# THE SIGNED TOTAL ROMAN $k$-DOMATIC NUMBER OF A GRAPH 

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#### Abstract

Let $k \geq 1$ be an integer. A signed total Roman $k$-dominating function on a graph $G$ is a function $f: V(G) \longrightarrow\{-1,1,2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for every $v \in V(G)$, where $N(v)$ is the neighborhood of $v$, and every vertex $u \in V(G)$ for which $f(u)=-1$ is adjacent to at least one vertex $w$ for which $f(w)=2$. A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct signed total Roman $k$-dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_{i}(v) \leq k$ for each $v \in V(G)$, is called a signed total Roman $k$-dominating family (of functions) on $G$. The maximum number of functions in a signed total Roman $k$-dominating family on $G$ is the signed total Roman $k$-domatic number of $G$, denoted by $d_{s t R}^{k}(G)$. In this paper we initiate the study of signed total Roman $k$-domatic numbers in graphs, and we present sharp bounds for $d_{s t R}^{k}(G)$. In particular, we derive some Nordhaus-Gaddum type inequalities. In addition, we determine the signed total Roman $k$-domatic number of some graphs.


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## 1. Terminology and Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [2]. Specifically, let $G$ be a simple graph with vertex set $V=$ $V(G)$ and edge set $E=E(G)$. The order $|V|$ of $G$ is denoted by $n=n(G)$. For every vertex $v \in V$, the open neighborhood $N_{G}(v)=N(v)$ is the set $\{u \in$ $V(G) \mid u v \in E(G)\}$. The degree of a vertex $v \in V$ is $d_{G}(v)=d(v)=|N(v)|$.

The minimum and maximum degree of a graph $G$ are denoted by $\delta=\delta(G)$ and $\Delta=\Delta(G)$, respectively. A graph $G$ is regular or $r$-regular if $\delta(G)=\Delta(G)=r$. The complement of a graph $G$ is denoted by $\bar{G}$. We write $K_{n}$ for the complete graph of order $n, K_{p, p}$ for the complete bipartite graph of order $2 p$, and $C_{n}$ for the cycle of length $n$.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. If $k \geq 1$ is an integer, then the signed total Roman $k$-dominating function (STRkDF) on a graph $G$ is defined in [6] as a function $f: V(G) \longrightarrow$ $\{-1,1,2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u)=-1$ is adjacent to at least one vertex $w$ for which $f(w)=2$. The weight of an STRkDF $f$ is the value $\omega(f)=\sum_{v \in V} f(v)$. The signed total Roman $k$-domination number of a graph $G$, denoted by $\gamma_{s t R}^{k}(G)$, equals the minimum weight of an STRkDF on $G$. A $\gamma_{s t R}^{k}(G)$-function is a signed total Roman $k$-dominating function of $G$ with weight $\gamma_{s t R}^{k}(G)$. The signed total Roman $k$-domination number exists when $\delta(G) \geq \frac{k}{2}$. However, for investigations of the signed total Roman $k$-dominating number it is reasonable to claim that $\delta(G) \geq k$. Thus we assume throughout this paper that $\delta(G) \geq k$. If $k=1$, then we write $\gamma_{s t R}^{1}(G)=\gamma_{s t R}(G)$. This case was introduced and studied in [5].

Wang [8] defined the signed total $k$-dominating function on a graph $G$ as a function $f: V(G) \longrightarrow\{-1,1\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for every vertex $v \in V(G)$. Thus a signed total Roman $k$-dominating function combines the properties of both a Roman dominating function and a signed total $k$-dominating function.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [1]. They have defined the domatic number $d(G)$ of a graph $G$ by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in $G$, is called a domatic partition. The maximum number of classes of a domatic partition of $G$ is the domatic number $d(G)$ of $G$. But Rall has defined a variant of the domatic number of $G$, namely the fractional domatic number of $G$, using functions on $V(G)$. (This was mentioned by Slater and Trees in [4].) Analogous to the fractional domatic number we may define the signed total Roman $k$-domatic number.

A set $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ of distinct signed total Roman $k$-dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_{i}(v) \leq k$ for each $v \in V(G)$, is called a signed total Roman $k$-dominating family (of functions) on $G$. The maximum number of functions in a signed total Roman $k$-dominating family (STRkD family) on $G$ is the signed total Roman $k$-domatic number of $G$, denoted by $d_{s t R}^{k}(G)$. If $k=1$, then we write $d_{s t R}^{1}(G)=d_{s t R}(G)$. This case was introduced and investigated in [7]. The signed total Roman $k$-domatic number is well-defined and $d_{s t R}^{k}(G) \geq 1$ for all graphs $G$ with $\delta(G) \geq k$, since the set consisting of any STRkDF forms an STRkD family on $G$.

