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CONSTANT 2-LABELLINGS AND AN APPLICATION TO (r, a, b)-COVERING CODES

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Abstract

We introduce the concept of constant 2-labelling of a vertex-weighted graph and show how it can be used to obtain perfect weighted coverings. Roughly speaking, a constant 2-labelling of a vertex-weighted graph is a black and white colouring of its vertex set which preserves the sum of the weights of black vertices under some automorphisms. We study constant 2-labellings on four types of vertex-weighted cycles. Our results on cycles allow us to determine (r, a, b)-codes in $\mathbb{Z}^2$ whenever $|a - b| > 4$, $r \geq 2$ and we give the precise values of a and b. This is a refinement of Axenovich's theorem proved in 2003.

Keywords: covering codes, weighted codes, infinite grid, vertex-weighted graphs.

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1. Introduction

Constant 2-labellings are particular 2-colourings of vertex-weighted graphs. For every composition of the colouring with an automorphism of a given group, the sum of the weights of the black vertices must be equal to a constant that depends on the colour of a given particular vertex.

The motivation about introducing constant 2-labellings comes from covering problems in graphs. Covering and packing problems are traditional issues in graph theory. A natural packing problem in a graph is to determine the maximal number of non-intersecting identical balls that can be placed in the graph; a covering problem is for example to determine the minimal number of r-balls that can be placed in such a way that every vertex of the graph is contained in at least one of them. Packing problems are fundamental in "error correction" while covering problems have application in mobile network. See [3] for many bibliographic pointers.

We consider in this paper coverings with balls of constant radius that satisfy certain multiplicity conditions. For positive integers r, a, b, an (r, a, b)-*covering code* or simply (r, a, b)-*code* of a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices such that every element of S belongs to exactly a balls of radius r with elements of S as centers and every element of $V \setminus S$ belongs to exactly b balls of radius r with elements of S as centers. We can view an (r, a, b)-code as a particular colouring c with two colours, black and white, where the black vertices are the elements of the code. Such codes are also known as (a, b)-codes of radius r [9], as (r, a, b)-isotropic colourings [1] or as perfect colourings [14].

The notion of (r, a, b)-codes generalizes the notion of domination and perfect codes in graphs. Perfect codes were introduced in terms of graphs by Biggs [2]. An r-*perfect code* of a graph $G = (V, E)$ is a subset $C \subseteq V$ with the property that each vertex is within distance r of exactly one vertex of C. Hence, an r-perfect code is an $(r, 1, 1)$-code. Kratochvíl [12] showed that recognising existence of an r-perfect code in graphs is NP-complete.

Perfect codes have also been studied in infinite graphs. For example, Golomb and Welch [7, 8] considered the multidimensional rectangular grid $\mathbb{Z}^d$. They proved the existence of 1-perfect codes, i.e., $(1, 1, 1)$-codes, in $\mathbb{Z}^d$. Such codes can be considered as periodic tilings of the grid $\mathbb{Z}^d$ by balls of radius 1. Moreover, the authors conjectured that there do not exist r-perfect codes with $r > 1$ in $\mathbb{Z}^d$ [7, 8]. Horak [11] wrote a survey of results that support the conjecture. For more information about perfect codes, see [3, Chapter 11].

The (r, a, b)-codes have already been studied in some graphs under the name of *weighted covering codes* by Cohen *et al.* [4]. Their work corresponds to a study of these codes in the Hamming metric. For a subset C of vertices, they attach weights to different layers of the Hamming sphere and they consider weighted spheres centred at vertices of C. If several such spheres intersect in a vertex, they define the density of each vertex as the sum of the weights of the corresponding layers. The set C is called a *weighted covering* if the density at each vertex is at least one. When the density is exactly equal to one for all vertices, then C is called a *perfect weighted covering*. If the radius is equal to 1, a $(1, a, b)$-code is exactly a perfect weighted covering of radius one with weight $\left(\frac{b-a+1}{b}, \frac{1}{b}\right)$. For more details see [3, Chapter 13].

While Cohen *et al.* [4] studied weighted codes in Hamming metric, Telle considered a particular case of these codes in graphs in general [15]. For a subset C of vertices, he defines the state of a vertex $u \in C$ by

$$state(u) = \begin{cases} \sigma_i & \text{if } u \in C \text{ and } |N(u) \cap C| = i, \\ \rho_i & \text{if } u \notin C \text{ and } |N(u) \cap C| = i. \end{cases}$$

Then many properties of vertex subsets can be defined by allowing only a specific set L of states. For instance, the set C is a dominating set if the state ρ_0 is not allowed. In this setting, $(1, a, b)$-codes are equivalent to $[\sigma_{a-1}, \rho_b]$-dominating sets. Telle [15] proved that the following decidability problem was NP-complete: "Is it possible to decide whether a graph has an $[\sigma_a, \rho_b]$-dominating set ?". The problem is still NP-complete when restricted to planar bipartite graphs of maximum degree three.

The particular case where the radius is 1 has been studied a lot. For instance, $(1, a, b)$-codes are equivalent to equitable partitions with two cells [6, Chapter 5]. In the multidimensional grid, which corresponds to the Lee metric with an infinite alphabet, $(1, a, b)$-codes were studied by Dorbec *et al.* [5] and Gravier *et al.* [10]. For instance, the existence of $(1, 2, 1)$-codes[1] in $\mathbb{Z}^d$ is proved in both papers [5, 10]. In [5, Theorem 4], Dorbec *et al.* present a method to construct $(1, a, b)$-codes in $\mathbb{Z}^d$. This method is based on a one-dimensional pattern of finite length that is extended by translations to colour $\mathbb{Z}^d$. Hence, the code obtained satisfies periodic properties.

Theorem 1 (Dorbec *et al.* [5])**.** *Assume that* $1 \le k \le n, 1 \le d$ *and*

$$A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{Z}_n \ (\textit{where } a_i \neq a_j, \textit{ when } i \neq j)$$

and $w_1, \ldots, w_d$ *are (not necessarily distinct) elements of* $\mathbb{Z}_n$*. Consider the sums* $a_i + w_j$ *and the differences* $a_i - w_j$*. If these* $2kd$ *elements take each value in* A *exactly* a *times and each value in* $\mathbb{Z}_n \setminus A$ *exactly* b *times, then the set*

$$C = \Big\{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid x_1 w_1 + \cdots + x_d w_d \in A\Big\}$$

is a $(1, a+1, b)$*-code of* $\mathbb{Z}^d$*.*

In the two-dimensional grid, i.e., the usual infinite grid, Puzynina studied the periodicity of (r, a, b)-codes. For $r = 1$, there exist non-periodic (r, a, b)-codes but, all of them can be obtained from periodic ones [13]. That is to say, if r, a, b are such that there exists an (r, a, b)-code, then there exists a periodic colouring that is an (r, a, b)-code. Moreover when $r \ge 2$, the author proved that every

[1] The reader may notice a distinction of notation between this paper and [5]. They write (a, b)-codes for what we denote $(1, a+1, b)$-codes as they consider open neighbourhoods and we consider closed neighbourhoods.

(r, a, b)-code is periodic [14]. The notion of constant 2-labellings comes up as a natural translation of the periodicity of (r, a, b)-codes in the infinite grid $\mathbb{Z}^2$ (see Section 4).

When the difference between a and b is large enough, the precise type of the periodic colouring is known.

Theorem 2 (Axenovich [1]). *If a colouring is an (r, a, b)-code of $\mathbb{Z}^2$ with $r \geq 2$ and $|a - b| > 4$, then it is one of the following diagonal colourings*:

1. *q-periodic colouring where $q \in \{r, r+1\}$ is odd and the monochromatic diagonals are parallel.*
2. *q-anti-periodic colouring where $q \in \{r, r+1\}$ is even.*
3. *q-periodic colouring where $q \in \{r, r+1\}$ is even and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
4. *$(2r + 1)$-periodic colouring and for all horizontal or vertical interval I of length p the number of black vertices from the even sublattice and from the odd sublattice is the same.*
5. *2-periodic or 3-periodic colouring.*

Using Axenovich's characterization in terms of diagonal colourings of all (r, a, b)-codes in $\mathbb{Z}^2$ with $r \geq 2$ and $|a - b| > 4$ (Theorem 2), we show that the existence of (r, a, b)-codes in the infinite grid is linked with the existence of constant 2-labellings in particular cycles. It turns out that studying only four types of vertex-weighted cycles is sufficient to characterize all (r, a, b)-codes with $|a - b| > 4$ and to determine explicitly the possible values taken by the constants a and b. Hence, we obtain a refinement of Axenovich's theorem.

Outline. This paper is organized as follows. The first section is dedicated to the presentation of constant 2-labellings of vertex-weighted graphs in a general framework. Then we focus on the constant 2-labellings in four types of vertex-weighted cycles. In Section 3, we present projection and folding techniques that link constant 2-labellings to (r, a, b)-codes. Hopefully, these techniques can be applied to other problems involving periodic tilings. In Section 4, we apply the projection and folding method to obtain all possible values of constants a and b such that there exist (r, a, b)-codes of $\mathbb{Z}^2$ with $|a - b| > 4$ and $r \geq 2$. Note that to apply this method, the colouring of the grid must satisfy some specific properties. Finally, we suggest directions for future work.

2. Constant 2-Labellings

By abuse of language, we identify the colour *black* with the colour `1`. Similarly, *white* corresponds to the colour `0`. In the sequel, a *weighted graph* is understood as a vertex-weighted graph.

Given a graph $G = (V, E)$, a particular vertex $v \in V$, a map $w : V \to \mathbb{R}$ and a subgroup A of the set $Aut(G)$ of all automorphisms of G, a *constant* 2*-labelling* of G is a mapping $c : V \to \{\mathtt{0}, \mathtt{1}\}$ such that there exist constants a and b satisfying

$$a = \sum_{\{u \in V \mid c \circ \xi(u) = \mathtt{1}\}} w(u), \text{ for all } \xi \in A_{\mathtt{1}} \text{ and } b = \sum_{\{u \in V \mid c \circ \xi'(u) = \mathtt{1}\}} w(u), \text{ for all } \xi' \in A_{\mathtt{0}},$$

where $A_{\mathtt{1}} = \{\xi \in A \mid c \circ \xi(v) = \mathtt{1}\}$, $A_{\mathtt{0}} = \{\xi \in A \mid c \circ \xi(v) = \mathtt{0}\}$ and $c \circ \xi$ denotes the composition of c and ξ, i.e., $c \circ \xi(v) = c(\xi(v))$.

