delta(G)-2)/3}\right)$. ■

Since any planar graph has a vertex of degree at most five, we obtain the following.

Corollary 15. *If $G \in \mathcal{R}_{UVR}$ is planar graph of order n with $\delta(G) \geq 4$ and $\Delta(G) \geq 10$, then $\gamma_R(G) \leq \frac{2n}{3}\left(\frac{\Delta(G)-1/n}{\Delta(G)+(\delta(G)-2)/3}\right)$.*

5. Concluding Remarks

Samodivkin [10] gave a constructive characterization for all trees in $\mathcal{R}_{UVR}$. He constructed a family $\mathcal{T}$ of trees and proved that for a tree T, $T \in \mathcal{R}_{UVR}$ if and only if $T \in \mathcal{T}$. Assume that $G \in \mathcal{R}_{UVR}$ is a graph with $\delta(G) = 1$ and $\gamma_R(G) = \frac{2n}{3}$. If $\Delta(G) > 2$ then by the argument given in the proof of Theorem 7, we obtain a forest $F \in \mathcal{R}_{UVR}$ with $\gamma_R(F) = \frac{2n}{3}$. Thus each component of F belongs to $\mathcal{T}$. We propose the following problem.

Problem 16. Characterize all graphs $G \in \mathcal{R}_{UVR}$ with $\gamma_R(G) = \frac{2n}{3}$ and $\delta(G) = 1$.

It can be seen that for any graph G, $H = G \circ K_2 \in \mathcal{R}_{UVR}$, and note that $\gamma_R(H) = \frac{2|V(H)|}{3}$ and $\delta(H) = 1$. We also remark that we do not know the sharpness of bounds of Theorems 13 and 14, and thus we propose the problem of showing the sharpness of them or improving them.

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