Our purpose in this paper is to initiate the study of signed total Roman $k$ domatic numbers in graphs. We first derive basic properties and bounds for the signed total Roman $k$-domatic number of a graph. In particular, we derive the Nordhaus-Gaddum type result

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-1,
$$

and we discuss the equality in this inequality. In addition, we determine the signed total Roman $k$-domatic number of some classes of graphs.

We make use of the following results in this paper.
Proposition 1 ([6]). If $G$ is a graph of order $n$ with $\delta(G) \geq k$, then $\gamma_{s t R}^{k}(G) \leq n$.
Proposition 2 ([6]). If $G$ is a graph of order n with $\delta(G) \geq k+2$, then $\gamma_{s t R}^{k}(G) \leq$ $n-1$.
Proposition 3 ([6]). If $n \geq k+2$, then $\gamma_{s t R}^{k}\left(K_{n}\right)=k+2$.
Proposition 4 ([6]). If $G$ is a $\delta$-regular graph of order $n$ with $\delta \geq k$, then

$$
\gamma_{s t R}^{k}(G) \geq\left\lceil\frac{k n}{\delta}\right\rceil
$$

Proposition 5 ([6]). If $k \geq 1$ and $p \geq k$ are integers, then $\gamma_{s t R}^{k}\left(K_{p, p}\right)=2 k$, with exception of the case that $k=1$ and $p=3$, in which case $\gamma_{s t R}^{1}\left(K_{3,3}\right)=4$.

## 2. Bounds on the Signed Total Roman $k$-Domatic Number

In this section we present basic properties of $d_{s t R}^{k}(G)$ and sharp bounds on the signed total Roman $k$-domatic number of a graph.

Theorem 6. For every graph $G$ with $\delta(G) \geq k$,

$$
d_{s t R}^{k}(G) \leq \delta(G) .
$$

Moreover, if $d_{s t R}^{k}(G)=\delta(G)$, then for each STRkD family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $G$ with $d=d_{s t R}^{k}(G)$ and each vertex $v$ of minimum degree, $\sum_{x \in N(v)} f_{i}(x)=k$ for each function $f_{i}$ and $\sum_{i=1}^{d} f_{i}(x)=k$ for all $x \in N(v)$.
Proof. Let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be an STRkD family on $G$ such that $d=d_{s t R}^{k}(G)$. If $v$ is a vertex of minimum degree $\delta(G)$, then we deduce that

$$
k d \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_{i}(x)=\sum_{x \in N(v)} \sum_{i=1}^{d} f_{i}(x) \leq \sum_{x \in N(v)} k=k \delta(G)
$$

and thus $d_{s t R}^{k}(G) \leq \delta(G)$.

If $d_{s t R}^{k}(G)=\delta(G)$, then the two inequalities occurring in the proof become equalities. Hence for the STRkD family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $G$ and for each vertex $v$ of minimum degree, $\sum_{x \in N(v)} f_{i}(x)=k$ for each function $f_{i}$ and $\sum_{i=1}^{d} f_{i}(x)=k$ for all $x \in N(v)$.

Theorem 7. If $G$ is a graph of order $n$, then

$$
\gamma_{s t R}^{k}(G) \cdot d_{s t R}^{k}(G) \leq k n .
$$

Moreover, if $\gamma_{s t R}^{k}(G) \cdot d_{s t R}^{k}(G)=k n$, then for each STRkD family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $G$ with $d=d_{s t R}^{k}(G)$, each function $f_{i}$ is a $\gamma_{s t R}^{k}(G)$-function and $\sum_{i=1}^{d} f_{i}(v)=k$ for all $v \in V(G)$.

