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RAMSEY PROPERTIES OF RANDOM GRAPHS AND FOLKMAN NUMBERS

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Abstract

For two graphs, G and F, and an integer $r \geq 2$ we write $G \to (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. In 1995, the first two authors established a threshold edge probability for the Ramsey property $G(n,p) \to (F)_r$, where $G(n,p)$ is a random graph obtained by including each edge of the complete graph on n vertices, independently, with probability p. The original proof was based on the regularity lemma of Szemerédi and this led to tower-type dependencies between the involved parameters. Here, for $r = 2$, we provide a self-contained proof of a quantitative version of the Ramsey threshold theorem with only double exponential dependencies between the constants. As a corollary we obtain a double exponential upper bound on the 2-color Folkman numbers. By a different proof technique, a similar result was obtained independently by Conlon and Gowers.

Keywords: Ramsey property, random graph, Folkman number .

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1. Introduction

For two graphs, G and F, and an integer $r \geq 2$ we write $G \to (F)_r$ if every r-coloring of the edges of G results in a monochromatic copy of F. By a copy we mean here a subgraph of G isomorphic to F. Let $G(n,p)$ be the binomial random graph, where each of $\binom{n}{2}$ possible edges is present, independently, with probability p. In [6] the first two authors established a threshold edge probability for the Ramsey property $G(n,p) \to (F)_r$.

For a graph F, let v_F and e_F stand for, respectively, the number of vertices and edges of F. Assuming $e_F \geq 1$, define

$$d_F = \begin{cases} \frac{e_F - 1}{v_F - 2} & \text{if} \quad e_F > 1 \\ \frac{1}{2} & \text{if} \quad e_F = 1 \end{cases}, \tag{1}$$

and

$$m_F = \max\{d_H \colon H \subseteq F \text{ and } e_H \geq 1\}. \tag{2}$$

Let $\Delta(F)$ be the maximum vertex degree in F. Observe that $m_F = \frac{1}{2}$ for every F with $\Delta(F) = 1$, while for every F with $\Delta(F) \geq 2$ we have $m_F \geq 1$. Moreover, for every k-vertex graph F,

$$m_F \leq m_{K_k} = \frac{k+1}{2}.$$

We now state the main result of [6] in a slightly abridged form.

Theorem 1 [6]**.** *For every integer $r \geq 2$ and a graph F with $\Delta(F) \geq 2$ there exists a constant $C_{F,r}$ such that if $p = p(n) \geq C_{F,r} n^{-1/m_F}$ then*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)_r) = 1.$$

The original proof of Theorem 1 was based on the regularity lemma of Szemerédi [10] and this led to tower-type dependencies on the involved parameters. In [7] it was noticed that for *two* colors the usage of the regularity lemma could be replaced by a simple Ramsey-type argument. Here we follow that thread and for $r = 2$ prove a quantitative version of Theorem 1 with only double exponential dependencies between the constants.

In order to state the result, we first define inductively four parameters indexed by the number of edges of a k-vertex graph F. For fixed $k \geq 3$ we set

$$a_1 = \frac{1}{2}, \quad b_1 = \frac{1}{8}, \quad C_1 = 1, \quad \text{and} \quad n_1 = 1 \tag{3}$$

and for each $i = 1, \dots, \binom{k}{2} - 1$, define

$$(4) \quad a_{i+1} = \frac{a_i^{19k^4}}{2^{55k^6}}, \; b_{i+1} = \frac{a_i^{37k^2}}{2^{118k^4}} b_i^4, \; C_{i+1} = \frac{2^{122k^4}}{b_i^4 a_i^{37k^2}} C_i, \; \text{and} \; n_{i+1} = \frac{2^{14k^3}}{a_i^{4k}} n_i \, .$$

Note that a_i and b_i decrease with i, while C_i and n_i increase. Finally, for a graph F on k vertices, denote by

$$\mu_F = \binom{n}{k} \frac{k!}{\operatorname{aut}(F)} p^{e_F}$$

the expected number of copies of F in $G(n,p)$ and note that

$$(5) \qquad \binom{n}{k} p^{e_F} \le \mu_F \le n^k p^{e_F} = n^{v_F} p^{e_F} .$$

For a real number $\lambda > 0$ we write $G \xrightarrow{\lambda} F$ if every 2-coloring of the edges of G produces at least λ monochromatic copies of F. We call a graph F *k-admissible* if $v_F = k$ and either $e_F = 1$ or $\Delta(F) \ge 2$. Now, we are ready to state a quantitative version of Theorem 1.

Theorem 2. *For every $k \ge 3$, every k-admissible graph F, and for all $n \ge n_{e_F}$ and $p \ge C_{e_F} n^{-1/m_F}$,*

$$\mathbb{P}\left(G(n,p) \xrightarrow{a_{e_F}\mu_F} F \right) \ge 1 - \exp\left(-b_{e_F} p \tbinom{n}{2} \right) .$$

Note that, for $r = 2$, Theorem 1 is an immediate corollary of Theorem 2.

Another consequence of Theorem 2 concerns Folkman numbers. Given an integer $k \ge 3$, the Folkman number $f(k)$ is the smallest integer n for which there exists an n-vertex graph G such that $G \to (K_k)_2$ but $G \not\supset K_{k+1}$. In the special case of $F = K_k$ and $r = 2$, Theorem 2, with $p = C_{\binom{k}{2}} n^{-\frac{2}{k+1}}$, provides a lower bound on $\mathbb{P}(G(n,p) \to (K_k)_2)$. In Section 4, by a standard application of the FKG inequality, we also estimate from below $\mathbb{P}(G(n,p) \not\supset K_{k+1})$, so that the sum of the two probabililities is strictly greater than 1. This, after a careful analysis of the involved constants, provides a self-contained derivation of a double exponential bound for $f(k)$.

Corollary 3. *There exists an absolute constant $c > 0$ such that for every $k \ge 3$*

$$f(k) \le 2^{k^{ck^2}} .$$

Independently, a similar double exponential bound (with arbitrarily many colors) was obtained by Conlon and Gowers [2]. The method used in [2] is quite different from ours and allows for a further generalization to hypergraphs. After Theorem 2 as well as the result in [2] had been proved, we learned that Nenadov and Steger [5] have found a new proof of Theorem 1 by means of the celebrated containers' method (see [1] and [9] for more on that). In [8], we used the ideas from [5] to obtain the bound $f(k) \leq 2^{O(k^4 \log k)}$ which, at least for large k, supersedes Corollary 3. However, the advantage of our approach here is that the proofs of both Theorem 2 and Corollary 3, as opposed to those in [8], are self-contained and, in case of Theorem 2, incorporate the original ideas from [6].

The paper is organized as follows. In Section 3 we prove our main result, Theorem 2. This is preceded by Section 2 collecting preliminary results needed in the main body of the proof. Section 4 is devoted to a proof of Corollary 3.

2. Preliminary Results

Before we start with the proof of Theorem 2, we need to recall abridged versions of two useful facts from [4, Lemmas 2.52 and 2.51] (see also [6, 7]), which we formulate as Propositions 4 and 5 below.

Given a set Γ and a real number p, $0 \leq p \leq 1$, let Γ_p be the random binomial subset of Γ, that is, a subset obtained by independently including each element of Γ with probability p. Further, given an increasing family $\mathcal{Q}$ of subsets of a set Γ and an integer h, we denote by $\mathcal{Q}_h$ the subfamily of $\mathcal{Q}$ consisting of the sets $A \in \mathcal{Q}$ having the property that all subsets of A with at least $|A| - h$ elements still belong to $\mathcal{Q}$.