Example 3. Let $G = (V, E)$ be the graph with $V = \{v_0, \ldots, v_4\}$ represented in Figure 1. Take $v = v_0, A = Aut(G), w : V \to \mathbb{R}$ and $c : V \to \{\mathtt{0}, \mathtt{1}\}$ defined by $w(v_0) = 3$, $w(v_1) = w(v_3) = 2$, $w(v_2) = w(v_4) = 5$ and $c(v_0) = c(v_3) = c(v_4) = \mathtt{0}, c(v_1) = c(v_2) = \mathtt{1}$. It is clear that c is a constant 2-labelling since A contains only two automorphisms, *id* and

$$\sigma : v_0 \mapsto v_0;\ v_1 \mapsto v_4;\ v_2 \mapsto v_3;\ v_3 \mapsto v_2;\ v_4 \mapsto v_1.$$

We have $A_{\mathtt{1}} = \emptyset, A_{\mathtt{0}} = A, b = 8$ and a is not defined.

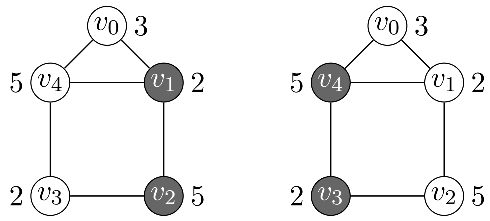

Figure 1. A colouring of a graph G (on the left) and its composition with the automorphism σ (on the right).

We can make some straightforward observations about constant 2-labellings. The following proposition allows us to consider either a colouring c or its complement colouring $\overline{c}$.

Proposition 4 (Complementary property)**.** *Let $G = (V, E)$ be a weighted graph, $w : V \to \mathbb{R}$ be the weight map, $v \in V$ and $A \leq Aut(G)$. Set $\omega := \sum_{u \in V} w(u)$. A colouring c is a constant 2-labelling of G with respective constants a and b if and only if the colouring $\overline{c}$ is a constant 2-labelling with respective constants $\omega - b$ and $\omega - a$.*

It is clear that, for a weighted graph $G = (V, E)$ with $v \in V$, any monochromatic colouring of V is a constant 2-labelling for any weight map and any subgroup of $Aut(G)$. Such constant 2-labellings are called *trivial*. If c is monochromatic black, then the constants are such that $a = \sum_{u \in V} w(u)$ and b is not defined. If c is monochromatic white, a is not defined and $b = 0$.

Given the definition of constant 2-labellings, one can ask whether there exist non-trivial constant 2-labellings for some classes of weighted graphs. We answer that question in the case of four types of weighted cycles in the next subsection.

Remark 5. Consider the complete graph K_n and let $w : V(K_n) \to \mathbb{R}$, $v \in V(K_n)$, $A = Aut(K_n)$. It is straightforward to show that there exists a non-trivial constant 2-labelling of K_n if and only if $w(v_1) = w(v_2)$ for all $v_1, v_2 \in V \setminus \{v\}$. See [16, Remark 6.3] for details.

2.1. Constant 2-labellings in particular weighted cycles

We consider some particular weighted cycles $\mathcal{C}_p$ with at most 4 different weights on the vertices $0, \ldots, p-1$. If the weights are $w(0), \ldots, w(p-1)$, then we represent the cycle by the word $w(0) \cdots w(p-1)$. We will use the letters z, x, y and t to denote the weights of vertices. For instance, the cycle depicted in Figure 2 is represented by the word zx^{p-1}.

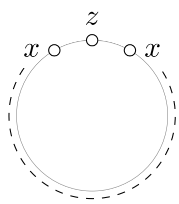

Figure 2. Weighted cycle $\mathcal{C}_p$ represented by the word zx^{p-1}.

We restrict our study of constant 2-labellings to only four types of weighted cycles (see Figure 3). We set $v := 0$ and $A := \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ to be the set of rotations with

$$\mathcal{R}_k : \{0, \ldots, p-1\} \to \{0, \ldots, p-1\}, i \mapsto i + 1 \bmod p.$$

This restriction is due to the initial motivation behind this work: to know the possible values of the constants in (r, a, b)-code of the infinite grid. The four types of weighted cycles on p vertices depend on $p \bmod 4$.

- If $p \equiv 1 \pmod 4$, the cycle represented by the word $z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}}$ is called *Type* 1.
- If $p \equiv 2 \pmod 4$, the cycle represented by the word $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ is called *Type* 2.
- If $p \equiv 3 \pmod 4$, the cycle represented by the word $z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}}$ is called *Type* 3.
- If $p \equiv 0 \pmod 4$, the cycle represented by the word $z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$ is called *Type* 4.

Hence, we only consider weighted cycles with an axial symmetry in the distribution of weights. It seems to play an important role for the existence of constant 2-labellings. For instance, a weighted cycle $\mathcal{C}_p$ represented by the word $z(xy)^{\frac{p-1}{2}}$ with $x \neq y$, has only monochromatic colourings as constant 2-labellings. See [16, Lemma B.3] for a proof.

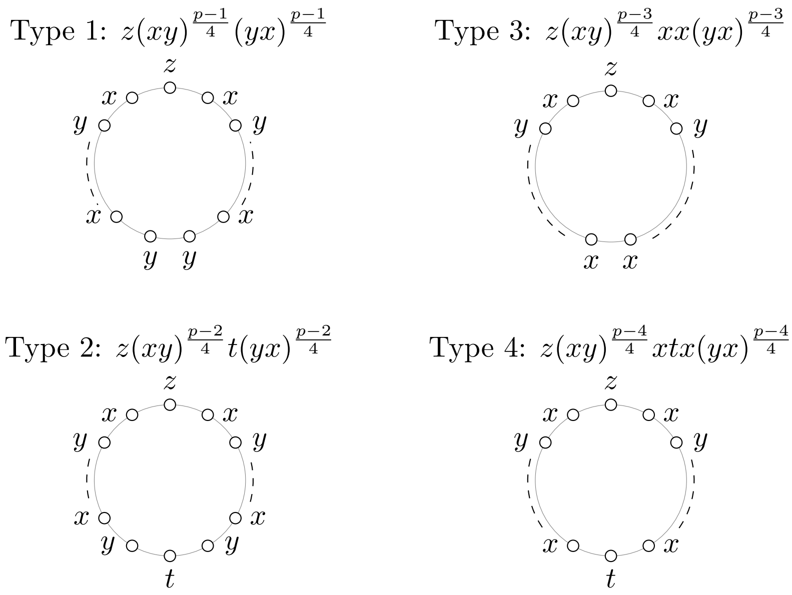

Figure 3. Types of weighted cycles $\mathcal{C}_p$.

Note that the cycle in Figure 2 is a particular case of all of these types. Such cycles are called *Type* 0. As we see in the next lemma, the case of Type 0 cycles is easy to handle.

Lemma 6. *For cycles $\mathcal{C}_p$ of Type* 0*, represented by zx^{p-1} with $1 < p \in \mathbb{N}$, all colourings are constant* 2*-labellings.*

Proof. Let c be a colouring of $\mathcal{C}_p$. We set α_x to be the number of black vertices with weight x. If $c(0) = \mathtt{1}$, we have $\alpha_x + 1$ black vertices and it is clear that c is a constant 2-labelling where the weighted sum of black vertices is equal to $a = \alpha_x x + z$ (respectively $b = (\alpha_x + 1)x$) if the vertex 0 is black (respectively white). ■

We now turn our attention to cycles of Type 1 which are not of Type 0.

Lemma 7. *Let $p > 2$ be an integer such that $p \equiv 1 \pmod 4$. For cycles $\mathcal{C}_p$ of Type* 1*, represented by $z(xy)^{\frac{p-1}{4}}(yx)^{\frac{p-1}{4}}$, with $x \neq y$, there exists a non-trivial* *constant* 2*-labelling* c *if and only if* $p \equiv 0 \pmod 3$. *In which case,* c *is* 3*-periodic of pattern period* $\texttt{110}$.

Proof. Let $p > 2$ be an integer such that $p \equiv 1 \pmod 4$ and let $\mathcal{C}_p$ be a cycle of Type 1 with $x \neq y$. Assume that c is a non-trivial constant 2-labelling of $\mathcal{C}_p$. The colouring c is not alternate since p is odd. Hence, without loss of generality, we can assume that there exist two consecutive black vertices. Moreover, we can suppose that these vertices are the vertices 0 and 1 of $\mathcal{C}_p$. Indeed, c is a constant 2-labelling if and only if $c \circ \mathcal{R}_j$ is a constant labelling for all $j \in \mathbb{Z}$.

For the colouring c, we let α_x, α_y denote respectively the number of black vertices with weight x and y. We have $a = \alpha_x x + \alpha_y y + z$. We consider the colour of the vertex $\frac{p+1}{2}$.

Assume first that $c(\frac{p+1}{2}) = \texttt{1}$. Then, for the colouring $c \circ \mathcal{R}_1$, the sum of the weights of black vertices is

$$a = (\alpha_x - 1)y + z + (\alpha_y - 1)x + y = \alpha_y x + \alpha_x y + z,$$

since under a 1-rotation, any black vertex with weight x becomes a black vertex of weight y, except for the vertex 1 which becomes the vertex with weight z, and similarly any black vertex with weight y becomes a black vertex of weight x except for the vertex $\frac{p+1}{2}$ which becomes a vertex of weight y. As the weights x and y are distinct, it implies that $\alpha_x = \alpha_y$ (in order to have a sum of black vertices constant and equal to a). We set $\alpha := \alpha_x = \alpha_y$ for a shorter notation.

Let i be the smallest integer in $\{0, \ldots, \frac{p-1}{2} - 1\}$ such that $c(i+1) = \texttt{0}$ and assume $c(\frac{p+1}{2} + \ell) = \texttt{1}$ for any $\ell \in \{0, \ldots, i\}$ (otherwise, consider the colouring $c \circ \mathcal{R}_{\frac{p+1}{2}}$ instead of c). Then Figure 4 shows that assuming $c(\frac{p+1}{2} + i + 1) = \texttt{1}$ leads to a contradiction about the value of a. So $c(\frac{p+1}{2} + i + 1) = \texttt{0}$. With the colouring $c \circ \mathcal{R}_{i+1}$, we obtain a sum of the weights of black vertices equal to $b = \alpha x + (\alpha + 1)y$ (Figure 5). To conclude this case, consider the vertex $i+2$ and observe that whatever value is assigned to $c(i+2)$, we obtain a contradiction (Figure 6).