Proof. Let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be an STRkD family on $G$ such that $d=d_{s t R}^{k}(G)$ and let $v \in V(G)$. Then

$$
d \cdot \gamma_{s t R}^{k}(G)=\sum_{i=1}^{d} \gamma_{s t R}^{k}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_{i}(v)=\sum_{v \in V(G)} \sum_{i=1}^{d} f_{i}(v) \leq \sum_{v \in V(G)} k=k n
$$

If $\gamma_{s t R}^{k}(G) \cdot d_{s t R}^{k}(G)=k n$, then the two inequalities occurring in the proof become equalities. Hence for the STRkD family $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ on $G$ and for each $i, \sum_{v \in V(G)} f_{i}(v)=\gamma_{s t R}^{k}(G)$. Thus each function $f_{i}$ is a $\gamma_{s t R}^{k}(G)$-function, and $\sum_{i=1}^{d} f_{i}(v)=k$ for all $v \in V(G)$.
Example 8. If $k \geq 3$ is an integer, then $d_{s t R}^{k}\left(K_{k, k}\right)=k$. In addition, $d_{s t R}^{1}\left(K_{1,1}\right)=$ $d_{s r R}^{2}\left(K_{2,2}\right)=1$.
Proof. Clearly, $d_{s t R}^{1}\left(K_{1,1}\right)=d_{s r R}^{2}\left(K_{2,2}\right)=1$. Let now $k \geq 3$. According to Theorem 6, we have $d_{s t R}^{k}\left(K_{k, k}\right) \leq k$. Let $A=\left\{a_{1}, a_{2}, \ldots, a_{k}\right\}$ and $B=\left\{b_{1}, b_{2}, \ldots, b_{k}\right\}$ be a bipartition of $K_{k, k}$. Now we distinguish three cases.

Case 1. Let $k=3 t$ for an integer $t \geq 1$. Define $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=-1$ for $1 \leq i \leq t$ and $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=2$ for $t+1 \leq i \leq 3 t$. For $2 \leq j \leq k$ and $1 \leq i \leq k$ define $f_{j}\left(a_{i}\right)=f_{j}\left(b_{i}\right)=f_{1}\left(a_{i+j-1}\right)$, where the indices are taken modulo $k$. It is easy to see that $f_{i}$ is a signed total Roman $k$-dominating function of $K_{k, k}$ for $1 \leq i \leq k$ and $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ is a signed total Roman $k$-dominating family on $K_{k, k}$. Hence $d_{s t R}^{k}\left(K_{k, k}\right) \geq k$ and thus $d_{s t R}^{k}\left(K_{k, k}\right)=k$ in this case.

Case 2. Let $k=3 t+1$ for an integer $t \geq 1$. Define $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=-1$ for $1 \leq i \leq t, f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=2$ for $t+1 \leq i \leq 3 t$ and $f_{1}\left(a_{3 t+1}\right)=f_{1}\left(b_{3 t+1}\right)=1$. For $2 \leq j \leq k$ and $1 \leq i \leq k$ define $f_{j}\left(a_{i}\right)=f_{j}\left(b_{i}\right)=f_{1}\left(a_{i+j-1}\right)$, where the indices are taken modulo $k$. It is easy to see that $f_{i}$ is a signed total Roman $k$-dominating function of $K_{k, k}$ for $1 \leq i \leq k$ and $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ is a signed total Roman $k$-dominating family on $K_{k, k}$. Hence $d_{s t R}^{k}\left(K_{k, k}\right) \geq k$ and thus $d_{s t R}^{k}\left(K_{k, k}\right)=k$.

Case 3. Let $k=3 t+2$ for an integer $t \geq 1$. Define $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=-1$ for $1 \leq i \leq t, f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=2$ for $t+1 \leq i \leq 3 t$ and $f_{1}\left(a_{3 t+1}\right)=f_{1}\left(a_{3 t+2}\right)=$ $f_{1}\left(b_{3 t+1}\right)=f_{1}\left(b_{3 t+2}\right)=1$. For $2 \leq j \leq k$ and $1 \leq i \leq k$ define $f_{j}\left(a_{i}\right)=$ $f_{j}\left(b_{i}\right)=f_{1}\left(a_{i+j-1}\right)$, where the indices are taken modulo $k$. It is easy to see that $f_{i}$ is a signed total Roman $k$-dominating function of $K_{k, k}$ for $1 \leq i \leq k$ and $\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ is a signed total Roman $k$-dominating family on $K_{k, k}$. Hence $d_{s t R}^{k}\left(K_{k, k}\right) \geq k$ and thus $d_{s t R}^{k}\left(K_{k, k}\right)=k$.

Example 9. If $k, p$ are integers such that $p \geq k+1 \geq 2$, then $d_{s t R}^{k}\left(K_{p, p}\right)=p$, with exception of the case $k=1$ and $p=3$, in which case $d_{s t R}^{1}\left(K_{3,3}\right)=1$.
Proof. Theorem 6 implies that $d_{s t R}^{k}\left(K_{p, p}\right) \leq p$. Now let $A=\left\{a_{1}, a_{2}, \ldots, a_{p}\right\}$ and $B=\left\{b_{1}, b_{2}, \ldots, b_{p}\right\}$ be a bipartition of $K_{p, p}$. We distinguish two cases.