Proposition 4. *Let $0 < c < 1$, $\delta = c^2/9$, $Np \geq 72/\delta^2 = 2^3 3^6/c^4$, and $h = \delta Np/2$. Then for every increasing family $\mathcal{Q}$ of subsets of an N-element set Γ the following holds. If*

$$\mathbb{P}\left(\Gamma_{(1-\delta)p} \notin \mathcal{Q}\right) \leq \exp(-cNp),$$

then

$$\mathbb{P}\left(\Gamma_p \notin \mathcal{Q}_h\right) \leq \exp(-\delta^2 Np/9)\,.$$

Proof. We want to apply [4, Lemma 2.52], which is very similar to Proposition 4. Lemma 2.52 from [4] states that if c and $\delta > 0$ satisfy

$$\delta(3 + \log(1/\delta)) \leq c \tag{6}$$

and

$$\mathbb{P}(\Gamma_{(1-\delta)p} \notin \mathcal{Q}) \leq \exp(-cNp),$$

then

$$\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) \leq 3\sqrt{Np}\exp(-cNp/2) + \exp(-\delta^2 Np/8)\,. \tag{7}$$

To this end we first note that by assumption of Proposition 4 we have $\delta < 1/9$. Since $\sqrt{x}(\log(1/x))$ is increasing for $x \in (0, 1/\mathrm{e}^2]$ it follows for every $\delta \leq 1/9$ that

$$\sqrt{\delta}\log(1/\delta) \leq \frac{\log(9)}{3} \leq 2.$$

Consequently, $\sqrt{\delta}(3 + \log(1/\delta)) \leq 3$ and owing to the assumption $\delta = c^2/9$ this is equivalent to (6). Moreover, since $Np \geq 2^3 3^6/c^4 > (12/c)^2$ we have

$$3\sqrt{Np} \leq \exp(3\sqrt{Np}) \leq \exp(cNp/4)\,.$$

Hence, (7) yields

$$\begin{aligned}\mathbb{P}(\Gamma_p \notin \mathcal{Q}_h) &\leq \exp(-cNp/4) + \exp(-\delta^2 Np/8) \leq 2\exp(-\delta^2 Np/8)\\ &\leq \exp(-\delta^2 Np/8 + 1) \leq \exp(-\delta^2 Np/9),\end{aligned}$$

where the last inequality follows by our assumption $Np \geq 72/\delta^2$. ■

The following result has appeared in [4] as Lemma 2.51. We state it here for $t = 2$ only.

Proposition 5 [4]**.** *Let $\mathcal{S} \subseteq \binom{\Gamma}{s}$, $0 \leq p \leq 1$, and $\lambda = |\mathcal{S}|p^s$. Then for every nonnegative integer h, with probability at least $1 - \exp(-\frac{h}{2s})$, there exists a subset $E_0 \subseteq \Gamma_p$ of size h such that $\Gamma_p \setminus E_0$ contains at most 2λ sets from $\mathcal{S}$.*

In the proof of Theorem 2 we will also use an elementary fact about (ϱ, d)-dense graphs. For constants ϱ and d with $0 < d, \varrho \leq 1$ we call an n-vertex graph Γ (ϱ, d) *-dense* if every induced subgraph on $m \geq \varrho n$ vertices contains at least $d(m^2/2)$ edges. It follows by an easy averaging argument that it suffices to check the above inequality only for $m = \lceil \varrho n \rceil$. Note also that every induced subgraph of a (ϱ, d)-dense n-vertex graph on at least cn vertices is $(\frac{\varrho}{c}, d)$-dense.

It turns out that for a suitable choice of the parameters, (ϱ, d)-dense graphs enjoy a Ramsey-like property. For a two-coloring of (the edges of) Γ we call a sequence of vertices $(v_1, \dots, v_\ell)$ *canonical* if for each $i = 1, \dots, \ell - 1$ all the edges $\{v_i, v_j\}$, for $j > i$ are of the same color.

Proposition 6. *For every $\ell \geq 2$ and $d \in (0, 1)$, if $n \geq 2(4/d)^{\ell-2}$ and $0 < \varrho \leq (d/4)^{\ell-2}/2$, then every two-colored n-vertex (ϱ, d)-dense graph Γ contains at least*

$$f_n(\ell) := \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} n^\ell$$

canonical sequences of length ℓ.

Proof. First, note that as long as $\varrho \le 1/2$ every (ϱ, d)-dense graph contains at least $n/2$ vertices with degrees at least $dn/2$. Indeed, otherwise a set of $m = \lceil (n+1)/2 \rceil$ vertices of degrees smaller than $dn/2$ would induce less than $mdn/4 \le d(m^2/2)$ edges, a contradiction.

We prove Proposition 6 by induction on ℓ. For $\ell = 2$, every ordered pair of adjacent vertices is a canonical sequence and there are at least $2d\binom{n}{2} > f_n(2)$ such pairs if $n \ge 2$. Assume that the proposition is true for some $\ell \ge 2$ and consider an n-vertex (ϱ, d)-dense graph Γ, where $\varrho \le (d/4)^{\ell-1}/2$ and $n \ge 2(4/d)^{\ell-1}$. As observed above, there is a set U of at least $n/2$ vertices with degrees at least $dn/2$. Fix one vertex $u \in U$ and let M_u be a set of at least $dn/4$ neighbors of u connected to u by edges of the same color. Let $\Gamma_u = \Gamma[M_u]$ be the subgraph of Γ induced by the set M_u. Note that Γ_u has $n_u \ge dn/4 \ge 2(4/d)^{\ell-2}$ vertices and is (ϱ_u, d)-dense with $\varrho_u \le (d/4)^{\ell-2}/2$. Hence, by the induction assumption, there are at least

$$f_{n_u}(\ell) \ge \left(\frac{1}{4}\right)^{\binom{\ell+1}{2}} d^{\binom{\ell}{2}} \left(\frac{dn}{4}\right)^{\ell} = \left(\frac{1}{4}\right)^{\binom{\ell}{2}+\ell} d^{\binom{\ell+1}{2}} n^{\ell}$$

canonical sequences of length ℓ in Γ_u. Each of these sequences preceded by the vertex u makes a canonical sequence of length $\ell+1$ in Γ. As there are at least $n/2$ vertices in U, there are at least

$$\frac{n}{2} f_{n_u}(\ell) \ge \left(\frac{1}{4}\right)^{\binom{\ell+2}{2}} d^{\binom{\ell+1}{2}} n^{\ell+1}$$

canonical sequences of length $\ell+1$ in Γ. This completes the inductive proof of Proposition 6. ∎

Corollary 7. *For every $k \ge 2$, every graph F on k vertices, and every $d \in (0,1)$, if $n \ge (4/d)^{2k}$ and $0 < \varrho \le (d/4)^{2k}$, then every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least γn^k monochromatic copies of F, where $\gamma = d^{2k^2} 2^{-5k^2}$.*

Proof. Every canonical sequence $(v_1, \dots, v_{2k-2})$ contains a monochromatic copy of K_k. Indeed, among the vertices $v_1, \dots, v_{2k-3}$, some $k-1$ have the same color on all the "forward" edges. Therefore, these vertices together with vertex v_{2k-2} form a monochromatic copy of K_k. On the other hand, every such copy is contained in no more than $k!\binom{2k-2}{k} n^{k-2} = (2k-2)_k n^{k-2}$ canonical sequences of length $2k-2$. Finally, every copy of K_k contains at least one copy of F, and different copies of K_k contain different copies of F. Consequently, by Proposition 6, every two-colored n-vertex, (ϱ, d)-dense graph Γ contains at least

$$\frac{f_n(2k-2)}{(2k-2)_k n^{k-2}} = \frac{1}{(2k-2)_k} \left(\frac{1}{4}\right)^{\binom{2k-1}{2}} d^{\binom{2k-2}{2}} n^k > \frac{d^{2k^2}}{2^{5k^2}} n^k$$

monochromatic copies of F. ∎

3. Proof of Theorem 2

3.1. Preparations and outline

For given $n \in \mathbb{N}$, $p \in (0,1)$, and a k-vertex graph F we denote by X_F the random variable counting the number of copies of F in $G(n,p)$. We also recall that $\mu_F = \mathbb{E}X_F$.

For fixed $k \geq 3$ we prove Theorem 2 by induction on e_F. We may assume $n \geq k$, as for $n < k$ we have $\mu_F = 0$ and there is nothing to prove.