Therefore, we have $c(\frac{p+1}{2}) = \texttt{0}$ and $a = \alpha_x x + \alpha_y y + z$ as in the beginning. Observe that the previous reasoning means that for any integer j, we have

$$c \circ \mathcal{R}_j(0) = \texttt{1} = c \circ \mathcal{R}_j(1) \Rightarrow c \circ \mathcal{R}_j\left(\frac{p+1}{2}\right) = \texttt{0}. \tag{1}$$

With the colouring $c \circ \mathcal{R}_1$, the sum of the weights of black vertices is

$$a = (\alpha_x - 1)y + z + \alpha_y x + x = (\alpha_y + 1)x + (\alpha_x - 1)y + z.$$

Since $x \neq y$, we get $\alpha_x = \alpha_y + 1$. We set $\alpha := \alpha_y$ for a shorter notation.

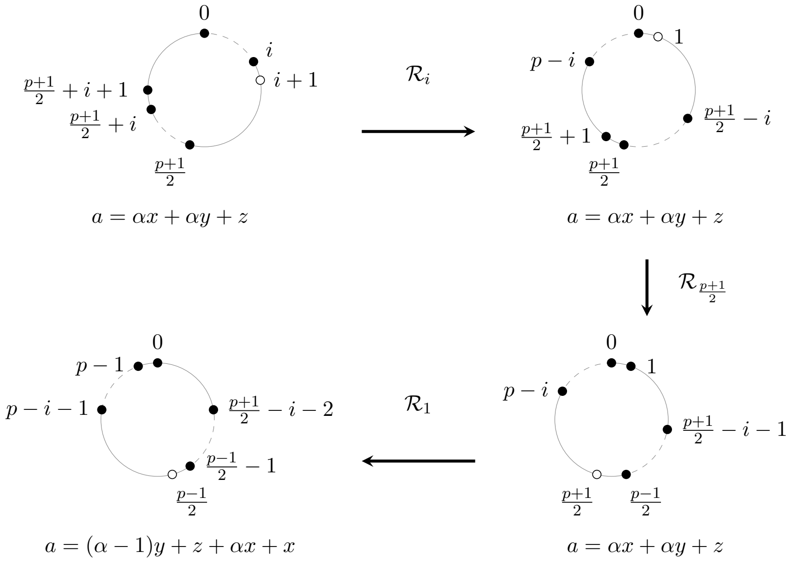

Figure 4. Rotations of the colouring c of a Type 1 cycle with $c(\frac{p+1}{2} + i + 1) = \mathtt{1}$, and their corresponding weighted sums of black vertices which are not all equal.

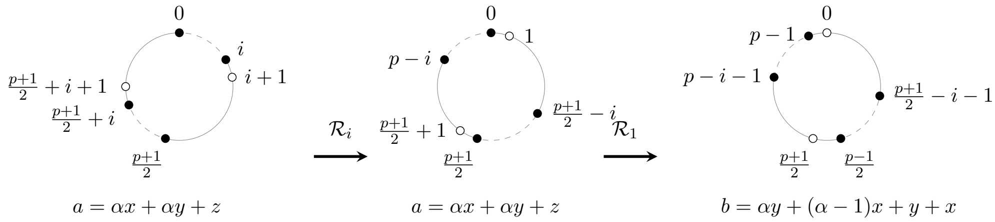

Figure 5. Rotations of the colouring c of a Type 1 cycle with $c(\frac{p+1}{2} + i + 1) = \mathtt{0}$, and their corresponding weighted sums of black vertices.

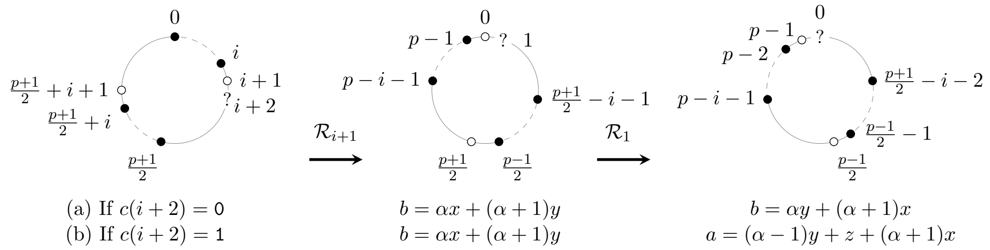

Figure 6. Rotations of the colouring c of a Type 1 cycle and their corresponding weighted sums of black vertices depending on the colour $c(i + 2)$.

Let i be the smallest integer in $\{0,\ldots,\frac{p-1}{2}-1\}$ such that $c(i+1)=\mathtt{0}$. From equation (1), we have $c(\frac{p+1}{2}+\ell)=\mathtt{0}$ for any $\ell\in\{0,\ldots,i-1\}$. Moreover, we have $c(\frac{p+1}{2}+i)=\mathtt{1}$. Indeed, assume that $c(\frac{p+1}{2}+i)=\mathtt{0}$ (Figure 7), then with the colouring $c\circ\mathcal{R}_{i+1}$ we obtain a sum of the weights of black vertices equal to $b=(\alpha+1)x+(\alpha+1)y$. As c is a constant 2-labelling, with the colouring $c\circ\mathcal{R}_{\frac{p+1}{2}}$ we have the same weighted sum b. Then it implies that the weighted sum b with the colouring $c\circ\mathcal{R}_{\frac{p+1}{2}+1}$ has a different value, which is a contradiction. So $c(\frac{p+1}{2}+i)=\mathtt{1}$ and with the colouring $c\circ\mathcal{R}_{i+1}$ we have a sum of the weights of black vertices equal to $b=\alpha x+(\alpha+2)y$ (see Figure 8).

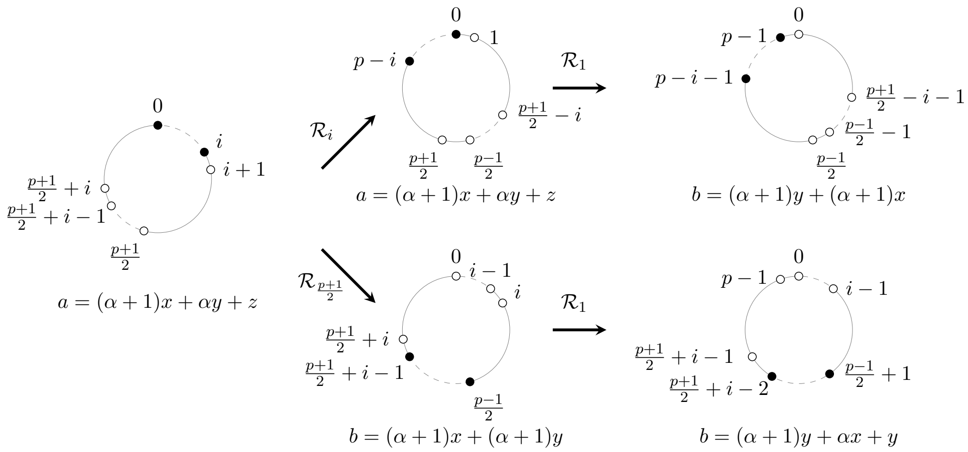

Figure 7. Rotations of the colouring c of a Type 1 cycle with $c(\frac{p+1}{2}+i)=\mathtt{0}$, and their corresponding weighted sums b of black vertices which are not all equal.

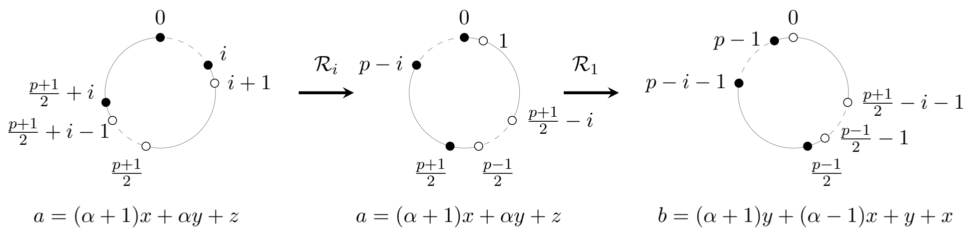

Figure 8. Rotations of the colouring c of a Type 1 cycle with $c(\frac{p+1}{2}+i)=\mathtt{1}$, and their corresponding weighted sums of black vertices.

From $b=\alpha x+(\alpha+2)y$, it follows that i must be equal to 2, otherwise the colouring $c\circ\mathcal{R}_{\frac{p+1}{2}}$ leads to a different sum of the weights of black vertices (Figure 9). Then we have $c(3)=\mathtt{1}$ (Figure 10). Similarly, $c\big(\frac{p+1}{2}+2\big)=\mathtt{1}$ (Figure 11).

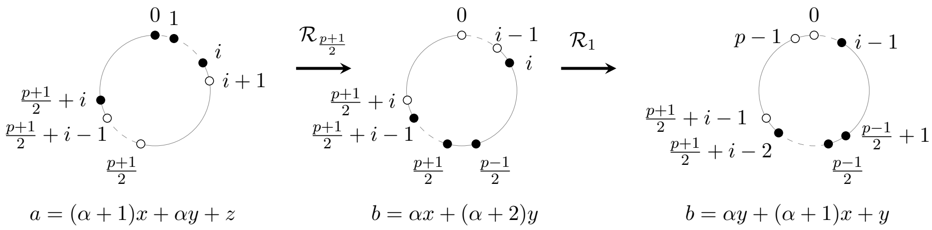

Figure 9. Rotations of the colouring c of a Type 1 cycle with $c(j) = \mathtt{1}$ for all $0 \leq j \leq i$ with $i > 1$, and their corresponding weighted sums of black vertices distinct which are not all equal.

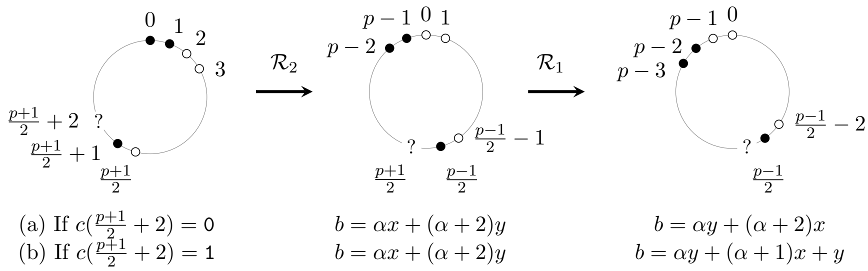

Figure 10. Rotations of the colouring c of a Type 1 cycle with $c(0) = c(1) = \mathtt{1}$, $c(\frac{p+1}{2}+1) = \mathtt{1}$ and $c(3) = c(\frac{p+1}{2}) = \mathtt{0}$, and their corresponding weighted sums of black vertices depending on the colour $c(\frac{p+1}{2}+2)$.