Case 1. Assume that $p-k$ is odd. Define $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=1$ for $1 \leq i \leq$ $\frac{p+k-3}{2}, f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=-1$ for $\frac{p+k-1}{2} \leq i \leq p-1$ and $f_{1}\left(a_{p}\right)=f_{1}\left(b_{p}\right)=2$. For $2 \leq j \leq p$ and $1 \leq i \leq p$ define $f_{j}\left(a_{i}\right)=f_{j}\left(b_{i}\right)=f_{1}\left(a_{i+j-1}\right)$, where the indices are taken modulo $p$. It is easy to see that $f_{i}$ is a signed total Roman $k$-dominating function of $K_{p, p}$ for $1 \leq i \leq p$ and $\left\{f_{1}, f_{2}, \ldots, f_{p}\right\}$ is a signed total Roman $k$ dominating family on $K_{p, p}$. Hence $d_{s t R}^{k}\left(K_{p, p}\right) \geq p$ and thus $d_{s t R}^{k}\left(K_{p, p}\right)=p$ in this case.

Case 2. Assume that $p-k$ is even. If $k=1$ and $p=3$, then Theorem 7 and Proposition 5 imply that $d_{s t R}^{1}\left(K_{3,3}\right) \leq 3 / 2$ and thus $d_{s t R}^{1}\left(K_{3,3}\right)=1$.

Let now $p+k \geq 6$. Define $f_{1}\left(a_{i}\right)=f_{1}\left(b_{i}\right)=1$ for $1 \leq i \leq \frac{p+k-6}{2}, f_{1}\left(a_{i}\right)=$ $f_{1}\left(b_{i}\right)=-1$ for $\frac{p+k-4}{2} \leq i \leq p-2$ and $f_{1}\left(a_{p-1}\right)=f_{1}\left(a_{p}\right)=f_{1}\left(b_{p-1}\right)=f_{1}\left(b_{p}\right)=2$. For $2 \leq j \leq p$ and $1 \leq i \leq p$ define $f_{j}\left(a_{i}\right)=f_{j}\left(b_{i}\right)=f_{1}\left(a_{i+j-1}\right)$, where the indices are taken modulo $p$. It is easy to see that $f_{i}$ is a signed total Roman $k$-dominating function of $K_{p, p}$ for $1 \leq i \leq p$ and $\left\{f_{1}, f_{2}, \ldots, f_{p}\right\}$ is a signed total Roman $k$ dominating family on $K_{p, p}$. Hence $d_{s t R}^{k}\left(K_{p, p}\right) \geq p$ and thus $d_{s t R}^{k}\left(K_{p, p}\right)=p$.

Examples 8 and 9 demonstrate that Theorem 6 is sharp. If $k \geq 3$, then Example 8 and Proposition 5 show that Theorem 7 is sharp too.

Corollary 10. If $K_{n}$ is the complete graph of order $n$ such that $n \geq k+2$, then

$$
d_{s t R}^{k}\left(K_{n}\right) \leq \frac{k}{k+2} n .
$$

Proof. It follows from Theorem 7 and Proposition 3 that

$$
(k+2) d_{s t R}^{k}\left(K_{n}\right)=\gamma_{s t R}^{k}\left(K_{n}\right) \cdot d_{s t R}^{k}\left(K_{n}\right) \leq k n .
$$

This yields the desired bound immediately.
Example 11. If $n \geq 6$ is an even integer, then $d_{s t R}^{2}\left(K_{n}\right)=\frac{n}{2}$.

Proof. Corollary 10 implies $d_{s t R}^{2}\left(K_{n}\right) \leq \frac{n}{2}$. Let $n=2 p$ for an integer $p \geq 3$, and let $V\left(K_{n}\right)=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be the vertex set of $K_{n}$. For the opposite inequality we distinguish two cases.