Base case. Let F_1 be a graph consisting of one edge and $k-2$ isolated vertices. Note that $m_{F_1} = 1/2$ (see (2)) and for every two-coloring of the edges of $G(n,p)$ every copy of F_1 in $G(n,p)$ is monochromatic. Clearly,

$$X_{F_1} = \binom{n-2}{k-2} X_{K_2} \quad \text{and} \quad \mu_{F_1} = \binom{n-2}{k-2}\mu_{K_2} = \binom{n-2}{k-2}\binom{n}{2}p\,.$$

Thus, by Chernoff's bound (see, e.g., [4, ineq. (2.6)]) we have

$$\mathbb{P}\left(X_{F_1} \leq \frac{1}{2}\mu_{F_1}\right) = \mathbb{P}\left(X_{K_2} \leq \frac{1}{2}\mu_{K_2}\right) \leq \exp\left(-\frac{1}{8}\binom{n}{2}p\right),$$

which holds for any values of p and n. Hence, Theorem 2 follows for $F = F_1$ and with the constants $a_1 = 1/2$, $b_1 = 1/8$, and $C_1 = n_1 = 1$ as given in (3).

Inductive step. Given a graph G, an edge f of G and a nonedge e, that is an edge of the complement of G, we denote by $G - f$ a graph obtained from G by removing f, and by $G + f$ a graph obtained by adding e to G. Let F_{i+1} be a graph with $i+1 \geq 2$ edges and maximum degree $\Delta(F_{i+1}) \geq 2$. If $i+1 \geq 3$, then we can remove one edge from F_{i+1} in such a way that the resulting graph F_i still contains at least one vertex of degree at least two, i.e., $\Delta(F_i) \geq 2$. If $i+1 = 2$, the graph $F_{i+1} = F_2$ consists of a path of length two and $k-3$ isolated vertices and removing any of the two edges results in the graph $F_i = F_1$. In either case, we may fix an edge $f \in E(F_{i+1})$ such that the graph $F_i = F_{i+1} - f$ is k-admissible. Hence, we can assume that Theorem 2 holds for F_i and for the constants a_i, b_i, C_i, and n_i inductively defined by (3) and (4).

We have to show that Theorem 2 holds for F_{i+1} and constants a_{i+1}, b_{i+1}, C_{i+1}, and n_{i+1} given in (4). To this end, let $n \geq n_{i+1}$ and $p \geq C_{i+1}n^{-1/m_{F_{i+1}}}$. We will expose the random graph $G(n,p)$ in two independent rounds $G(n,p_{\mathrm{I}})$ and $G(n,p_{\mathrm{II}})$ and have $G(n,p) = G(n,p_{\mathrm{I}}) \cup G(n,p_{\mathrm{II}})$. For that, we will fix p_{I} and p_{II} as follows. First we fix auxiliary constants[4]

[4]The proof requires several auxiliary constants which at first may appear a bit unmotivated. For example, we now define δ_{II}, while δ_{I} is to be defined only later. Both δ's will be used in applications of Proposition 4.

$$(8)\quad d=\frac{a_i^2}{64^{k^2}}\,,\quad \varrho=\left(\frac{d}{4}\right)^{2k}\,,\quad \gamma=\frac{d^{2k^2}}{2^{5k^2}}\,,\quad \delta_{\mathrm{II}}=\frac{\gamma^4}{9\cdot 16^{k^2}}\,,\quad \text{and}\quad \alpha=\frac{\delta_{\mathrm{II}}^2\gamma}{36}\,.$$

Then p_{I} and $p_{\mathrm{II}}\in(0,1)$ are defined by the equations

$$(9)\qquad p=p_{\mathrm{I}}+p_{\mathrm{II}}-p_{\mathrm{I}}p_{\mathrm{II}}\qquad\text{and}\qquad p_{\mathrm{I}}=\alpha p_{\mathrm{II}}\,.$$

Clearly, we have

$$(10)\qquad p\geq p_{\mathrm{II}}\geq\frac{p}{2}\geq\alpha p\geq\alpha p_{\mathrm{II}}=p_{\mathrm{I}}\geq\alpha\frac{p}{2}\,.$$

We continue with a short outline of the main ideas of the forthcoming proof.

Outline. First we consider a two-coloring χ, with colors red and blue, of the edges of $G(n,p_{\mathrm{I}})$ (first round). Owing to the induction assumption (Theorem 2 for F_i) we note that with high probability the coloring χ yields many monochromatic copies of F_i. We will say that an unordered pair of vertices $e=\{u,v\}$ is *χ-rich* if $G(n,p_{\mathrm{I}})+e$ possesses "many" (to be defined later) copies of F_{i+1}, in which e plays the role of the edge f and the rest is a monochromatic copy of F_i. Let Γ_χ be an auxiliary graph of all χ-rich pairs. We will show that with 'high' probability (to be specified later), Γ_χ is, in fact, (ϱ,d)-dense for d and ϱ as in (8) (Claim 8).

To this end, note that if the monochromatic copies of F_i were clustered at relatively few pairs, then we might fall short of proving Claim 8. However, we will show that in the random graph $G(n,p_{\mathrm{I}})$ it is unlikely that many copies of F_i share the same pair of vertices. For that, we will consider the distribution of the graphs T consisting of two copies of F_i which share the vertices of a missing edge f (and possibly other vertices). We will show that the number of those copies is of the same order of magnitude as its expectation (Fact 9), and will also require that this holds with high probability. Such a sharp concentration result is known to be false, but Proposition 5 asserts that it can be obtained on the cost of removing a few edges of $G(n,p_{\mathrm{I}})$.

The auxiliary graph Γ_χ is naturally two-colored (by azure and pink), since every χ-rich pair closes either many blue or many red copies of F_i (or both and then we pick the color for that edge, azure or blue, arbitrarily). Consequently, Corollary 7 yields many monochromatic copies of F_{i+1} in Γ_χ and at least half of them are colored, say, pink. That is, there are many copies of F_{i+1} in Γ_χ such that each of their edges closes many red copies of F_i in $G(n,p_{\mathrm{I}})$ under the coloring χ. By Janson's inequality combined with Proposition 4, with high probability, many pink copies will be still present in $\Gamma_\chi\cap G(n,p_{\mathrm{II}})$ (second round) even after a fraction of edges is deleted. Thus, we are facing a 'win-win' scenario. Namely, if an extension of χ colors only few pink edges of $\Gamma_\chi\cap G(n,p_{\mathrm{II}})$ red then, by the above, many copies of F_{i+1} in $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ have to be colored completely blue. Otherwise, many pink edges of $\Gamma_\chi \cap G(n, p_{\mathrm{II}})$ are red, which, by the definition of a pink edge, results in many red copies of F_{i+1} in $G(n,p)$.

Useful estimates. For the verification of several inequalities in the proof, it will be useful to appeal to the following lower bounds for γ, α, and ϱ in terms of powers of a_i and 2. From the definitions in (8), for sufficiently large k, one obtains the following bounds.

$$\begin{aligned} \gamma &= \frac{a_i^{4k^2}}{2^{12k^4+5k^2}} \geq \frac{a_i^{4k^2}}{2^{13k^4}}\,, \\ \alpha &= \frac{a_i^{36k^2}}{3^6 \cdot 2^{108k^4+53k^2+2}} \geq \frac{a_i^{36k^2}}{2^{109k^4}}\,, \\ \varrho &= \frac{a_i^{4k}}{2^{12k^3+4k}} \geq \frac{a_i^{4k}}{2^{13k^3}}\,. \end{aligned} \tag{11}$$

We will also make use of the inequalities

$$np \geq C_{i+1}, \tag{12}$$

valid because $m_{F_{i+1}} \geq 1$, and, for every subgraph H of F_{i+1} with $v_H \geq 3$,

$$n^{v_H-2}p^{e_H-1} \geq C_{i+1}^{e_H-1}, \tag{13}$$

valid because

$$m_{F_{i+1}} \geq d_H = \frac{e_H - 1}{v_H - 2}.$$

Of course, (12) follows from (13), by taking H with $d_H = m_{F_{i+1}}$.

3.2. Details

First round. As outlined above, in the first round we want to show that with high probability the random graph $G(n, p_{\mathrm{I}})$ has the property that for every two-coloring χ the auxiliary graph Γ_χ (defined below) is (ϱ, d)-dense. For that we set

$$\delta_{\mathrm{I}} = \frac{b_i^2}{36} \tag{14}$$

and for a two-coloring χ call a pair $\{u,v\}$ of vertices χ-*rich* if it closes at least

$$\ell = \frac{a_i}{4^{k^2}}(\varrho n)^{k-2}p_{\mathrm{I}}^i \tag{15}$$

monochromatic copies of F_i in $G(n, p_{\mathrm{I}})$ to a copy of F_{i+1}. Then Γ_χ is an auxiliary n-vertex graph with the edge set being the set of χ-rich pairs.