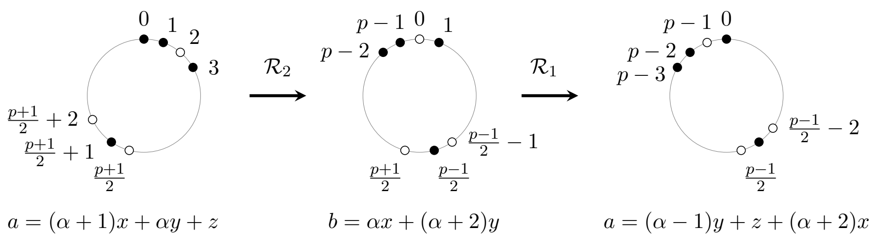

Figure 11. Rotations of the colouring c of a Type 1 cycle with $c(0) = c(1) = \mathtt{1}$, $c(\frac{p+1}{2}+1) = \mathtt{1}$ and $c(3) = c(\frac{p+1}{2}) = c(\frac{p+1}{2}+2) = \mathtt{0}$, and their corresponding weighted sums of black vertices which are not all equal.

Therefore, the colouring $c \circ \mathcal{R}_{\frac{p+1}{2}+1}$ has the same configuration as the colouring c, i.e., the vertices $0, 1$ are black and the vertex $\frac{p+1}{2}$ is white. We can apply the same argument as before. Hence, the colouring c must be 3-periodic of pattern period `110` and the number p of vertices is such that $p \equiv 0 \pmod 3$. ■

Remark 8. In the previous proof, we used the following fact. Let c be a constant 2-labelling of a cycle of Type 1 with $x \neq y$. If the sum of the weights of black vertices is equal to $a = \alpha_x x + \alpha_y y + z$ with the colouring c, where α_x, α_y respectively denote the black vertices with weight x and weight y, then for any colouring $c \circ \mathcal{R}_j$ such that $c \circ \mathcal{R}_j(0) = \mathtt{1}$, the weighted sum is $a = \alpha_x x + \alpha_y y + z$ and α_x, α_y respectively denote the numbers of black vertices of weight x and weight y with respect to the colouring $c \circ \mathcal{R}_j$. In other words, the number of black vertices of weight x (respectively y) is the same for the colouring $c \circ \mathcal{R}_j$ such that $c \circ \mathcal{R}_j(0) = \mathtt{1}$. This fact follows from the uniqueness of the solution (λ, μ) of the system

$$\begin{cases} a - z = \lambda x + \mu y \\ n - 1 = \lambda + \mu \end{cases}$$

where n denotes the total number of black vertices (which is known from the colouring c).

The same argument holds when the weighted sum is $b = \alpha_x x + \alpha_y y$ and the colouring $c \circ \mathcal{R}_j$ such that $c \circ \mathcal{R}_j(0) = \mathtt{0}$.

Cycles of Type 1 and Type 3 share some similarities. Both types have at most 3 distinct weights and their non-trivial constant 2-labellings are the same as shown in the next lemma. We omit the proof here since it follows exactly the same lines as the proof of Lemma 7, but the details can be found in [16, Appendix B].

Lemma 9. *For cycles $\mathcal{C}_p$ of Type* 3, *i.e., $z(xy)^{\frac{p-3}{4}}xx(yx)^{\frac{p-3}{4}}$ with $x \neq y$ and $3 < p \in \mathbb{N}$, if c is a non-trivial constant* 2*-labelling, then $p \equiv 0 \pmod 3$ and c is* 3*-periodic of pattern period* `110`.

Now for cycles of Type 2 and Type 4 with weights z, x, y, t, if the number n of black vertices and the values $a := z + \alpha_x x + \alpha_y y + \alpha_t t$ and $b := \beta_x x + \beta_y y + \beta_t t$ are known, then the following system

$$\begin{cases} a = \lambda x + \mu y + \nu t + z \\ n = \lambda + \mu + \nu + 1 \end{cases} \qquad \left(\text{respectively } \begin{cases} b = \lambda x + \mu y + \nu t \\ n = \lambda + \mu + \nu \end{cases}\right)$$

does not necessarily have a unique solution $(\lambda, \mu, \nu) = (\alpha_x, \alpha_y, \alpha_t)$ (respectively $(\lambda, \mu, \nu) = (\beta_x, \beta_y, \beta_t)$). Hence for these cycles, it is important to make a distinction between the colouring c and its rotations.

We first deal with an easy particular case of these cycles that corresponds to a Type 2 cycle with $t = x \neq y$ or to a Type 4 cycle with $t = y \neq x$.

Lemma 10. *Let $p \geq 4$ be an integer such that $p \equiv 0 \pmod 2$. Let $\mathcal{C}_p$ be a cycle of Type 2 with $t = x \neq y$ or of Type 4 with $t = y \neq x$, i.e., $\mathcal{C}_p$ is a cycle represented by $z(xy)^{\frac{p-2}{2}}x$ with $x \neq y$. Any non-trivial constant 2-labelling c of $\mathcal{C}_p$ is either the alternate colouring, or a colouring such that the number α_x of black vertices of weight x is equal to $\alpha_y + c(0)$ where α_y is the number of black vertices of weight y.*

Proof. Let $p \geq 4$ be an integer such that $p \equiv 0 \pmod 2$ and let $\mathcal{C}_p$ be a cycle represented by $z(xy)^{\frac{p-2}{2}}x$ with $x \neq y$. Clearly, the alternate colouring is a constant 2-labelling with $a = (\frac{p}{2} - 1)y + z$ and $b = \frac{p}{2}x$.

Now assume that c is a non-trivial constant 2-labelling of $\mathcal{C}_p$ which is not the alternate colouring. Without loss of generality, we assume that the vertices 0 and 1 are both coloured in black. Let α_x, α_y denote respectively the number of black vertices with weight x and y for the colouring c. We have $a = \alpha_x x + \alpha_y y + z$ as the sum of the weights of black vertices. For the colouring $c \circ \mathcal{R}_1$, the weighted sum is equal to

$$a = (\alpha_x - 1)y + z + \alpha_y x + x = (\alpha_y + 1)x + (\alpha_x - 1)y + z.$$

As $x \neq y$, we get $\alpha_x = \alpha_y + 1$ and we set $\alpha := \alpha_y$.

Let i be the smallest integer in $\{0, \ldots, p-2\}$ such that $c(i+1) = \mathtt{0}$. The weighted sum for the colouring $c \circ \mathcal{R}_i$ is $a = (\alpha+1)x + \alpha y + z$ by hypothesis. Therefore, the weighted sum for the colouring $c \circ \mathcal{R}_{i+1}$ is equal to $b = (\alpha+1)y + \alpha x + x = (\alpha+1)x + (\alpha+1)y$. Moreover, the weighted sum is preserved for the colouring $c \circ \mathcal{R}_{i+2}$ regardless to the colour of the vertex $i+2$:

$$\begin{cases} b = (\alpha+1)y + (\alpha+1)x & \text{if } c(i+2) = \mathtt{0} \\ a = (\alpha)y + z + (\alpha+1)y & \text{if } c(i+2) = \mathtt{1}. \end{cases}$$

It follows that the only condition on the constant 2-labelling c is to be a colouring with $\alpha_x = \alpha_y + 1$ if $c(0) = \mathtt{1}$. Similarly, the condition is $\alpha_x = \alpha_y$ if $c(0) = \mathtt{0}$. ∎

We now consider the Type 2 cycles in general.

Lemma 11. *Let $p \equiv 2 \pmod 4$ with $p > 2$ and let $\mathcal{C}_p$ be a weighted cycle of Type 2 represented by $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ where the weights x, y, t are not all equal. If c is a non-trivial constant 2-labelling, then c is one of the following colourings*

- *alternate,*
- *$\frac{p}{2}$-periodic,*
- *if $x = y$, $\frac{p}{2}$-anti-periodic,*
- *if $t = x$, any colouring such that the number of black vertices of weight x is equal to the sum of $c(0)$ and the number of black vertices of weight y.*

Proof. Let $p \equiv 2 \pmod 4$ with $p > 2$ and let $\mathcal{C}_p$ be a weighted cycle of Type 2 represented by $z(xy)^{\frac{p-2}{4}}t(yx)^{\frac{p-2}{4}}$ where the weights x, y and t are not all equal. Clearly, the alternate colouring is a constant 2-labelling with $a = (\frac{p}{2} - 1)y + z$ and $b = (\frac{p}{2} - 1)x + t$.

The case where the weights t and x are equal follows from Lemma 10. Hence, we suppose from now on that $t \neq x$. Consider a non-trivial constant 2-labelling c of $\mathcal{C}_p$ that is not the alternate colouring. Without loss of generality, we may assume that $c(0) = c(1) = \mathtt{1}$. We let $\alpha_x, \alpha_y, \alpha_t$ denote respectively the number of black vertices of weight x and y for the colouring c. The sum of the weights of the black vertices is then equal to $a = \alpha_x x + \alpha_y y + \alpha_t t + z$. We consider the colour of the vertex $\frac{p}{2} + 1$.

Assume first that $c(\frac{p}{2}+1) = \mathtt{1}$. It follows that $c(\frac{p}{2}) = \mathtt{1}$ and $\alpha_t = 1$, otherwise the weighted sum is not preserved (Figure 12). Then for the colouring $c \circ \mathcal{R}_1$, the sum of the weights of the black vertices is

$$a = (\alpha_x - 1)y + z + (\alpha_y - 1)x + t + x + y = \alpha_y x + \alpha_x y + t + z$$

as under a 1-rotation, any black vertex with weight x becomes a black vertex of weight y, except for the vertex 1 which becomes the vertex 0 with weight z, and similarly any black vertex with weight y becomes a black vertex of weight x, except for the vertex $\frac{p}{2} + 1$ which becomes the vertex $\frac{p}{2}$ with weight t. If the weights x and y are distinct, then $\alpha_x = \alpha_y$ and we set $\alpha := \alpha_x$. Otherwise, we denote by β the number $\alpha_x + \alpha_y$ of black vertices with weight $x = y$.

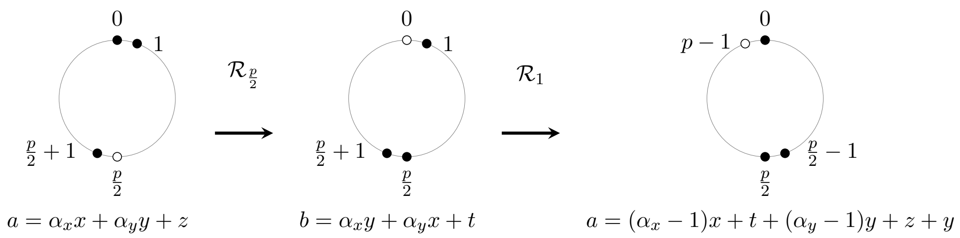

Figure 12. Rotations of the colouring c of a Type 2 cycle with $c(\frac{p}{2} + 1) = \mathtt{1}$, and their corresponding weighted sums of black vertices which are not all equal as $x \neq t$.