Case 1. Assume that $p=2 t+1$ for an integer $t \geq 1$ and thus $n=4 t+2$. Define the function $f_{1}$ by $f_{1}\left(x_{i}\right)=-1$ for $1 \leq i \leq 2 t, f_{1}\left(x_{i}\right)=1$ for $2 t+1 \leq i \leq 4 t$ and $f_{1}\left(x_{4 t+1}\right)=f_{1}\left(x_{4 t+2}\right)=2$. For $2 \leq j \leq 2 t+1$ and $1 \leq i \leq 4 t+2$ define $f_{j}\left(x_{i}\right)=f_{1}\left(x_{i+2 j-2}\right)$, where the indices are taken modulo $n$. It is easy to see that $f_{i}$ is a signed total Roman 2-dominating function of $K_{n}$ for $1 \leq i \leq 2 t+1$ and $\left\{f_{1}, f_{2}, \ldots, f_{2 t+1}\right\}$ is a signed total Roman 2 -dominating family on $K_{n}$. Hence $d_{s t R}^{2}\left(K_{n}\right) \geq 2 t+1=\frac{n}{2}$ and thus $d_{s t R}^{2}\left(K_{n}\right)=\frac{n}{2}$ in this case.

Case 2. Assume that $p=2 t$ for an integer $t \geq 2$ and thus $n=4 t$. Define the functions $f_{1}$ by $f_{1}\left(x_{i}\right)=-1$ for $1 \leq i \leq 2 t, f_{1}\left(x_{i}\right)=1$ for $2 t+1 \leq i \leq 4 t-4$ and $f_{1}\left(x_{i}\right)=2$ for $4 t-3 \leq i \leq 4 t$. For $2 \leq j \leq 2 t$ and $1 \leq i \leq 4 t$ define $f_{j}\left(x_{i}\right)=f_{1}\left(x_{i+2 j-2}\right)$, where the indices are taken modulo $n$. It is easy to see that $f_{i}$ is a signed total Roman 2-dominating function of $K_{n}$ for $1 \leq i \leq 2 t$ and $\left\{f_{1}, f_{2}, \ldots, f_{2 t}\right\}$ is a signed total Roman 2 -dominating family on $K_{n}$. Hence $d_{s t R}^{2}\left(K_{n}\right) \geq 2 t=\frac{n}{2}$ and thus $d_{s t R}^{2}\left(K_{n}\right)=\frac{n}{2}$.

Example 12. If $n \geq 5$, then $d_{s t R}^{n-2}\left(K_{n}\right)=n-2$.
Proof. Corollary 10 implies $d_{s t R}^{n-2}\left(K_{n}\right) \leq n-2$. Let $V\left(K_{n}\right)=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be the vertex set of $K_{n}$. Define the function $f_{1}$ by $f_{1}\left(x_{1}\right)=-1, f_{1}\left(x_{2}\right)=f_{2}\left(x_{3}\right)=2$ and $f_{1}\left(x_{i}\right)=1$ for $4 \leq i \leq n$. For $2 \leq j \leq n-2$ and $1 \leq i \leq n-2$ define $f_{j}\left(x_{i}\right)=$ $f_{1}\left(x_{i+j-1}\right)$, where the indices are taken modulo $n-2$ and $f_{j}\left(x_{n-1}\right)=f_{j}\left(x_{n}\right)=1$. It is easy to see that $f_{i}$ is a signed total Roman ( $n-2$ )-dominating function of $K_{n}$ for $1 \leq i \leq n-2$ and $\left\{f_{1}, f_{2}, \ldots, f_{n-2}\right\}$ is a signed total Roman ( $n-2$ )-dominating family on $K_{n}$. Hence $d_{s t R}^{n-2}\left(K_{n}\right) \geq n-2$ and thus $d_{s t R}^{n-2}\left(K_{n}\right)=n-2$.

These are further examples showing the sharpness of Theorem 7 and Corollary 10. For some regular graphs we will improve the upper bound given in Theorem 6.

Theorem 13. Let $G$ be a $\delta$-regular graph of order $n$ with $\delta \geq \max \{2, k\}$ such that $n=p \delta+r$ with integers $p \geq 1$ and $1 \leq r \leq \delta-1$ and $k r=t \delta+s$ with integers $t \geq 0$ and $1 \leq s \leq \delta-1$. Then $d_{s t R}^{k}(G) \leq \delta-1$.

Proof. Let $\left\{f_{1}, f_{2}, \ldots, f_{d}\right\}$ be an STRkD family on $G$ such that $d=d_{s t R}^{k}(G)$. It follows that

$$
\sum_{i=1}^{d} \omega\left(f_{i}\right)=\sum_{i=1}^{d} \sum_{v \in V(G)} f_{i}(v)=\sum_{v \in V(G)} \sum_{i=1}^{d} f_{i}(v) \leq \sum_{v \in V(G)} k=k n
$$