Let $\mathcal{E}$ be the event (defined on $G(n, p_{\mathrm{I}})$) that for every two-coloring χ of $G(n, p_{\mathrm{I}})$ the graph Γ_χ is (ϱ, d)-dense.

Claim 8.

$$\mathbb{P}(\mathcal{E}) \geq 1 - \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2\right).$$

Before giving the proof of Claim 8 we need one more fact. Let $\{T_1, T_2, \ldots, T_t\}$ be the family of all pairwise non-isomorphic graphs which are unions of two copies of F_i, say $F_i' \cup F_i''$, with the property that adding a single edge completes both, F_i' and F_i'' to a copy of F_{i+1}. We will refer to these graphs as *double creatures* (of F_i). Clearly, with some foresight of future applications,

$$t \leq 2^{\binom{2k-2}{2}} \leq 2^{2k^2-4k} \leq \frac{2^{2k^2-1}}{4\binom{k}{2}}. \tag{16}$$

Let X_j be the number of copies of T_j in $G(U, p_{\mathrm{I}})$, $j = 1, \ldots, t$.

Fact 9. For every $j = 1, \ldots, t$

$$\mathbb{E}X_j \leq (\varrho n)^{2k-2} p_{\mathrm{I}}^{2i}.$$

Proof. Let $T := T_j = F_i' \cup F_i''$ be a double creature and set $S = F_i' \cap F_i''$. Then the expected number of copies of T is bounded from above by

$$\mathbb{E}X_T \overset{(5)}{\leq} (\varrho n)^{v_T} p_{\mathrm{I}}^{e_T} = \frac{(\varrho n)^{2k} p_{\mathrm{I}}^{2i}}{(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S}},$$

and it remains to show that

$$(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S} \geq (\varrho n)^2.$$

There is nothing to prove when $v_S = 2$ (and thus $e_S = 0$). Otherwise, pick a pair of vertices f in T such that both, $F_i' + f$ and $F_i'' + f$, are isomorphic to F_{i+1}. Then $J := S + f \subseteq F_{i+1}$. Note that $e_J = e_S + 1$ and $3 \leq v_J = v_S \leq k$. Since $C_{i+1} \geq 2/\alpha$,

$$\begin{aligned}(\varrho n)^{v_S} p_{\mathrm{I}}^{e_S} &\overset{(10)}{\geq} (\varrho n)^{v_J}\left(\frac{\alpha}{2}\right)^{e_S} p^{e_J-1} \overset{(13)}{\geq} \varrho^{v_S-2}\left(\frac{\alpha}{2}\right)^{e_S} C_{i+1}^{e_S}(\varrho n)^2 \\ &\geq \varrho^k \frac{\alpha}{2} C_{i+1}(\varrho n)^2 \overset{(11)}{\geq} \frac{1}{2}\frac{a_i^{4k^2}}{2^{13k^4}}\frac{a_i^{36k^2}}{2^{109k^4}} C_{i+1}(\varrho n)^2 \overset{(4)}{\geq} \frac{2^{13k^4-1}}{b_i^4} C_i(\varrho n)^2 \geq (\varrho n)^2.\end{aligned} \tag{17}$$

■

***Proof of Claim* 8.** Let χ be a two-coloring of $G(n, p_{\mathrm{I}})$. Fix a set $U \subseteq [n]$ with $|U| = \varrho n$ (throughout we assume that ϱn is an integer) and consider the random graph $G(n, p_{\mathrm{I}})$ induced on U

$$G(U, p_{\mathrm{I}}) := G(n, p_{\mathrm{I}})[U].$$

By the induction assumption, if $\varrho n \geq n_i$ and $p_i \geq C_i(\varrho n)^{-1/m_{F_i}}$ then, with high probability, there are many monochromatic copies of F_i in $G(U, p_{\mathrm{I}})$. For technical reasons that will become clear only later, we want to strengthen the above Ramsey property so that it is resilient to deletion of a small fraction of edges. For that we apply the induction assumption to the random graph $G(U,(1-\delta_{\mathrm{I}})p_{\mathrm{I}})$, followed by an application of Proposition 4. We begin by verifying the assumptions of Theorem 2 with respect to F_i and $G(U,(1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. First, note that

$$|U| = \varrho n \geq \varrho n_{i+1} \overset{(11)}{\geq} \frac{a_i^{4k}}{2^{13k^3}} n_{i+1} \overset{(4)}{=} \frac{a_i^{4k}}{2^{13k^3}} \cdot \frac{2^{14k^3}}{a_i^{4k}} n_i = 2^{k^3} n_i \geq n_i\,. \tag{18}$$

It remains to check that

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \geq C_i(\varrho n)^{-1/m_{F_i}}\,. \tag{19}$$

To this end, we simply note that using $\delta_{\mathrm{I}} \leq 1/2$, $\varrho \leq 1$, and $m_{F_{i+1}} \geq \max(1, m_{F_i})$ we have

$$(1-\delta_{\mathrm{I}})p_{\mathrm{I}} \overset{(10)}{\geq} \frac{\alpha p}{4} \geq \frac{\alpha}{4} C_{i+1} \varrho^{1/m_{F_{i+1}}} (\varrho n)^{-1/m_{F_{i+1}}} \geq \frac{\alpha}{4} C_{i+1} \varrho (\varrho n) n^{-1/m_{F_i}}.$$

Furthermore, we have

$$\frac{\alpha\varrho}{4} C_{i+1} \overset{(11)}{\geq} \frac{a_i^{36k^2+4k}}{2^{109k^4+13k^3+2}} \cdot C_{i+1} \overset{(4)}{=} \frac{a_i^{37k^2}}{2^{110k^4}} \cdot \frac{2^{122k^4} C_i}{b_i^4 a_i^{37k^2}} = \frac{2^{12k^2} C_i}{b_i^4} \geq C_i\,. \tag{20}$$

and (19) follows. Thus, we are in position to apply the induction assumption to $G(U,(1-\delta_{\mathrm{I}})p_{\mathrm{I}})$ and F_i. Let

$$\mu := \mu_{F_i}^{\varrho,\delta_{\mathrm{I}}} := \binom{\varrho n}{k} \frac{k!}{\mathrm{aut}(F_i)} ((1-\delta_{\mathrm{I}})p_{\mathrm{I}})^i \geq \frac{1}{4^{k^2}} (\varrho n)^k p_{\mathrm{I}}^i \tag{21}$$

denote the expected number of copies of F_i in $G(U,(1-\delta_{\mathrm{I}})p_{\mathrm{I}})$. By Theorem 2 we infer that

$$\begin{aligned} \mathbb{P}\left(G(U,(1-\delta_{\mathrm{I}})p_{\mathrm{I}}) \xrightarrow{a_i\mu} F_i\right) &\geq 1 - \exp\left(-b_i(1-\delta_{\mathrm{I}})p_{\mathrm{I}}\tbinom{\varrho n}{2}\right) \\ &\geq 1 - \exp\left(-\tfrac{b_i}{2} p_{\mathrm{I}} \tbinom{\varrho n}{2}\right). \end{aligned} \tag{22}$$

Next we head for an application of Proposition 4 with $c = b_i/2$, $\delta = \delta_{\mathrm{I}}$, $N = \binom{\varrho n}{2}$, and p_{I}. Note that, indeed, $\delta_{\mathrm{I}} = b_i^2/36 = c^2/9$ (see (14)). Moreover, using $\varrho n \geq 3$ (see (18)) and (12) we see that

$$p_{\mathrm{I}} \binom{\varrho n}{2} \overset{(10)}{\geq} \frac{\alpha p}{2} \cdot \varrho n \geq \frac{\alpha\varrho}{2} \cdot C_{i+1} \overset{(20)}{\geq} \frac{2^{12k^3+1}}{b_i^4} \geq \frac{72}{\delta_{\mathrm{I}}^2}$$

and the assumptions of Proposition 4 are verified. From (22) we infer by Proposition 4 that with probability at least

$$1-\exp\left(-\frac{\delta_{\mathrm{I}}^2}{9}\binom{\varrho n}{2}p_{\mathrm{I}}\right) \tag{23}$$