Let i be the smallest integer in $\{0, \ldots, \frac{p}{2} - 1\}$ such that $c(i + 1) = \mathtt{0}$ and assume that $c(\frac{p}{2} + i) = \mathtt{1}$ for any $\ell \in \{0, \ldots, i\}$ (otherwise, consider the colouring $c \circ \mathcal{R}_{\frac{p}{2}}$ instead of c). Then $c(\frac{p}{2} + i + 1) = \mathtt{0}$ as depicted in Figure 13.

With the colouring $c \circ \mathcal{R}_{i+1}$ we obtain (Figure 14) a sum of the weights of black vertices equal to

$$b = \begin{cases} (\alpha + 1)(x + y) & \text{if } x \neq y, \\ (\beta + 2)x & \text{if } x = y \end{cases}$$

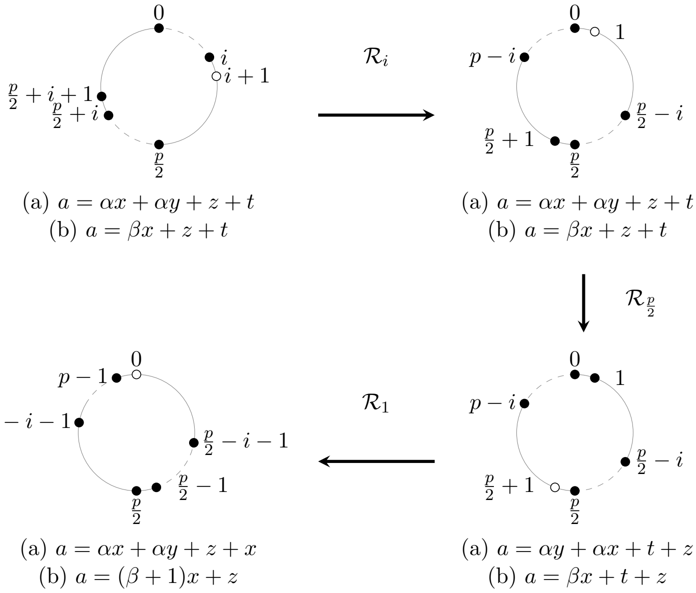

Figure 13. Rotations of the colouring c of a Type 2 cycle $\mathcal{C}_p$ with $c(\frac{p}{2}+i+1) = \mathtt{1}$, and their corresponding weighted sums of black vertices which are not all equal, where the line (a) corresponds to the case $x \neq y$ and the line (b) to the case $x = y$.

and the number of black vertices of weight x for the colouring $c \circ \mathcal{R}_{i+1}$ is actually $\alpha + 1$ (respectively $\beta + 2$) when $x \neq y$ (respectively $x = y$). Observe that if $c(i+2) = \mathtt{0} = c(\frac{p}{2}+i+2)$, then the weighted sum b for the colouring $c \circ \mathcal{R}_{i+2}$ is preserved.

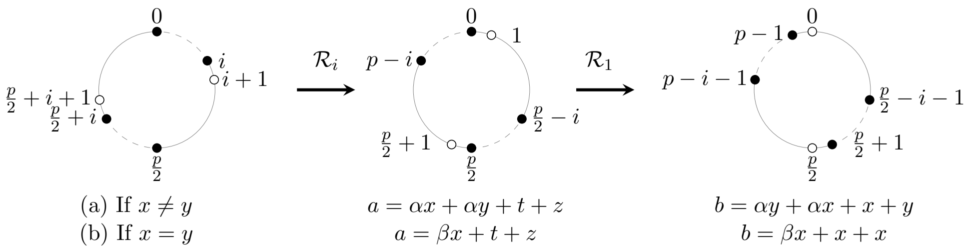

Figure 14. Rotations of the colouring c of a Type 2 cycle and their corresponding weighted sums of black vertices depending on the equality of the weights x and y.

Therefore, let j be the smallest integer in $\{i+1, \ldots, \frac{p}{2}-1\}$ such that $c(j+1) = \mathtt{1}$. Without loss of generality, we assume that $c(\frac{p}{2}+\ell) = \mathtt{0}$ for all $\ell \in \{i+1, \ldots, j\}$. Then $c\left(\frac{p}{2}+j+1\right) = \mathtt{1}$, otherwise it implies that $x = t$ which is a contradiction (Figure 15).

Consequently, the sum of the weights of the black vertices for the colouring $c \circ \mathcal{R}_{j+1}$ is $a = \alpha x + \alpha y + z + t$ (respectively $a = \beta x + t + z$) if the weights x

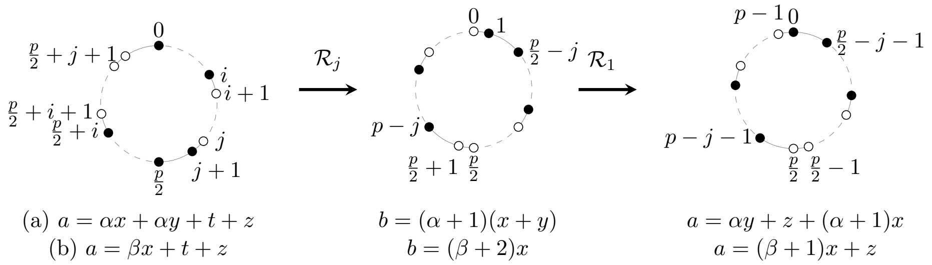

Figure 15. Rotations of the colouring c of a Type 2 cycle with $c(j+1) \neq c(\frac{p}{2}+j+1) = \mathtt{0}$, and their corresponding weighted sums of black vertices which are not all equal, where the line (a) corresponds to the case $x \neq y$ and the line (b) to the case $x = y$.

and y are distinct (respectively equal). Moreover, the colourings c and $c \circ \mathcal{R}_{j+1}$ present the same configuration as $c(j+1) = \mathtt{1} = c(\frac{p}{2}+j+1)$ and as the weighted sums are equal. Hence, we can apply the same reasoning given before for c to the colouring $c \circ \mathcal{R}_{j+1}$. It follows that the colouring c is $\frac{p}{2}$-periodic. In particular, we have the following weighted sums

$$\begin{cases} a = \alpha(x+y) + z + t \text{ and } b = (\alpha+1)(x+y) & \text{if } x \neq y \\ a = \beta x + t + z \text{ and } b = (\beta+2)x & \text{if } x = y \end{cases}$$

with β even.

Assume now that $c(\frac{p}{2}+1) = \mathtt{0}$. It follows that $c(\frac{p}{2}) = \mathtt{0}$. Indeed, suppose that $c(\frac{p}{2}) = \mathtt{1}$, i.e., $\alpha_t = 1$. If the weights x and y are distinct, then the weighted sum $a = \alpha_x x + \alpha_y y + t + z$ for the colouring c implies that the weighted sum for the colouring $c \circ \mathcal{R}_{\frac{p}{2}}$ is equal to $a = \alpha_x y + \alpha_y x + t + z$. Hence, $\alpha_x = \alpha_y$ as c is a constant 2-labelling. Then the weighted sum for the colouring $c \circ \mathcal{R}_1$ is given by

$$a = (\alpha_x - 1)y + z + \alpha_x x + y + x = (\alpha_x + 1)x + \alpha_x y + z$$

which is not equal to the initial weighted sum as $t \neq x$. This is a contradiction. Now, if the weights x and y are equal, then the weighted sum $a = (\alpha_x + \alpha_y)x + t + z$ for the colouring c implies that the weighted sum for the colouring $c \circ \mathcal{R}_1$ is equal to

$$a = (\alpha_x + \alpha_y - 1)x + z + x + x = (\alpha_x + \alpha_y + 1)x + z$$

which is a contradiction (as $t \neq x$).

So $c(\frac{p}{2}+1) = \mathtt{0} = c(\frac{p}{2})$ and $\alpha_t = 0$. For the colouring $c \circ \mathcal{R}_1$, we obtain the weighted sum $a = (\alpha_x - 1)y + z + \alpha_y x + x$ as depicted in Figure 16. Hence, α_x must be equal to $\alpha_y + 1$ if the weights x and y are distinct. In this case, we set $\alpha = \alpha_y$. In the case where $x = y$, we simply set $\beta = \alpha_x + \alpha_y$. Hence,

$$\begin{cases} a = (\alpha+1)x + \alpha y + z & \text{if } x \neq y \\ a = \beta x + z & \text{if } x = y. \end{cases}$$

We obtain the following weighted sum (Figure 16) for the colouring $c \circ \mathcal{R}_{\frac{p}{2}}$

$$\begin{cases} b = (\alpha + 1)y + \alpha x + t & \text{if } x \neq y \\ b = \beta x + t & \text{if } x = y. \end{cases}$$

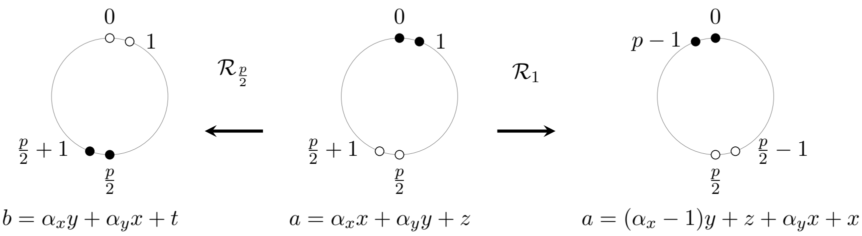

Figure 16. Rotations of the colouring c of a Type 2 cycle with $c(\frac{p}{2}) = \mathtt{0} = c(\frac{p}{2} + 1)$, and their corresponding weighted sums of black vertices.

Let i be the smallest integer in $\{0, \ldots, \frac{p}{2} - 1\}$ such that $c(i+1) = \mathtt{0}$. We may assume that $c\left(\frac{p}{2} + \ell\right) = \mathtt{0}$ for all $\ell \in \{0, \ldots, i\}$. Then $c(\frac{p}{2} + i + 1) = \mathtt{1}$, otherwise we obtain a contradiction as $x \neq t$ (Figure 17).

