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THE SIGNED TOTAL ROMAN k-DOMATIC NUMBER OF A GRAPH

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Abstract

Let $k \geq 1$ be an integer. A *signed total Roman k-dominating function* on a graph G is a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for every $v \in V(G)$, where $N(v)$ is the neighborhood of v, and every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed total Roman k-dominating family* (of functions) on G. The maximum number of functions in a signed total Roman k-dominating family on G is the *signed total Roman k-domatic number* of G, denoted by $d^k_{stR}(G)$. In this paper we initiate the study of signed total Roman k-domatic numbers in graphs, and we present sharp bounds for $d^k_{stR}(G)$. In particular, we derive some Nordhaus-Gaddum type inequalities. In addition, we determine the signed total Roman k-domatic number of some graphs.

Keywords: signed total Roman k-dominating function, signed total Roman k-domination number, signed total Roman k-domatic number.

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1. Terminology and Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [2]. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$.

The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $\delta(G) = \Delta(G) = r$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,p}$ for the *complete bipartite graph* of order $2p$, and C_n for the *cycle* of length n.

In this paper we continue the study of Roman dominating functions in graphs and digraphs. If $k \geq 1$ is an integer, then the *signed total Roman k-dominating function* (STRkDF) on a graph G is defined in [6] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an STRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed total Roman k-domination number* of a graph G, denoted by $\gamma^k_{stR}(G)$, equals the minimum weight of an STRkDF on G. A *$\gamma^k_{stR}(G)$-function* is a signed total Roman k-dominating function of G with weight $\gamma^k_{stR}(G)$. The signed total Roman k-domination number exists when $\delta(G) \geq \frac{k}{2}$. However, for investigations of the signed total Roman k-dominating number it is reasonable to claim that $\delta(G) \geq k$. Thus we assume throughout this paper that $\delta(G) \geq k$. If $k = 1$, then we write $\gamma^1_{stR}(G) = \gamma_{stR}(G)$. This case was introduced and studied in [5].

Wang [8] defined the *signed total k-dominating function* on a graph G as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for every vertex $v \in V(G)$. Thus a signed total Roman k-dominating function combines the properties of both a Roman dominating function and a signed total k-dominating function.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [1]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition. The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [4].) Analogous to the fractional domatic number we may define the signed total Roman k-domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed total Roman k-dominating family* (of functions) on G. The maximum number of functions in a signed total Roman k-dominating family (STRkD family) on G is the *signed total Roman k-domatic number* of G, denoted by $d^k_{stR}(G)$. If $k = 1$, then we write $d^1_{stR}(G) = d_{stR}(G)$. This case was introduced and investigated in [7]. The signed total Roman k-domatic number is well-defined and $d^k_{stR}(G) \geq 1$ for all graphs G with $\delta(G) \geq k$, since the set consisting of any STRkDF forms an STRkD family on G.

Our purpose in this paper is to initiate the study of signed total Roman k-domatic numbers in graphs. We first derive basic properties and bounds for the signed total Roman k-domatic number of a graph. In particular, we derive the Nordhaus-Gaddum type result

$$d_{stR}^k(G) + d_{stR}^k(\overline{G}) \leq n - 1,$$

and we discuss the equality in this inequality. In addition, we determine the signed total Roman k-domatic number of some classes of graphs.

We make use of the following results in this paper.

Proposition 1 ([6]). *If G is a graph of order n with $\delta(G) \geq k$, then $\gamma_{stR}^k(G) \leq n$.*

Proposition 2 ([6]). *If G is a graph of order n with $\delta(G) \geq k+2$, then $\gamma_{stR}^k(G) \leq n - 1$.*

Proposition 3 ([6]). *If $n \geq k+2$, then $\gamma_{stR}^k(K_n) = k+2$.*

Proposition 4 ([6]). *If G is a δ-regular graph of order n with $\delta \geq k$, then*

$$\gamma_{stR}^k(G) \geq \left\lceil \frac{kn}{\delta} \right\rceil.$$

Proposition 5 ([6]). *If $k \geq 1$ and $p \geq k$ are integers, then $\gamma_{stR}^k(K_{p,p}) = 2k$, with exception of the case that $k = 1$ and $p = 3$, in which case $\gamma_{stR}^1(K_{3,3}) = 4$.*

2. Bounds on the Signed Total Roman k-Domatic Number

In this section we present basic properties of $d_{stR}^k(G)$ and sharp bounds on the signed total Roman k-domatic number of a graph.

Theorem 6. *For every graph G with $\delta(G) \geq k$,*

$$d_{stR}^k(G) \leq \delta(G).$$

Moreover, if $d_{stR}^k(G) = \delta(G)$, then for each STRkD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_{stR}^k(G)$ and each vertex v of minimum degree, $\sum_{x \in N(v)} f_i(x) = k$ for each function f_i and $\sum_{i=1}^d f_i(x) = k$ for all $x \in N(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STRkD family on G such that $d = d_{stR}^k(G)$. If v is a vertex of minimum degree $\delta(G)$, then we deduce that

$$kd \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) = \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N(v)} k = k\delta(G)$$

and thus $d_{stR}^k(G) \leq \delta(G)$.