$G(U,p_{\mathrm{I}})$ has the property that for every subgraph $G'\subseteq G(U,p_{\mathrm{I}})$ with

$$\left|E(G(U,p_{\mathrm{I}}))\setminus E(G')\right|\le\frac{\delta_{\mathrm{I}}}{2}\binom{\varrho n}{2}p_{\mathrm{I}}$$

we have

$$G'\xrightarrow{a_i\mu}F_i\,. \tag{24}$$

Our goal is to show that, with high probability, any two-coloring χ of $G(U,p_{\mathrm{I}})$ yields at least $d(|U|^2/2)$ χ-rich edges, and ultimately, by repeating this argument for every set $U\subseteq[n]$ with ϱn vertices, that Γ_χ is (ϱ,d)-dense. The above 'robust' Ramsey property (24) means that after applying Proposition 5 to $G(U,p_{\mathrm{I}})$ the resulting subgraph of $G(U,p_{\mathrm{I}})$ will still have the Ramsey property with high probability.

Let Y be the random variable counting the number of double creatures in $G(U,p_{\mathrm{I}})$. It follows from Fact 9 that

$$\mathbb{E}Y\le t(\varrho n)^{2k-2}p_{\mathrm{I}}^{2i}\,. \tag{25}$$

Hence, by Proposition 5, applied for every $j=1,\dots,t$ to the families $\mathcal{S}_j$ of all copies of T_j in $G(U,p_{\mathrm{I}})$ with

$$h_{\mathrm{I}}=\frac{\delta_{\mathrm{I}}}{2t}\binom{\varrho n}{2}p_{\mathrm{I}} \tag{26}$$

we conclude that with probability at least

$$1-\sum_{j=1}^{t}\exp\left(-\frac{h_{\mathrm{I}}}{2e(T_j)}\right)\ge1-t\exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right) \tag{27}$$

there exists a subgraph $G_0\subseteq G(U,p_{\mathrm{I}})$ with $|E(G(U,p_{\mathrm{I}})\setminus E(G_0)|\le th_{\mathrm{I}}$ such that G_0 contains at most $2\mathbb{E}Y$ double creatures. Since

$$th_{\mathrm{I}}\overset{(26)}{=}\frac{\delta_{\mathrm{I}}}{2}\binom{\varrho n}{2}p_{\mathrm{I}},$$

the robust Ramsey property (24) holds with $G'=G_0$.

Recall that a two-coloring χ of $G(n, p_{\mathrm{I}})$ is fixed. For $\{u, v\} \subset U$, let x_{uv} be the number of monochromatic copies of F_i in G_0 which together with the pair $\{u, v\}$ form a copy of F_{i+1}. Owing to (24), we have

$$\sum_{\{u,v\}\in\binom{U}{2}} x_{uv} \geq a_i \mu. \tag{28}$$

By the above application of Proposition 5 we infer that

$$\sum_{\{u,v\}\in\binom{U}{2}} x_{uv}^2 \leq 2 \cdot \binom{k}{2} \cdot |DC(G_0)| \leq 4\binom{k}{2}\mathbb{E}Y \overset{(25),(16)}{\leq} 2^{2k^2-1}(\varrho n)^{2k-2} p_{\mathrm{I}}^{2i}\,, \tag{29}$$

where $DC(G_0)$ is the set of all double creatures in G_0. Recall that $\{u, v\} \in E(\Gamma_\chi)$ if it is χ-rich, which is implied by $x_{uv} \geq \ell$, where ℓ is defined in (15). We want to show that $e(\Gamma_\chi[U]) \geq d(\varrho n)^2/2$. Since $\ell \leq a_i\mu/(\varrho n)^2$ (compare (15) and (21)), it follows from (28) that

$$\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq \ell}} x_{uv} \geq \frac{a_i\mu}{2} \overset{(21)}{\geq} \frac{1}{2} \cdot \frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\,.$$

Squaring the last inequality and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left(\frac{1}{2} \cdot \frac{a_i}{4^{k^2}}(\varrho n)^k p_{\mathrm{I}}^i\right)^2 \leq \left(\sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq \ell}} x_{uv}\right)^2 &\leq\ e(\Gamma_\chi[U]) \sum_{\substack{\{u,v\}\in\binom{U}{2}\\ x_{uv}\geq \ell}} x_{uv}^2 \\ &\overset{(29)}{\leq}\ e(\Gamma_\chi[U]) \cdot 2^{2k^2-1}(\varrho n)^{2k-2} p_{\mathrm{I}}^{2i}\,.\end{aligned}$$

Consequently,

$$e(\Gamma_\chi[U]) \geq \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \geq \frac{a_i^2}{64^{k^2}}(\varrho n)^2/2 \overset{(8)}{=} d(\varrho n)^2/2\,.$$

Summarizing the above, we have shown that if $G(U, p_{\mathrm{I}})$ has the robust Ramsey property for F_i (24) and if the conclusion of Proposition 5 holds for all $j = 1, \ldots, t$, then $e(\Gamma_\chi[U]) \geq d(\varrho n)^2/2$. The probability that at least one of these events fails is at most (see (23) and (27))

$$\exp\left(-\frac{\delta_{\mathrm{I}}^2}{9}\binom{\varrho n}{2}p_{\mathrm{I}}\right) + t\exp\left(-\frac{h_{\mathrm{I}}}{2k^2}\right).$$

Recalling that $t \leq 4^{k^2}$ (see (16)) and the definition of h_{I} in (26), Claim 8 now follows by summing up these probabilities over all choices of $U \subseteq [n]$ with

$|U| = \varrho n$. More precisely, using the union bound and the estimate $\binom{n}{\varrho n} \le 2^n$, we conclude that the probability that there is a coloring χ for which the graph Γ_χ is not (ϱ, d)-dense is

$$\begin{aligned}\mathbb{P}(\neg\mathcal{E}) &\le 2^n \exp\left(-\frac{1}{9}\delta_{\mathrm{I}}^2\binom{\varrho n}{2}p_{\mathrm{I}}\right) + 2^n 4^{k^2}\exp\left(-\frac{1}{k^2 4^{k^2}}\delta_{\mathrm{I}}\binom{\varrho n}{2}p_{\mathrm{I}}\right)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2\right).\end{aligned}$$

□

This ends the analysis of the first round.

Second round. Let $\mathcal{B}$ be the conjunction of $\mathcal{E}$ and the event that $|G(n,p_{\mathrm{I}})| \le n^2 p_{\mathrm{I}}$. In the second round we will condition on the event $\mathcal{B}$ and sum over all two-colorings χ of $G(n,p_{\mathrm{I}})$. Formally, let $\mathcal{A}$ be the (bad) event that there is a two-coloring of the edges of $G(n,p)$ with fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}. (That is, $\neg\mathcal{A}$ is the Ramsey property $G(n,p) \xrightarrow{a_{i+1}\mu_{F_{i+1}}} F_{i+1}$.) Further, given a two-coloring χ of $G(n,p_{\mathrm{I}})$, let $\mathcal{A}_\chi$ be the event that there exists an extension of χ to a coloring $\bar{\chi}$ of $G(n,p)$ yielding altogether fewer than $a_{i+1}\mu_{F_{i+1}}$ monochromatic copies of F_{i+1}.