Observe that if $c(i + 2) = \mathtt{0}$ and $c(\frac{p}{2} + i + 2) = \mathtt{1}$, then the weighted sum b for the colouring $c \circ \mathcal{R}_{i+2}$ is preserved. Therefore, let j be the smallest integer in $\{i+1, \ldots, \frac{p}{2}+i\}$ such that $c(j+1) = \mathtt{1}$. Without loss of generality, we can suppose that $c\left(\frac{p}{2} + \ell\right) = \mathtt{1}$ for all $\ell \in \{i + 1, \ldots, j\}$. It follows that $c\left(\frac{p}{2} + j + 1\right) = \mathtt{0}$. Indeed, $c\left(\frac{p}{2} + j + 1\right) = \mathtt{1}$ leads to a contradiction as $x \neq t$ (Figure 18).

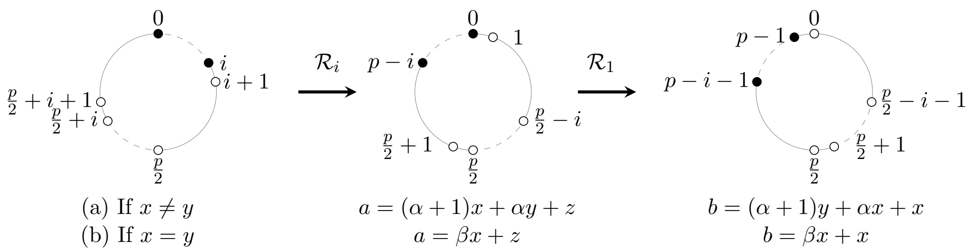

Figure 17. Rotations of the colouring c of a Type 2 cycle and their corresponding weighted sums of black vertices depending on the equality of the weights x and y.

Therefore, the sum of the weights of the black vertices for the colouring $c \circ \mathcal{R}_{j+1}$ is $a = (\alpha + 1)x + \alpha y + z$ (respectively $a = \beta x + z$) if the weights x and y are distinct (respectively equal). Hence, the colourings c and $c \circ \mathcal{R}_{j+1}$ present the same configuration as $c(j + 1) = \mathtt{1}$ and $c(\frac{p}{2} + j + 1) = \mathtt{0}$ and as the weighted sums are equal. It follows that the colouring c is $\frac{p}{2}$-anti-periodic.

If the weights x and y are distinct, then the number of black vertices is equal to $2\alpha + 2 = \frac{p}{2}$. It means that $\frac{p}{2}$ is even which is a contradiction as $p \equiv 2 \pmod 4$. Thus, there doest not exist a constant 2-labelling in this case.

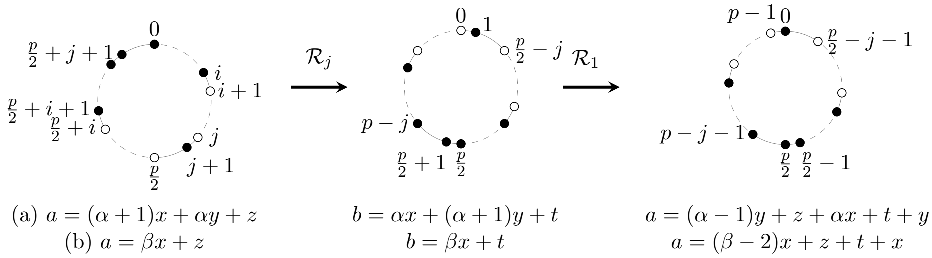

Figure 18. Rotations of the colouring c of a Type 2 cycle with $c(j+1) \neq c(\frac{p}{2}+j+1) = \mathtt{0}$, and their corresponding weighted sums of black vertices which are not all equal, where the line (a) corresponds to the case $x \neq y$ and the line (b) to the case $x = y$.

If the weights x and y are equal, then any $\frac{p}{2}$-anti-periodic colouring is a constant 2 labelling with

$$a = \left(\frac{p}{2} - 1\right) x + z \text{ and } b = \left(\frac{p}{2} - 1\right) x + t.$$

■

The last type of cycles is similar to Type 2 cycles. Hence, the proof of the following lemma is similar to the proof of Lemma 11. Details of the proof are available in [16, Appendix B].

Lemma 12. *Let $p \equiv 4 \pmod 4$ with $p > 4$ and let $\mathcal{C}_p$ be a weighted cycle of Type 4 represented by $z(xy)^{\frac{p-4}{4}}xtx(yx)^{\frac{p-4}{4}}$ where the weights x, y, t are not all equal. If c is a non-trivial constant 2-labelling, then c is one of the following colourings*

- *alternate,*
- *$\frac{p}{2}$-anti-periodic,*
- *$\frac{p}{2}$-periodic if $x = y$; $\frac{p}{2}$-periodic and such that the numbers of black vertices of weight x and y are equal when $c(0) = \mathtt{0}$ if $y \neq x$,*
- *a colouring c' satisfying $c'(i) = c'(i+\frac{p}{2}) = \mathtt{1}$ for all even $i \in \{0, \ldots, \frac{p}{2}-1\}$ and $c'(i) \neq c'(i+\frac{p}{2})$ for all odd $i \in \{0, \ldots, \frac{p}{2}-1\}$ (up to a 1-rotation), if $t = \frac{p}{4}x + (1-\frac{p}{4})y$.*

Using all the previous lemmas, we can now prove our main theorem.

Theorem 13. *Let c be a non-trivial constant 2-labelling of a cycle $\mathcal{C}_p$ of Type 0, Type 1, Type 2, Type 3 or Type 4 with $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and $v = 0$. Let $a = \sum_{\{u \in V \mid c \circ \xi(u) = \mathtt{1}\}} w(u)$ and $b = \sum_{\{u \in V \mid c \circ \xi'(u) = \mathtt{1}\}} w(u)$ for $\xi \in A_{\mathtt{1}}, \xi' \in A_{\mathtt{0}}$. Then the possible values of the constants a and b are given in the following table.*

Type	*Value of a*	*Value of b*	*Condition on parameters*
0	$\alpha x + z$	$(\alpha+1)x$	$\alpha \in \{0, \dots, p-2\}$
1	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}-1)x + (\frac{p}{3}+1)y$	$p \equiv 0 \pmod 3$
3	$\frac{p}{3}x + (\frac{p}{3}-1)y + z$	$(\frac{p}{3}+1)x + (\frac{p}{3}-1)y$	$p \equiv 0 \pmod 3$
2	$(\frac{p}{2}-1)y + z$	$(\frac{p}{2}-1)x + t$	
	$\alpha(x+y) + t + z$	$(\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{2}-1\}$
4	$(\frac{p}{2}-2)y + z + t$	$\frac{p}{2}x$	
	$(2\alpha+2)x + 2\alpha y + z + t$	$(2\alpha+2)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{4}-1\}$
	$\frac{p}{4}x + (\frac{p}{4}-1)y + z$	$\frac{p}{4}x + (\frac{p}{4}-1)y + t$	
	$\frac{p}{2}x + (\frac{p}{4}-1)y + z$	$\frac{3p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$
	$(\frac{p}{4}-1)y + z$	$\frac{p}{4}x$	$t = \frac{p}{4}x + (1-\frac{p}{4})y$
	$2\alpha x + t + z$	$2(\alpha+1)x$	$\alpha \in \{0, \dots, \frac{p}{2}-2\}$, $x = y$
	$(\alpha+1)x + \alpha y + z$	$(\alpha+1)(x+y)$	$\alpha \in \{0, \dots, \frac{p}{2}-2\}$, $t = y$

3. Projection and Folding Method

In this section, we present a method that allows us to translate specific colouring problems of the infinite grid in terms of constant 2-labellings of weighted cycles. We first give the instance of such problems. Let t and p be integers and let $\mathbf{t} = (t, 1)$, $\mathbf{p} = (p, 0)$. This particular choice of $\mathbf{t}$ and $\mathbf{p}$ is due to the initial motivation behind this work: to study the values of the constants in (r, a, b)-codes of the infinite grid (see Section 4). A *frame* is a set of vertices of a given shape where one of the vertices plays a special role and therefore is called the *center of the frame*. Let a and b be non-negative integers. We consider the problem of deciding whether there exists a 2-colouring c of the infinite grid such that the colouring is periodic with $c(\mathbf{y}+\mathbf{t}) = c(\mathbf{y}) = c(\mathbf{y}+\mathbf{p})$ for any $\mathbf{y} \in \mathbb{Z}^2$, and that each frame contains

- a black vertices if the center of the frame is black,
- b black vertices if the center of the frame is white.

Clearly, if the frames are the balls of radius r, then the problem is the same as determining if there exists an (r, a, b)-covering code of the infinite grid that is periodic of periods $\mathbf{t}$ and $\mathbf{p}$.

Now, for $\mathbf{t} = (t, 1)$, $\mathbf{p} = (p, 0)$, consider a 2-colouring of $\mathbb{Z}^2$ that is periodic of periods $\mathbf{t}$ and $\mathbf{p}$. Since c is periodic of period $\mathbf{t}$, the colouring of a line is obtained by doing a translation $\mathbf{t} = (t, 1)$ (respectively -$\mathbf{t} = (-t, -1)$) of the colouring of the line below (respectively above). In this case, if we know the colouring of one line and the translation $\mathbf{t}$, then the colouring of the whole grid $\mathbb{Z}^2$ is known.

Projection

Let $\mathbf{y} \in \mathbb{Z}^2$. Using the translation $\mathbf{t} = (t, 1)$, we can project the frame with center $\mathbf{y}$ on the line L containing $\mathbf{y}$. We assume $\mathbf{y} = (0,0)$ to simplify the notation. Let $Trans$ denote the set of all the translated frames of the frame with center $\mathbf{y}$ by a multiple of $\mathbf{t}$. Let $h : L \to \mathbb{N}$ be a map defined by

$$h((i,0)) = \#\{T \in Trans \mid (i,0) \in T\}.$$

The image of the line L by the mapping h, denoted by $h(L)$, is called the *projection* of the frame with center $\mathbf{y}$ with translation $\mathbf{t} = (t, 1)$. An example is given in the following section. Observe that $h((i,0))$ is a finite number for any $i \in \mathbb{N}$, h takes a non-zero value only finitely many times and the number of vertices of a frame is equal to $\sum_{i\in\mathbb{Z}} h((i,0))$. The map h is introduced to count the number of occurrences in the frame with center $\mathbf{y}$ of vertices of L, up to translation $\mathbf{t}$.

Folding

Using the translation $(p, 0)$, we can fold a projection on a cycle of p weighted vertices. Let L be the line containing $\mathbf{y} = (0,0)$ and $\{0, \ldots, p-1\}$ be the set of vertices of the cycle $\mathcal{C}_p$. We define a map $w : \{0, \ldots, p-1\} \to \mathbb{N}$ such that, for $i \in \{0, \ldots, p-1\}$,

$$w(i) := \sum_{k\in\mathbb{Z}} h((i + kp, 0)).$$

The *folding* of the projection $h(L)$ is the cycle $\mathcal{C}_p$ with vertices $0, \ldots, p-1$ of respective weights $w(0), \ldots, w(p-1)$.