Proposition 4 implies

$$
\begin{aligned}
\omega\left(f_{i}\right) & \geq \gamma_{s t R}^{k}(G) \geq\left\lceil\frac{k n}{\delta}\right\rceil=\left\lceil\frac{k p \delta+k r}{\delta}\right\rceil=k p+\left\lceil\frac{k r}{\delta}\right\rceil \\
& =k p+\left\lceil\frac{t \delta+s}{\delta}\right\rceil=k p+t+1
\end{aligned}
$$

for each $i \in\{1,2, \ldots, d\}$. If we suppose to the contrary that $d=\delta$, then the above inequality chains lead to the contradiction

$$
\begin{aligned}
k n & \geq \sum_{i=1}^{d} \omega\left(f_{i}\right) \geq d(k p+t+1)=\delta(k p+t+1)=k p \delta+\delta t+\delta \\
& =k p \delta+k r-s+\delta>k p \delta+k r=k(p \delta+r)=k n .
\end{aligned}
$$

Thus $d \leq \delta-1$, and the proof is complete.
Examples 8 and 9 demonstrate that Theorem 13 is not valid in general.
Theorem 14. If $G$ is a graph of order $n$ with $\delta(G) \geq k$, then

$$
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq n+k
$$

Proof. If $d_{s R}^{k}(G) \leq k$, then Proposition 1 implies $\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq n+k$ immediately. Let now $d_{s t R}^{k}(G) \geq k$. It follows from Theorem 7 that

$$
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq \frac{k n}{d_{s t R}^{k}(G)}+d_{s t R}^{k}(G)
$$

According to Theorem 6 , we have $k \leq d_{s t R}^{k}(G) \leq n$. Using these bounds, and the fact that the function $g(x)=x+(k n) / x$ is decreasing for $k \leq x \leq \sqrt{k n}$ and increasing for $\sqrt{k n} \leq x \leq n$, we obtain

$$
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq \frac{k n}{d_{s t R}^{k}(G)}+d_{s t R}^{k}(G) \leq \max \{n+k, k+n\}=n+k
$$

and the desired bound is proved.
Proposition 5 and Example 8 demonstrate that Theorem 14 is sharp for $k \geq 3$. For $\delta(G) \geq k+2$, we will improve the bound in Theorem 14 slightly.

Theorem 15. If $G$ is a graph of order $n$ with $\delta(G) \geq k+2$, then

$$
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq n+k-1
$$

Proof. If $d_{s R}^{k}(G) \leq k$, then Proposition 2 implies $\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq n+k-1$. Let now $d_{s t R}^{k}(G) \geq k+1$. It follows from Theorem 7 that

$$
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) \leq \frac{k n}{d_{s t R}^{k}(G)}+d_{s t R}^{k}(G)
$$

According to Theorem 6 , we have $k+1 \leq d_{s t R}^{k}(G) \leq n-1$. Using these bounds, and the fact that the function $g(x)=x+(k n) / x$ is decreasing for $k+1 \leq x \leq \sqrt{k n}$ and increasing for $\sqrt{k n} \leq x \leq n-1$, we obtain

$$
\begin{aligned}
\gamma_{s t R}^{k}(G)+d_{s t R}^{k}(G) & \leq \frac{k n}{d_{s t R}^{k}(G)}+d_{s t R}^{k}(G) \\
& \leq \max \left\{\frac{k n}{k+1}+k+1, \frac{k n}{n-1}+n-1\right\}<n+k
\end{aligned}
$$

and this leads to the desired bound.
The special case $k=1$ of Theorems 6, 7, 13 and 14 can be found in [7].

## 3. Nordhaus-Gaddum Type Results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their current classical paper [3], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the signed total Roman $k$-domatic number.

Theorem 16. If $G$ is a graph of order $n$ such that $\delta(G), \delta(\bar{G}) \geq k$, then

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-1
$$

Furthermore, if $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G})=n-1$, then $G$ is regular.
Proof. It follows from Theorem 6 that

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq \delta(G)+\delta(\bar{G})=\delta(G)+(n-\Delta(G)-1) \leq n-1
$$

If $G$ is not regular, then $\Delta(G)-\delta(G) \geq 1$, and hence the above inequality chain implies the better bound $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$.

In the special case $k=1$, we have proved the following theorem in [7].
Theorem 17. If $G$ is a graph of order $n$ such that $\delta(G), \delta(\bar{G}) \geq 1$, then

$$
d_{s t R}(G)+d_{s t R}(\bar{G}) \leq n-1
$$

with equality if and only if $G=C_{4}$.