The following pair of inequalities exhibit the skeleton of our proof of Theorem 2.

$$\mathbb{P}(\mathcal{A}) \le \mathbb{P}(\neg\mathcal{B}) + \sum_{G\in\mathcal{B}} \mathbb{P}\Big(\mathcal{A}|G(n,p_{\mathrm{I}}) = G\Big)\mathbb{P}\Big(G(n,p_{\mathrm{I}}) = G\Big) \tag{30}$$

and

$$\begin{aligned}\mathbb{P}(\mathcal{A}|G(n,p_{\mathrm{I}}) = G) &= \mathbb{P}\left(\bigcup_{\chi}\mathcal{A}_\chi\middle|G(n,p_{\mathrm{I}}) = G\right)\\ &\le 2^{n^2 p_{\mathrm{I}}}\max_{\chi}\mathbb{P}(\mathcal{A}_\chi|G(n,p_{\mathrm{I}}) = G).\end{aligned} \tag{31}$$

By Claim 8 and Chernoff's inequality (see, e.g., [4, ineq. (2.5)])

$$\begin{aligned}\mathbb{P}(\neg\mathcal{B}) &\le \mathbb{P}(\neg\mathcal{E}) + \mathbb{P}\Big(|G(n,p_{\mathrm{I}})| > n^2 p_{\mathrm{I}}\Big)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2\right) + \exp\left(-\frac{1}{3}\binom{n}{2}p_{\mathrm{I}}\right)\\ &\le \exp\left(-\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} + n + 2k^2 + 1\right) =: q_{\mathrm{I}}.\end{aligned} \tag{32}$$

To complete the proof of Theorem 2 it is thus crucial to find an upper bound on $\mathbb{P}(\mathcal{A}_\chi|G(n,p_{\mathrm{I}}) = G)$ which substantially beats the factor $2^{n^2 p_{\mathrm{I}}}$.

Claim 10. *For every $G \in \mathcal{B}$ and every two-coloring χ of G,*

$$\mathbb{P}(\mathcal{A}_\chi | G(n, p_{\mathrm{I}}) = G) \leq \exp\left(-\frac{\delta_{\mathrm{II}}^2 \gamma}{9} n^2 p_{\mathrm{II}}\right).$$

The edges of Γ_χ are naturally two-colored according to the majority color among the monochromatic copies of F_i attached to them. We color an edge of Γ_χ *pink* if it closes at least $\ell/2$ red copies of F_i and we color it *azure* otherwise. Subsequently, we apply Corollary 7 to Γ_χ for F_{i+1} and d (chosen in (8)). Note that in (8) we chose ϱ to facilitate such an application. Moreover, the required lower bound on n is equivalent to $\varrho n \geq 1$ and this follows from (18). Hence, by Corollary 7 and the choice of γ in (8), we may assume without loss of generality, that there are at least $\gamma n^k/2$ pink copies of F_{i+1} in Γ_χ. In particular, all these copies of F_{i+1} consist entirely of edges closing each at least $\ell/2$ red copies of F_i (from the first round). Let us denote by $\mathcal{F}_\chi$ the family of these copies of F_{i+1}, and let $\Gamma_\chi^{\mathrm{pink}}$ be the subgraph of Γ_χ containing the pink edges. Since every edge may belong to at most n^{k-2} copies of F_{i+1}, we have

$$e\big(\Gamma_\chi^{\mathrm{pink}}\big) \geq \frac{(i+1)\cdot|\mathcal{F}_\chi|}{n^{k-2}} \geq \frac{(i+1)\cdot \gamma n^k/2}{n^{k-2}} \geq \gamma n^2 \,. \tag{33}$$

In the proof of Claim 10 we intend to use again Proposition 4, this time with $\Gamma = \Gamma_\chi^{\mathrm{pink}}$ and $\mathcal{Q}$ — the property of containing at least

$$\frac{\gamma}{2^{k^2}} n^k p_{\mathrm{II}}^{i+1} \tag{34}$$

copies of F_{i+1} belonging to $\mathcal{F}_\chi$. For this, however, we need the following fact.

Fact 11. With δ_{II} chosen in (8),

$$\mathbb{P}\big(\big(\Gamma_\chi^{\mathrm{pink}}\big)_{(1-\delta_{\mathrm{II}})p_{\mathrm{II}}} \notin \mathcal{Q}\big) \leq \exp\left(-\frac{\gamma^2}{4^{k^2}} e\big(\Gamma_\chi^{\mathrm{pink}}\big) p_{\mathrm{II}}\right).$$

Proof. Consider a random variable Z counting the number of copies F_{i+1} belonging to $\mathcal{F}_\chi$ which are subgraphs of $G(n, (1-\delta_{\mathrm{II}})p_{\mathrm{II}})$. We have

$$\mathbb{E}Z = |\mathcal{F}_\chi|((1-\delta_{\mathrm{II}})p_{\mathrm{II}})^{i+1} \geq \frac{1}{2}\gamma n^k ((1-\delta_{\mathrm{II}})p_{\mathrm{II}})^{i+1} \geq \frac{1}{2}\cdot\frac{1}{2^{\binom{k}{2}}}\gamma n^k p_{\mathrm{II}}^{i+1}\,, \tag{35}$$

where we used the bound $\delta_{\mathrm{II}} \leq 1/2$.

By Janson's inequality (see, e.g., [4, Theorem 2.14]),

$$\mathbb{P}\big(\big(\Gamma_\chi^{\mathrm{pink}}\big)_{(1-\delta_{\mathrm{II}})p_{\mathrm{II}}} \notin \mathcal{Q}\big) \leq \mathbb{P}\left(Z \leq \frac{1}{2}\mathbb{E}Z\right) \leq \exp\left(-\frac{(\mathbb{E}Z)^2}{8\bar{\Delta}}\right),$$

where

$$\bar{\Delta} = \sum_{F' \in \mathcal{F}_\chi} \sum_{F'' \in \mathcal{F}_\chi} \mathbb{P}(F' \cup F'' \subseteq G(n, (1-\delta_{\mathrm{II}})p_{\mathrm{II}}))\,,$$

with the double sum ranging over all ordered pairs $(F', F'') \in \mathcal{F}_\chi \times \mathcal{F}_\chi$ with $E(F') \cap E(F'') \neq \emptyset$. The quantity $\bar{\Delta}$ can be bounded from above by

$$\bar{\Delta} \leq \sum_{\widetilde{F} \subseteq F_{i+1}} n^{2k - v(\widetilde{F})} p_{\mathrm{II}}^{2(i+1) - e(\widetilde{F})}\,, \tag{36}$$

where the sum is taken over all subgraphs $\widetilde{F}$ of F_{i+1} with at least one edge. If $e(\widetilde{F}) = 1$ then

$$n^{v(\widetilde{F})} p_{\mathrm{II}}^{e(\widetilde{F})} = n^{v(\widetilde{F})} p_{\mathrm{II}} \geq n^2 p_{\mathrm{II}}\,. \tag{37}$$

Otherwise,

$$n^{v(\widetilde{F})} p_{\mathrm{II}}^{e(\widetilde{F})} \geq \frac{n^{v(\widetilde{F})} p^{e(\widetilde{F})}}{2^{e(\widetilde{F})}} \overset{(13)}{\geq} \frac{n^2 p C_{i+1}^{e(\widetilde{F})-1}}{2^{e(\widetilde{F})}} \geq n^2 p \overset{(10)}{\geq} n^2 p_{\mathrm{II}}\,, \tag{38}$$

where we also used the fact that $C_{i+1} \geq 4$ (see (4)). Combining (36) with the bounds (37) and (38) yields

$$\bar{\Delta} \leq 2^{i+1} n^{2k-2} p_{\mathrm{II}}^{2i+1} \leq 2^{\binom{k}{2}} n^{2k-2} p_{\mathrm{II}}^{2i+1}\,.$$

Finally, plugging this estimate for $\bar{\Delta}$ and (35) into Janson's inequality we obtain

$$\mathbb{P}\big((\Gamma_\chi^{\mathrm{pink}})_{(1-\delta_{\mathrm{II}})p_{\mathrm{II}}} \notin \mathcal{Q}\big) \leq \exp\left(-\frac{\gamma^2 n^2 p_{\mathrm{II}}}{32 \cdot 2^{2\binom{k}{2}} \cdot 2^{\binom{k}{2}}}\right) \leq \exp\left(-\frac{\gamma^2}{4^{k^2}} e\big(\Gamma_\chi^{\mathrm{pink}}\big) p_{\mathrm{II}}\right).$$