4. Application to (r, a, b)-Codes of $\mathbb{Z}^2$

The projection and folding method can be used to find (r, a, b)-codes that are periodic. We first give an example, then we characterize the values of a and b of any (r, a, b)-code with $r \geq 2$ and $|a - b| > 4$.

Example 14. Consider frames that are balls of radius $r = 3$ and set $t = 2$, $p = 4$. Using the projection and folding method, there exists an (r, a, b)-code of the infinite grid that is periodic of periods $\mathbf{t} = (2, 1)$ and $\mathbf{p} = (4, 0)$ if and only if there exists a constant 2-labelling of the cycle $\mathcal{C}_4$ with weights $w(0) = 7, w(1) = w(2) = w(3) = 6$. The process of projecting and folding is depicted in the three first stages of Figure 19. If there is an (r, a, b)-code of the infinite grid that is periodic of periods $\mathbf{t} = (2, 1)$ and $\mathbf{p} = (4, 0)$, then we set the colours of the line obtained after projection as the colours of the line going through $(0,0)$ and we set the colours of the cycle as the colours of the vertices $(0,0)$ to $(3,0)$. The weight of a given vertex in the cycle corresponds to the number of occurrences of this vertex in the infinite grid up to translation $\mathbf{t}$ and $\mathbf{p}$. Hence, this colouring is a constant 2-labelling of the weighted cycle. The last three stages of Figure 19 represent the process of unfolding the cycle into a line using the translation $\mathbf{p} = (4, 0)$ and expanding the line to the infinite grid using the translation $\mathbf{t} = (2, 1)$. For instance, the colouring c defined by $c(0) = c(1) = \mathtt{1}$ and $c(2) = c(3) = \mathtt{0}$ is a constant 2-labelling and corresponds to an $(3, 13, 12)$-code of $\mathbb{Z}^2$, which is given at the bottom of Figure 19.

Let $r \geq 2$ and $a, b \in \mathbb{N}$ such that $|a - b| > 4$. Let c be an (r, a, b)-code of $\mathbb{Z}^2$. By Theorem 2, c is a diagonal colouring. Hence, c is determined by the colouring of any horizontal line, e.g. $\{(x_1, 0) \mid x_1 \in \mathbb{Z}\}$, and by the orientation of the monochromatic diagonals in the even and odd sublattices.

Assume first that the monochromatic diagonals are all parallel. Without loss of generality, we can suppose that they are of the type $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ with $c \in \mathbb{Z}$. Indeed, the case where the monochromatic diagonals are of type $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$ is similar since the grid is symmetric. In this case, if the colouring of a line of $\mathbb{Z}^2$ is known, then the colouring of the line above (respectively below) is obtained by doing a translation $\mathbf{t} = (1, 1)$ (respectively $-\mathbf{t}$) as $c(\mathbf{x}) = c(\mathbf{x} + \mathbf{t})$ for all $\mathbf{x} \in \mathbb{Z}^2$. So we can apply the projection method. Moreover, by Theorem 2, c is such that $c(\mathbf{x} + (m, 0)) = c(\mathbf{x})$ for some $m \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{Z}^2$. Hence, it is possible to apply the folding method.

Now assume that the monochromatic diagonals are not parallel. We may suppose that the even (respectively odd) sublattice is the union of monochromatic diagonals of type $\{(x_1, x_1 + c) \mid x_1 \in \mathbb{Z}\}$ (respectively $\{(x_1, -x_1 + c) \mid x_1 \in \mathbb{Z}\}$) with $c \in \mathbb{Z}$. We consider an r-ball $B_r(\mathbf{y})$ with center $\mathbf{y}$. Observe that a diagonal intersecting the ball contains either r or $r + 1$ elements of the ball. Moreover, two intersecting diagonals belong to the same sublattice. Hence, in terms of counting vertices of a particular colour appearing in the ball, it is equivalent to consider monochromatic diagonals that are parallel or not. So, we can apply the folding method in both cases.

Therefore, for $r \geq 2$ and $|a - b| > 4$, there exists an (r, a, b)-code of the infinite grid $\mathbb{Z}^2$ if and only if there exists a constant 2-labelling of some cycle $\mathcal{C}_p$, with $v = 0$, $A = \{\mathcal{R}_k \mid k \in \mathbb{Z}\}$ and the mapping w defined as before, such that

$$a = \sum_{\{u \in V \mid c \circ \xi(u) = \mathtt{1}\}} w(u) \text{ and } b = \sum_{\{u \in V \mid c \circ \xi'(u) = \mathtt{1}\}} w(u) \quad \text{for all } \ \xi \in A_{\mathtt{1}},\ \xi' \in A_{\mathtt{0}}.$$

Theorem 15. *Let $r, a, b \in \mathbb{N}$ such that $|a - b| > 4$ and $r \geq 2$. If there exists an (r, a, b)-code of $\mathbb{Z}^2$, then the values of a and b are given in the following table.*

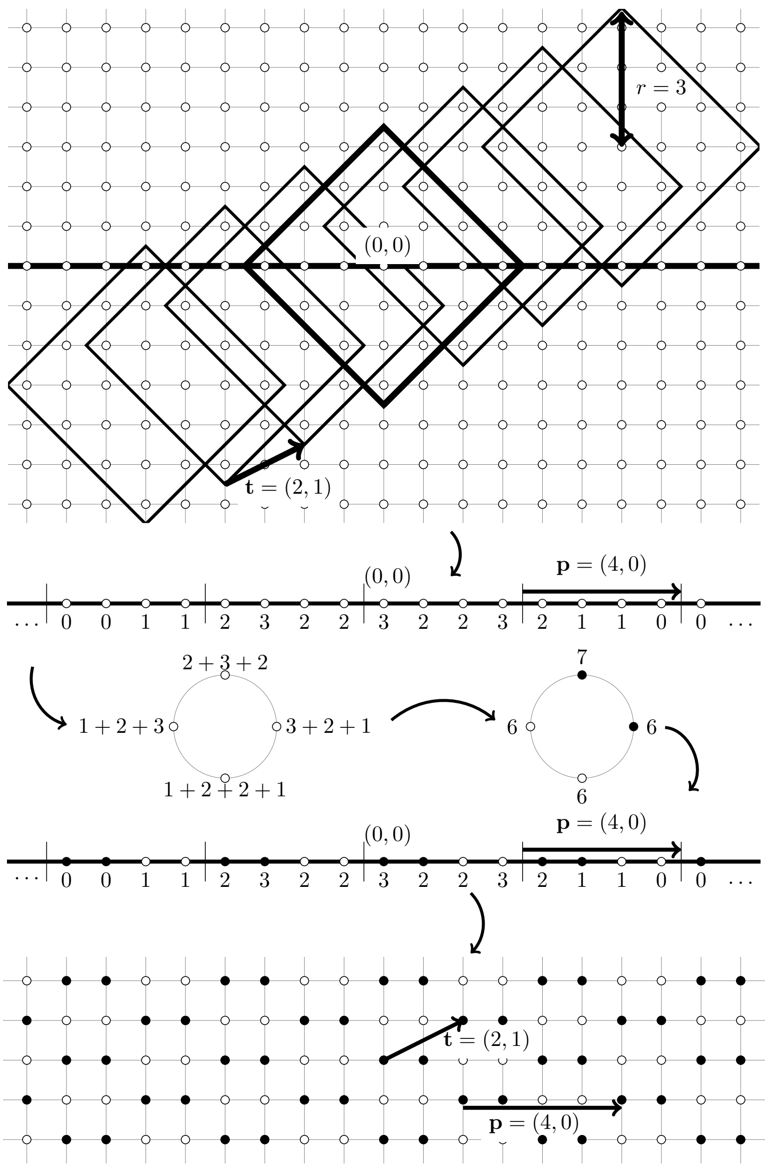

Figure 19. Projection and folding of a ball of radius 3 with respect to the translations $\mathbf{t} = (2, 1)$ and $\mathbf{p} = (4, 0)$.

a	b	*Condition on parameters*
$r+1+\alpha(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-1\}$, $r \equiv 0 \pmod 2$
$(r+1)^2-\alpha(\frac{3r}{2}+1)$	$r^2+\alpha(\frac{3r}{2}+1)$	$\alpha \in \{0,1\}$, $r \equiv 0 \pmod 2$
$r+1+(\alpha+1)(2r+1)$	$(\alpha+1)(2r+1)$	$\alpha \in \{0,\dots,r-2\}$, $r \equiv 1 \pmod 2$
$r^2+\alpha\frac{3r+1}{2}$	$(r+1)^2-\alpha\frac{3r+1}{2}$	$\alpha \in \{0,1\}$, $r \equiv 1 \pmod 2$
$(\alpha+1)\frac{2r^2+2r+2}{3}-1$	$(\alpha+1)\frac{2r^2+2r+2}{3}$	$\alpha \in \{0,1\}$, $r \equiv 1 \pmod 3$
$(\alpha+1)\frac{2r^2+2r}{3}-\frac{r+1}{3}+1$	$(\alpha+1)\frac{2r^2+2r}{3}+\frac{r+1}{3}$	$\alpha \in \{0,1\}$, $r \equiv 2 \pmod 3$
$(\alpha+1)\frac{2r^2+2r}{3}+\frac{r}{3}-1$	$(\alpha+1)\frac{2r^2+2r}{3}-\frac{r}{3}$	$\alpha \in \{0,1\}$, $r \equiv 0 \pmod 3$

Proof. For $r \geq 2$ and $|a-b| > 4$, Axenovich described all possible (r,a,b)-codes (see Theorem 2) in terms of diagonal colourings. Theorem 2 allows us to apply the projection and folding method in this case. Let $\mathbf{y} = (0,0)$. We project the ball $B_r(\mathbf{y})$ on the line L using the translation $\mathbf{t} = (1,1)$ and we obtain for an even radius r

$$h((i,0)) = \begin{cases} r & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ r+1 & \text{if } i \leq r \text{ and } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

and for an odd radius r

$$h((i,0)) = \begin{cases} r+1 & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ r & \text{if } i \leq r \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if r is even, then any diagonal of the even (respectively odd) sublattice intersecting the ball contains $r+1$ (respectively r) elements of $B_r(\mathbf{y})$. The other case can be treated similarly.

Consider now the colourings 1–5 given in Theorem 2. For each kind of colouring, we fold the projection of $B_r(\mathbf{y})$ on a cycle $\mathcal{C}_p$, with $p \in \{2,3,r,r+1,2r,2r+1,2r+2\}$, according to the parity of r (see Table 1). Then we use Theorem 13 to give the possible values of the constant weighted sums a and b.