As a supplement to Theorem 17, we prove the next result.
Theorem 18. Let $k \geq 2$ be an integer. Then there is only a finite number of graphs $G$ with $\delta(G), \delta(\bar{G}) \geq k$ such that

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G})=n(G)-1
$$

Proof. Let $n(G)=n, \delta(G)=\delta$ and $\delta(\bar{G})=\bar{\delta}$. The strategy of our proof is as follows. For a fixed integer $k \geq 2$, we show that $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$ or $n \leq 2 k^{3}+5 k^{2}-5 k+1$. Together with Theorem 16 this implies the desired result.

If $G$ is not regular, then it follows from Theorem 16 that $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq$ $n-2$. Assume now that $G$ is $\delta$-regular. Then $\bar{G}$ is $\bar{\delta}$-regular such that $\delta+\bar{\delta}+1=n$. Assume, without loss of generality, that $\bar{\delta} \leq \delta$. We distinguish three cases.

Case 1. Assume that $\bar{\delta}=\delta$. Then $n=2 \delta+1$ and $k=t \delta+s$ with integers $t \geq 0$ and $1 \leq s \leq \delta-1$. If $\delta=k$, then $n=2 k+1 \leq 2 k^{3}+5 k^{2}-5 k+1$. If $\delta \geq k+1$, then $t=0$ and $s=k \leq \delta-1$, and Theorem 13 implies that $d_{s t R}^{k}(G) \leq \delta-1$. Applying now Theorem 6, we conclude that

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq \delta-1+\bar{\delta} \leq n-2
$$

Case 2. Assume that $\bar{\delta}=\delta-1$. Then $n=2 \bar{\delta}+2$. If $\bar{\delta}=2$, then $n=6 \leq$ $3 k \leq 2 k^{3}+5 k^{2}-5 k+1$. If $\bar{\delta} \geq 3$, then let $2 k=t \bar{\delta}+s$ with integers $t \geq 0$ and $0 \leq s \leq \bar{\delta}-1$. Since $\bar{\delta} \geq k$, we observe that $0 \leq t \leq 2$. If $t=2$, then $2 k=2 \bar{\delta}$ and therefore $n=\delta+\bar{\delta}+1=2 k+2 \leq 2 k^{3}+5 k^{2}-5 k+1$. If $t=1$, then $2 k=\bar{\delta}+s$. If $s=0$, then $\bar{\delta}=2 k$ and thus $n=\delta+\bar{\delta}+1=4 k+2 \leq 2 k^{3}+5 k^{2}-5 k+1$. If $s \neq 0$, then accoring to Theorem $13, d_{s t R}^{k}(\bar{G}) \leq \bar{\delta}-1$ and Theorem 6 leads to $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$. If $t=0$, then $2 k=s \leq \bar{\delta}-1$, and again Theorems 6 and 13 yield $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$.

Case 3. Assume that $\bar{\delta} \leq \delta-2$. Then $n=\delta+\bar{\delta}+1=\delta+r$ with $1 \leq r=$ $\bar{\delta}+1 \leq \delta-1$. Let now $k r=k(\bar{\delta}+1)=t \delta+s$ with integers $t \geq 0$ and $0 \leq s \leq \delta-1$. If $s \neq 0$, then we deduce from Theorem 13 that $d_{s t R}^{k}(G) \leq \delta-1$, and Theorem 6 yields to $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$. If $s=0$, then the condition $\bar{\delta} \leq \delta-2$ shows that

$$
\begin{equation*}
k(\bar{\delta}+1)=t \delta \text { with } 1 \leq t \leq k-1 \tag{1}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\delta=\frac{k(\bar{\delta}+1)}{t} \tag{2}
\end{equation*}
$$

Let now
(3) $\quad n=p \bar{\delta}+r$ with integers $p \geq 1$ and $0 \leq r \leq \bar{\delta}-1$
and when $r \neq 0$

$$
\begin{equation*}
k r=a \bar{\delta}+b \text { with integers } a \geq 0 \text { and } 0 \leq b \leq \bar{\delta}-1 \tag{4}
\end{equation*}
$$