■

***Proof of Claim* 10.** We plan to apply Proposition 4 with $c = \gamma^2/4^{k^2}$, $\delta_{\mathrm{II}} = \gamma^4/(9 \cdot 16^{k^2})$ (see (8)), $N = e(\Gamma_\chi^{\mathrm{pink}})$, and p_{II}. Therefore, first we have to verify that $e(\Gamma_\chi^{\mathrm{pink}}) p_{\mathrm{II}} \geq 72/\delta_{\mathrm{II}}^2$. Indeed,

$$e\left(\Gamma_\chi^{\mathrm{pink}}\right) \cdot p_{\mathrm{II}} \overset{(10,33)}{\geq} \gamma n^2 \cdot \frac{p}{2} \overset{(12)}{\geq} \frac{\gamma}{2} n C_{i+1} \overset{(4)}{\geq} \frac{\gamma}{2} \cdot \frac{2^{122k^4}}{a_i^{37k^3}} \overset{(11)}{\geq} \frac{72 \cdot 81 \cdot 16^{2k^2}}{\gamma^8} = \frac{72}{\delta_{\mathrm{II}}^2}.$$

Consequently, by Proposition 4, we conclude that with probability at least

$$1 - \exp\left(-\frac{\delta_{\mathrm{II}}^2}{9} e\big(\Gamma_\chi^{\mathrm{pink}}\big) p_{\mathrm{II}}\right) \overset{(33)}{\geq} 1 - \exp\left(-\frac{\delta_{\mathrm{II}}^2 \gamma}{9} n^2 p_{\mathrm{II}}\right), \tag{39}$$

the random graph $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ has the property that for every subgraph $\Gamma' \subseteq (\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ with

$$\left| E\left(\left(\Gamma_\chi^{\text{pink}}\right)_{p_{\text{II}}} \right) \setminus E(\Gamma') \right| \le \frac{\delta_{\text{II}}\gamma}{2} n^2 p_{\text{II}} =: h_{\text{II}} \tag{40}$$

we have $\Gamma' \in \mathcal{Q}$, that is, Γ' contains at least $\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1}$ copies of F_{i+1} belonging to $\mathcal{F}_\chi$ (see (34)).

Consider now an extension $\bar{\chi}$ of the coloring χ from $G(n, p_{\text{I}})$ to $G(n,p)$. If in the coloring $\bar{\chi}$ fewer than h_{II} edges of $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ are colored red, then, by the above consequence of Proposition 4, the blue part of $(\Gamma_\chi^{\text{pink}})_{p_{\text{II}}}$ contains at least

$$\frac{\gamma}{2^{k^2}} n^k p_{\text{II}}^{i+1} \overset{(10)}{\ge} \frac{\gamma}{4^{k^2}} n^k p^{i+1}$$

copies of F_{i+1}. If, on the other hand, more than h_{II} edges of $(\Gamma_\chi^{pink})_{p_{\text{II}}}$ are colored red, then, by the definition of a pink edge, noting that $i \le k^2/2$, at least

$$\begin{aligned} h_{\text{II}} \cdot \frac{\ell}{2} \cdot \frac{1}{i+1} &\overset{(15,40)}{\ge} \frac{\delta_{\text{II}}\gamma}{2} n^2 p_{\text{II}} \cdot \frac{a_i}{4^{k^2} k^2} (\varrho n)^{k-2} p_{\text{I}}^i \\ &\overset{(10)}{\ge} \frac{\delta_{\text{II}}\gamma}{4} n^2 p \cdot \frac{a_i \varrho^k}{4^{k^2} k^2} \left(\frac{\alpha}{2}\right)^i n^{k-2} p^i \\ &\ge \frac{\delta_{\text{II}}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} n^k p^{i+1} \end{aligned}$$

red copies of F_{i+1} arise. Owing to (8), (11), and the choice of a_{i+1} in (4) we have

$$\frac{\gamma}{4^{k^2}} \overset{(11)}{\ge} \frac{a_i^{4k^2}}{2^{13k^4+2k^2}} \overset{(4)}{\ge} a_{i+1}$$

and

$$\frac{\delta_{\text{II}}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} \overset{(8)}{=} \frac{\gamma^5 \varrho^k \alpha^{k^2/2}}{9 \cdot 2^{8k^2}} a_i \overset{(11)}{\ge} \frac{a_i^{18k^4+24k^2}}{2^{55k^6}} \overset{(4)}{\ge} a_{i+1} \,.$$

Therefore, we have shown that with probability as in (39), indeed any extension $\bar{\chi}$ of χ yields at least

$$\min\left(\frac{\gamma}{4^{k^2}} \,, \frac{\delta_{\text{II}}\gamma a_i \varrho^k \alpha^{k^2/2}}{16^{k^2}} n^k p^{i+1} \right) \ge a_{i+1} n^k p^{i+1} \overset{(5)}{\ge} a_{i+1} \mu_{F_{i+1}}$$

monochromatic copies of F_{i+1}. $\square$

The final touch. To finish the proof of Theorem 2 it is left to verify that indeed $\mathbb{P}(\mathcal{A}) \le \exp(-b_{i+1}\binom{n}{2}p)$. The error probability of the first round is (see (32))

$$\mathbb{P}(\neg\mathcal{B}) \le q_{\text{I}}.$$

Turning to the second round, by Claim 10 and (31), for any $G \in \mathcal{B}$,

$$\begin{aligned}\mathbb{P}(\mathcal{A}|G(n,p_{\mathrm{I}})=G) &\le 2^{n^2 p_{\mathrm{I}}}\cdot\exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}}\right)\\ &\le \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}}+n^2p_{\mathrm{I}}\right) =: q_{\mathrm{II}}\,,\end{aligned} \tag{41}$$

and, consequently, by (30),

$$\mathbb{P}(\mathcal{A}) \le q_{\mathrm{I}} + q_{\mathrm{II}}.$$

Below we show (see Fact 12) that q_{I} and q_{II} are each upper bounded by $\exp(-b_{i+1}n^2p)$. Consequently,

$$\begin{aligned}\mathbb{P}(\mathcal{A}) &\le 2\exp\left(-b_{i+1}n^2p\right) \le \exp\left(1-b_{i+1}n^2p\right)\\ &\le \exp\left(-\tfrac{b_{i+1}}{2}n^2p\right) \le \exp\left(-b_{i+1}\tbinom{n}{2}p\right)\end{aligned}$$

because

$$\frac{b_{i+1}}{2}n^2p \overset{(12)}{\ge} \frac{b_{i+1}}{2}C_{i+1}n_{i+1} \overset{(4)}{\ge} C_i n_{i+1} \ge 1\,.$$

Fact 12.

$$\max(q_{\mathrm{I}},q_{\mathrm{II}}) \le \exp\left(-b_{i+1}n^2p\right).$$

Proof. We first bound q_{I}. Since $\varrho n \ge 3$ (see (18)),

$$\frac{\delta_{\mathrm{I}}^2}{16^{k^2}}\binom{\varrho n}{2}p_{\mathrm{I}} \overset{(10)}{\ge} \frac{\delta_{\mathrm{I}}^2\varrho^2\alpha}{16^{k^2}\cdot 6}n^2p \overset{(11,14)}{\ge} \frac{b_i^4a_i^{36k^2+8k}}{6^5\cdot 2^{109k^4+26k^3+4k^2}}np^2 \overset{(4)}{\ge} 2b_{i+1}n^2p$$

while, since $i+1\ge 2$,

$$n+2k^2+1 \le n+n_{i+1} \le 2n \overset{(4)}{\le} b_{i+1}C_{i+1}n \overset{(12)}{\le} b_{i+1}n^2p\,.$$

Consequently,

$$q_{\mathrm{I}} \le \exp\left(-2b_{i+1}n^2p+b_{i+1}n^2p\right) = \exp\left(-b_{i+1}n^2p\right).$$

Now we derive the same upper bound for q_{II}. Since

$$p_{\mathrm{I}} \overset{(10)}{\le} \alpha p \overset{(8)}{=} \frac{\delta_{\mathrm{II}}^2\gamma}{36}p$$

while $p_{\mathrm{II}} \overset{(10)}{\ge} p/2$,

$$q_{\mathrm{II}} = \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{9}n^2p_{\mathrm{II}}+n^2p_{\mathrm{I}}\right) \le \exp\left(-\frac{\delta_{\mathrm{II}}^2\gamma}{36}n^2p\right).$$

Therefore, the required bound follows from

$$\frac{\delta_{\mathrm{II}}^2\gamma}{36} \overset{(8)}{=} \frac{\gamma^9}{36\cdot 81\cdot 16^{2k^2}} \overset{(11)}{\geq} \frac{a_i^{36k^2}}{2^{118k^4}} \overset{(4)}{\geq} b_{i+1}\,.$$

■

This concludes the proof of the inductive step, i.e., the proof of Theorem 2 for F_{i+1}, given it is true for F_i, $i = 1,\dots,\binom{k}{2}-1$. The proof of Theorem 2 is thus completed.