The colouring 1 is p-periodic of odd period $p \in \{r, r+1\}$. Hence it gives two different weighted cycles. If r is even, then $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{r+1}$ of Type 0 with $z = r+1$ and $x = 2r+1$. The corresponding values of the constants are then

$$a = r+1+\alpha(2r+1) \text{ and } b = (\alpha+1)(2r+1)$$

with $\alpha \in \{0,\dots,r-1\}$. If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_r$ of Type 0 with $z = 3r+2$ and $x = 2r+1$. So the corresponding values of the

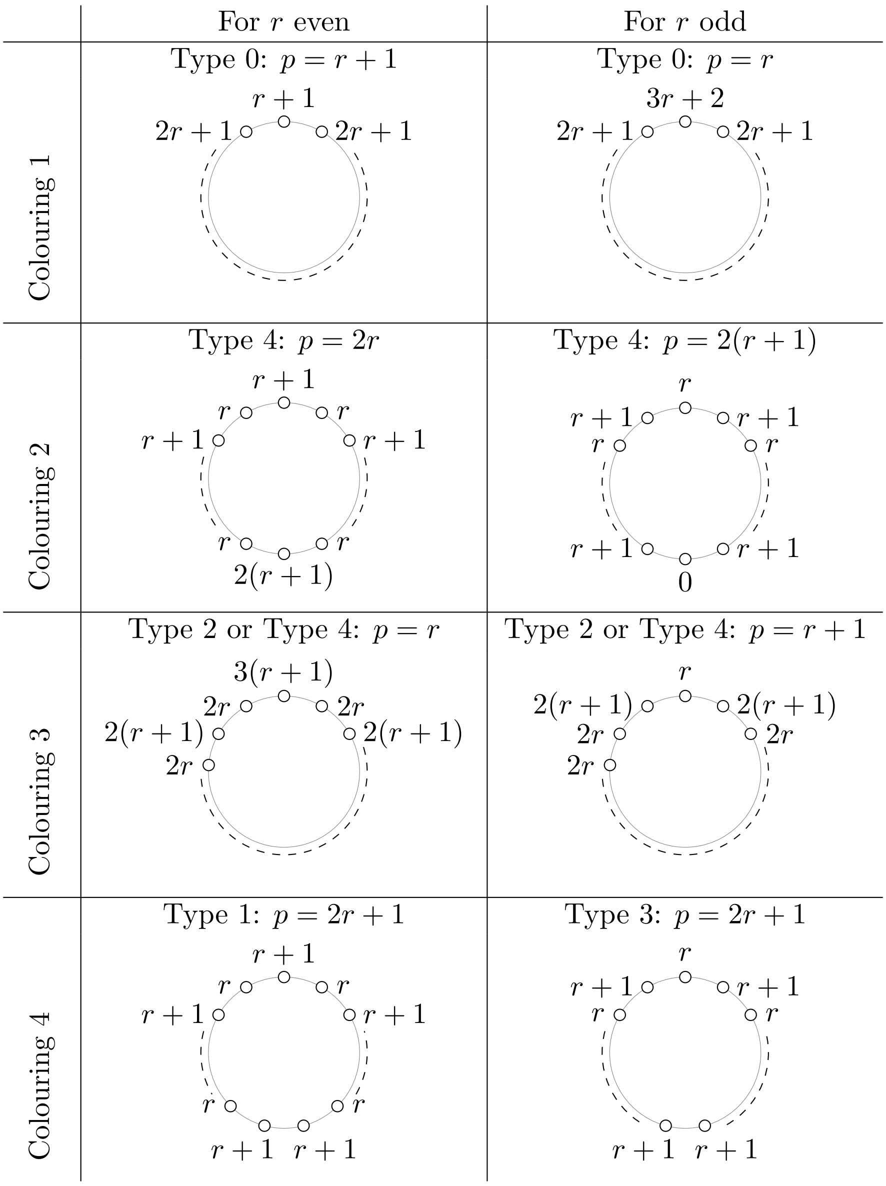

Table 1. Weighted cycles $\mathcal{C}_p$ corresponding to the colourings 1–4.

constants are

$$a = 3r + 2 + \alpha(2r + 1) \text{ and } b = (\alpha + 1)(2r + 1)$$

with $\alpha \in \{0, \ldots, r - 2\}$.

The colouring 2 is a p-anti-periodic colouring with $p \in \{r, r + 1\}$ and p even. It gives then two different weighted cycles with $2p$ vertices. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r}$ of Type 4 with $z = r + 1 = y$, $x = r$ and $t = 2r + 2$. Then the corresponding values of the constants are

$$a = \frac{2r}{4}r + \left(\frac{2r}{4} - 1\right)(r + 1) + r + 1 = (r + 1)^2 + \left(\frac{3r}{2} + 1\right),$$
$$b = \frac{2r}{4}r + \left(\frac{2r}{4} - 1\right)(r + 1) + 2(r + 1) = r^2 + \left(\frac{3r}{2} + 1\right).$$

If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+2}$ of Type 4 with $z = r = y$, $x = r + 1$ and $t = 0$. The corresponding values are

$$a = \frac{2r + 2}{4}(r + 1) + \frac{2r + 2}{4}r = r^2 + \frac{3r + 1}{2},$$
$$b = \frac{2r + 2}{4}(r + 1) + \left(\frac{2r + 2}{4} - 1\right)r = (r + 1)^2 + \frac{3r + 1}{2}.$$

The colouring 3 is p-periodic of period $p \in \{r, r+1\}$ with p even. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_r$. This cycle is a particular case of a Type 2 with $t = x$ or of a Type 4 with $t = y$, according to the value of r mod 4. So $\mathcal{C}_r$ is represented by $z(xy)^{\frac{r-2}{2}}x$ with $z = 3(r + 1), x = 2r$ and $y = 2(r + 1)$. The corresponding values of the constants are either $a = (r + 1)^2$ and $b = r^2$, or

$$a = 2(\alpha + 1)r + 2\alpha(r + 1) + 3(r + 1) \text{ and } b = 2(\alpha + 1)(2r + 1)$$

with $\alpha \in \{0, \ldots, \frac{r}{2} - 2\}$. Similarly, if r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{r+1}$ which is a particular case of a Type 2 or a Type 4 cycle, represented by $z(xy)^{\frac{r-1}{2}}x$ with $z = r, x = 2(r + 1)$ and $y = 2r$. The corresponding values of the constants are either $a = r^2$ and $b = (r + 1)^2$ or

$$a = 2(\alpha + 1)(r + 1) + 2\alpha r + r \text{ and } b = 2(\alpha + 1)(2r + 1)$$

with $\alpha \in \{0, \ldots, \frac{r-1}{2} - 1\}$.

The colouring 4 is $2r + 1$-periodic. If r is even, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+1}$ of Type 1 with $z = r + 1 = y$ and $x = r$. Such weighted cycle has a constant 2-labelling if $2r + 1 \equiv 0 \pmod 3$. Then the corresponding values of the constants are

$$a = \frac{2r+1}{3}r + \left(\frac{2r+1}{3} - 1\right)(r+1) + r + 1 = \frac{(2r+1)^2}{3},$$
$$b = \left(\frac{2r+1}{3} - 1\right)r + \left(\frac{2r+1}{3} + 1\right)(r+1) = \frac{(2r+1)^2}{3} + 1.$$

If r is odd, $B_r(\mathbf{y})$ is projected and folded on the cycle $\mathcal{C}_{2r+1}$ of Type 3 with $z = r = y$ and $x = r+1$. Hence, under the condition that $2r+1 \equiv 0 \pmod 3$, the corresponding values of the constants are

$$a = \frac{2r+1}{3}(r+1) + \left(\frac{2r+1}{3} - 1\right)r + r + 1 = \frac{(2r+1)^2}{3} + 1,$$
$$b = \left(\frac{2r+1}{3} + 1\right)(r+1) + \left(\frac{2r+1}{3} - 1\right)r = \frac{(2r+1)^2}{3} + 1.$$

Since the difference $|a-b| \leq 4$ in this case, we never obtain these values of a and b.

The colouring 5 is either 2-periodic or 3-periodic. Hence it gives five different weighted cycles. Let c be the colouring 5. If c is 2-periodic, then $B_r(\mathbf{y})$ is projected and folded on $\mathcal{C}_2$ of Type 0, represented by zx with

$$\begin{cases} z = (r+1)^2, x = r^2 & \text{for } r \text{ even} \\ z = r^2, x = (r+1)^2 & \text{for } r \text{ odd.} \end{cases}$$

So the corresponding values of the constants are $a = (r+1)^2$ and $b = r^2$ for r even, and $a = r^2, b = (r+1)^2$ for r odd. If c is 3-periodic, then $B_r(\mathbf{y})$ is projected and folded on $\mathcal{C}_3$ of Type 0. In that case, straightforward analysis give the weights z and x:

- $z = \frac{2r^2+2r-1}{3}$ and $x = \frac{2r^2+2r+2}{3}$ if $r = 3k+1$,
- $z = \frac{2r^2+2r}{3} - 2k + 1$ and $x = \frac{2r^2+2r}{3} + k$ if $r = 3k-1$,
- $z = \frac{2r^2+2r}{3} + 2k + 1$ and $x = \frac{2r^2+2r}{3} - k$ if $r = 3k$.

The corresponding values of the constants are then given by

$$a = \alpha x + z \text{ and } b = (\alpha+1)x \text{ with } \alpha \in \{0,1\}.$$

This concludes the proof of Theorem 15. ■

5. Conclusions and Perspectives

Constant 2-labellings in weighted cycles allow us to translate the periodicity of (r,a,b)-codes, with $r \geq 2$, of the 2-dimensional grid. Theorem 1 shows that many

$(1, a, b)$-codes of the multidimensional grid $\mathbb{Z}^d$ are periodic. It would be interesting to find the corresponding weighted graphs obtained with our projection and folding method and then to study constant 2-labellings in these graphs. Also, the projection and folding method is presented in general and can be applied to linear codes. It would be interesting to consider (r, a, b)-codes in other types of lattices as for example, in the king lattice.

The problem of finding a constant 2-labelling of a graph is interesting in and of itself. In Theorem 13, we only obtain a characterization of constant 2-labellings in four types of weighted cycles. It would be interesting to consider different weighted cycles. Moreover, we could study constant 2-labelling in graphs having a big automorphisms group, for instance, in circulant graphs or in vertex-transitive graphs. Finally, we could find a natural generalization of constant 2-labellings into constant k-labellings using k colours and then consider their links with distinguishing numbers and weighted codes with more than two values.

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