If $b, r \neq 0$, then we conclude from Theorem 13 that $d_{s t R}^{k}(\bar{G}) \leq \bar{\delta}-1$, and Theorem 6 implies that $d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2$. Now let $r \neq 0$ and $b=0$. Then (3) and (4) yield to

$$
k r=a \bar{\delta} \text { with } 1 \leq a \leq k-1
$$

and thus

$$
\begin{equation*}
\bar{\delta}=\frac{k r}{a} \tag{5}
\end{equation*}
$$

Using (2) and (3), we obtain

$$
\frac{k(\bar{\delta}+1)}{t}+\bar{\delta}+1=\delta+\bar{\delta}+1=n=p \bar{\delta}+\frac{a \bar{\delta}}{k}
$$

and thus

$$
p \bar{\delta}=\bar{\delta}\left(\frac{k}{t}+1-\frac{a}{k}\right)+\frac{k}{t}+1 \leq \bar{\delta}\left(\frac{k}{t}+2-\frac{a}{k}\right)+1 \leq \bar{\delta}\left(k+2-\frac{a}{k}\right)+1
$$

and so $p \leq k+2$. Combining (2) and (5), we obtain

$$
\delta=\frac{k}{t}\left(\frac{k r}{a}+1\right)
$$

and so

$$
\begin{equation*}
n=\delta+\bar{\delta}+1=\frac{k}{t}\left(\frac{k r}{a}+1\right)+\frac{k r}{a}+1 \tag{6}
\end{equation*}
$$

According to (3) and (5), we have

$$
\begin{equation*}
n=p \bar{\delta}+r=\frac{p k r}{a}+r \tag{7}
\end{equation*}
$$

Combining (6) and (7), we find that

$$
r\left(\frac{p k}{a}+1\right)=\frac{k r}{a}\left(\frac{k}{t}+1\right)+\frac{k}{t}+1
$$

and therefore

$$
\begin{equation*}
1+\frac{k}{t}=r\left(\frac{p k}{a}+1-\frac{k}{a}-\frac{k^{2}}{a t}\right) \tag{8}
\end{equation*}
$$

This equality shows that

$$
\frac{p k}{a}+1-\frac{k}{a}-\frac{k^{2}}{a t}>0
$$

and hence

$$
\frac{p k}{a}+1-\frac{k}{a}-\frac{k^{2}}{a t} \geq \frac{1}{a t}
$$

Using this and (8), we obtain

$$
1+\frac{k}{t} \geq \frac{r}{a t}
$$

and thus

$$
\begin{equation*}
r \leq a(t+k) \leq(k-1)(2 k-1) . \tag{9}
\end{equation*}
$$

In view of (5), it follows that

$$
\bar{\delta}=\frac{k r}{a} \leq \frac{k a(t+k)}{a}=k(t+k) \leq k(2 k-1) .
$$

Applying $p \leq k+2$, (3), (9) and the last inequality, we arrive at the desired bound

$$
n=p \bar{\delta}+r \leq(k+2) k(2 k-1)+(k-1)(2 k-1)=2 k^{3}+5 k^{2}-5 k+1
$$

It remains the case that $r=0$ and thus $n=p \bar{\delta}$ with an integer $p \geq 2$. Since $n=\delta+\bar{\delta}+1$, we deduce that

$$
\delta=(p-1) \bar{\delta}-1
$$

Using this identity and (1), we obtain

$$
\begin{equation*}
k(\bar{\delta}+1)=t \delta=t(p-1) \bar{\delta}-t \tag{10}
\end{equation*}
$$

and thus

$$
k+t=\bar{\delta}(t(p-1)-k) .
$$

It follows that $t(p-1)-k \geq 1$ and so $k+t=\bar{\delta}(t(p-1)-k) \geq \bar{\delta}$ and therefore $\bar{\delta} \leq k+t \leq 2 k-1$. Furthermore, (10) leads to

$$
k(\bar{\delta}+1)=t((p-1) \bar{\delta}-1) \geq(p-1) \bar{\delta}-1
$$

and consequently,

$$
p-1 \leq \frac{k \bar{\delta}+k+1}{\bar{\delta}}=k+\frac{k+1}{\bar{\delta}} \leq k+\frac{k+1}{2} \leq 2 k .
$$

Using $\bar{\delta} \leq 2 k-1$, we finally arrive at

$$
n=p \bar{\delta} \leq(2 k+1)(2 k-1) \leq 2 k^{3}+5 k^{2}-5 k+1 .
$$

This completes the proof.
Conjecture 19. Let $k \geq 2$ be an integer. If $G$ is a graph of order $n$ such that $\delta(G), \delta(\bar{G}) \geq k$, then

$$
d_{s t R}^{k}(G)+d_{s t R}^{k}(\bar{G}) \leq n-2 .
$$

If $n \geq 5$ is an integer, then Examples 9 and 12 show that

$$
d_{s t R}^{n-2}\left(K_{n, n}\right)+d_{s t R}^{n-2}\left(\overline{K_{n, n}}\right)=2 n-2=n\left(K_{n, n}\right)-2
$$

Thus Conjecture 19 would be tight for $k \geq 3$.

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