4. PROOF OF COROLLARY 3

In order to deduce Corollary 3 from Theorem 2, we first need to estimate the parameters a_i, b_i, C_i, n_i, $i = 1,\dots,\binom{k}{2}$, defined recursively in (4).

Proposition 13. *There exist positive constants* $c_1, c_2, c_3, c_4 > 0$ *such that for every* $k \geq 3$

$$a_{K_k} \geq 2^{-k^{(c_1\cdot k^2)}},\quad b_{K_k} \geq 2^{-k^{(c_2\cdot k^2)}},\quad C_{K_k} \leq 2^{k^{(c_3\cdot k^2)}},\quad n_{K_k} \leq 2^{k^{(c_4\cdot k^2)}}.$$

Proof. Throughout the proof we assume that $k \geq k_0$ for some sufficiently large constant k_0. Let $x = 19k^4$, $y = 55k^6$, and set $\alpha_i = \log a_i$, $i = 1,\dots,\binom{k}{2}$. Recall that $a_1 = \frac{1}{2}$. The recurrence relation (4) becomes now

$$\alpha_i = x\alpha_{i-1} - y,$$

whose solution can be easily found as

$$\alpha_i = -x^{i-1} - y\frac{x^{i-1}-1}{x-1}$$

(note that $\alpha_1 = -1$). Hence, for all $i = 1,\dots,\binom{k}{2}$, and some constant $c_1 > 0$,

$$-\alpha_i = x^{i-1} + y\frac{x^{i-1}-1}{x-1} \leq k^{c_1\cdot i}. \tag{42}$$

In particular,

$$a_{\binom{k}{2}} \geq 2^{-k^{c_1\cdot\binom{k}{2}}} \geq 2^{-k^{(c_1\cdot k^2)}}.$$

The recurrence relation for the b_i's is more complex. With $u = 37k^2$ and $v = 118k^4$, it reads as

$$b_i = b_{i-1}^4 a_{i-1}^u 2^{-v}.$$

Thus, recalling that $b_1 = \frac{1}{8}$,

$$b_i 8^{4^{i-1}} = \prod_{j=2}^{i} \left(\frac{b_j}{b_{j-1}^4} \right)^{4^{i-j}} = \prod_{j=2}^{i} \left(a_j^u 2^{-v} \right)^{4^{i-j}}.$$

Setting, $\beta_i = \log b_i$, and taking logarithms of both sides and using (42) we obtain, for some constant $c_2 > 0$,

$$\begin{aligned} -\beta_i = 3 \cdot 4^{i-1} + \sum_{j=2}^{i} 4^{i-j} \left(u(-\alpha_j) + v \right) &\leq 4^i + (i-1)4^{i-2} \left(u(-\alpha_i) + v \right) \\ &\leq 4^i \left[1 + i \left(uk^{(c_1 \cdot i)} + v \right) \right] \leq k^{c_2 \cdot i}, \end{aligned} \tag{43}$$

where in the last step above we used estimates $4^i \leq k^{2i}$ and $i \leq k^2$. In particular,

$$b_{\binom{k}{2}} \geq 2^{-k^{\left(c_2 \cdot k^2\right)}}.$$

The recurrence relation for C_i involves not only C_{i-1} and a_{i-1} but also b_{i-1}. Nevertheless, its solution follows the steps of that for b_i. Indeed, we have

$$\frac{C_i}{C_{i-1}} = \frac{2^z}{b_{i-1}^4 a_{i-1}^w},$$

where $z = 122k^4$ and $w = 37k^2$. Recalling that $C_1 = 1$,

$$C_i = \prod_{j=2}^{i} \frac{C_j}{C_{j-1}} = \prod_{j=2}^{i} \frac{2^z}{b_{j-1}^4 a_{j-1}^w}$$

and, consequently, by (42) and (43), for some constant $c_3 > 0$,

$$\begin{aligned} \log C_i \leq (i-1)z + \sum_{j=2}^{i} \left(4(-\beta_j) + w(-\alpha_j) \right) &\leq (i-1) \left(z + 4(-\beta_i) + w(-\alpha_i) \right) \\ &\leq k^2 \left(z + 4k^{(c_2 \cdot i)} + wk^{(c_1 \cdot i)} \right) \leq k^{c_3 \cdot i}. \end{aligned}$$

In particular,

$$C_{\binom{k}{2}} \leq 2^{k^{(c_3 \cdot k^2)}}.$$

Similarly, for some constant $c_4 > 0$,

$$n_i = \prod_{j=2}^{i} \frac{n_j}{n_{j-1}} = \prod_{j=2}^{i} \frac{2^{14k^3}}{a_{j-1}^{4k}} \leq 2^{k^{(c_4 \cdot i)}}$$

and, consequently,

$$n_{\binom{k}{2}} \leq 2^{k^{(c_4 \cdot k^2)}}. \qquad \blacksquare$$

We are going to prove Corollary 3 by the probabilistic method. We will show that for some $c > 0$, every $n \geq 2^{k^{c \cdot k^2}}$, and a suitable function $p = p(n)$, with positive probability, $G(n,p)$ has simultaneously two properties: $G(n,p) \to K_k$ and $G(n,p) \not\supset K_{k+1}$. The following simple lower bound on $\mathbb{P}(G(n,p) \not\supset K_{k+1})$ has been already proved in [8] (see lemma 3 therein). For the sake of completeness we reproduce that short proof here.

Lemma 14. *For all $k, n \geq 3$ and $C > 0$, if $p = Cn^{-2/(k+1)} \leq \frac{1}{2}$ then*

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left(-C^{\binom{k+1}{2}} n\right).$$

Proof. By applying the FKG inequality (see, e.g., [4, Theorem 2.12 and Corollary 2.13]), we obtain the bound

$$\begin{aligned} \mathbb{P}(G(n,p) \not\supset K_{k+1}) &\geq \left(1 - p^{\binom{k+1}{2}}\right)^{\binom{n}{k+1}} \geq \exp\left(-2C^{\binom{k+1}{2}} n^{-k} \tbinom{n}{k+1}\right) \\ &> \exp\left(-C^{\binom{k+1}{2}} n\right), \end{aligned}$$

where we used the inequalities $\binom{n}{k+1} < n^{k+1}/2$ and $1 - x \geq e^{-2x}$ for $0 < x < \frac{1}{2}$. ∎

Now, we are ready to complete the proof of Corollary 3. For convenience, set $\bar{b} = b_{\binom{k}{2}}$, $\bar{C} = C_{\binom{k}{2}}$, and $\bar{n} = n_{\binom{k}{2}}$. Let $n \geq \bar{n}$ and $p = \bar{C} n^{-2/(k+1)}$. By Theorem 2,

$$\mathbb{P}(G(n,p) \to K_k) \geq 1 - \exp\left\{-\bar{b}p\binom{n}{2}\right\}.$$

Let, in addition, $n \geq (2\bar{C})^{(k+1)/2}$. Then, by Lemma 14,

$$\mathbb{P}(G(n,p) \not\supset K_{k+1}) > \exp\left\{-\bar{b}p\binom{n}{2}\right\}$$

and, in turn,

$$\mathbb{P}(G(n,p) \to K_k \text{ and } G(n,p) \not\supset K_{k+1}) > 0.$$

Consequently, for every

$$n \geq n_0 := \max\left(\bar{n}, (2\bar{C})^{(k+1)/2}\right)$$

there exists a graph G with n vertices such that $G \to K_k$ but $G \not\supset K_{k+1}$. Finally, by Proposition 13, there exists $c > 0$ such that $n_0 \leq 2^{k^{c \cdot k^2}}$. This way we have proved that $f(k) \leq n_0 \leq 2^{k^{c \cdot k^2}}$.

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