If $d^k_{stR}(G) = \delta(G)$, then the two inequalities occurring in the proof become equalities. Hence for the STRkD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each vertex v of minimum degree, $\sum_{x\in N(v)} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N(v)$. ∎

Theorem 7. *If G is a graph of order n, then*

$$\gamma^k_{stR}(G) \cdot d^k_{stR}(G) \le kn.$$

Moreover, if $\gamma^k_{stR}(G) \cdot d^k_{stR}(G) = kn$, then for each STRkD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d^k_{stR}(G)$, each function f_i is a $\gamma^k_{stR}(G)$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(G)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STRkD family on G such that $d = d^k_{stR}(G)$ and let $v \in V(G)$. Then

$$d \cdot \gamma^k_{stR}(G) = \sum_{i=1}^{d} \gamma^k_{stR}(G) \le \sum_{i=1}^{d} \sum_{v\in V(G)} f_i(v) = \sum_{v\in V(G)} \sum_{i=1}^{d} f_i(v) \le \sum_{v\in V(G)} k = kn.$$

If $\gamma^k_{stR}(G) \cdot d^k_{stR}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the STRkD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each i, $\sum_{v\in V(G)} f_i(v) = \gamma^k_{stR}(G)$. Thus each function f_i is a $\gamma^k_{stR}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(G)$. ∎

Example 8. If $k \ge 3$ is an integer, then $d^k_{stR}(K_{k,k}) = k$. In addition, $d^1_{stR}(K_{1,1}) = d^2_{srR}(K_{2,2}) = 1$.

Proof. Clearly, $d^1_{stR}(K_{1,1}) = d^2_{srR}(K_{2,2}) = 1$. Let now $k \ge 3$. According to Theorem 6, we have $d^k_{stR}(K_{k,k}) \le k$. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ be a bipartition of $K_{k,k}$. Now we distinguish three cases.

Case 1. Let $k = 3t$ for an integer $t \ge 1$. Define $f_1(a_i) = f_1(b_i) = -1$ for $1 \le i \le t$ and $f_1(a_i) = f_1(b_i) = 2$ for $t+1 \le i \le 3t$. For $2 \le j \le k$ and $1 \le i \le k$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo k. It is easy to see that f_i is a signed total Roman k-dominating function of $K_{k,k}$ for $1 \le i \le k$ and $\{f_1, f_2, \ldots, f_k\}$ is a signed total Roman k-dominating family on $K_{k,k}$. Hence $d^k_{stR}(K_{k,k}) \ge k$ and thus $d^k_{stR}(K_{k,k}) = k$ in this case.

Case 2. Let $k = 3t+1$ for an integer $t \ge 1$. Define $f_1(a_i) = f_1(b_i) = -1$ for $1 \le i \le t$, $f_1(a_i) = f_1(b_i) = 2$ for $t+1 \le i \le 3t$ and $f_1(a_{3t+1}) = f_1(b_{3t+1}) = 1$. For $2 \le j \le k$ and $1 \le i \le k$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo k. It is easy to see that f_i is a signed total Roman k-dominating function of $K_{k,k}$ for $1 \le i \le k$ and $\{f_1, f_2, \ldots, f_k\}$ is a signed total Roman k-dominating family on $K_{k,k}$. Hence $d^k_{stR}(K_{k,k}) \ge k$ and thus $d^k_{stR}(K_{k,k}) = k$.

Case 3. Let $k = 3t + 2$ for an integer $t \geq 1$. Define $f_1(a_i) = f_1(b_i) = -1$ for $1 \leq i \leq t$, $f_1(a_i) = f_1(b_i) = 2$ for $t + 1 \leq i \leq 3t$ and $f_1(a_{3t+1}) = f_1(a_{3t+2}) = f_1(b_{3t+1}) = f_1(b_{3t+2}) = 1$. For $2 \leq j \leq k$ and $1 \leq i \leq k$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo k. It is easy to see that f_i is a signed total Roman k-dominating function of $K_{k,k}$ for $1 \leq i \leq k$ and $\{f_1, f_2, \ldots, f_k\}$ is a signed total Roman k-dominating family on $K_{k,k}$. Hence $d^k_{stR}(K_{k,k}) \geq k$ and thus $d^k_{stR}(K_{k,k}) = k$. ■

Example 9. If k, p are integers such that $p \geq k + 1 \geq 2$, then $d^k_{stR}(K_{p,p}) = p$, with exception of the case $k = 1$ and $p = 3$, in which case $d^1_{stR}(K_{3,3}) = 1$.

Proof. Theorem 6 implies that $d^k_{stR}(K_{p,p}) \leq p$. Now let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_p\}$ be a bipartition of $K_{p,p}$. We distinguish two cases.

Case 1. Assume that $p - k$ is odd. Define $f_1(a_i) = f_1(b_i) = 1$ for $1 \leq i \leq \frac{p+k-3}{2}$, $f_1(a_i) = f_1(b_i) = -1$ for $\frac{p+k-1}{2} \leq i \leq p - 1$ and $f_1(a_p) = f_1(b_p) = 2$. For $2 \leq j \leq p$ and $1 \leq i \leq p$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo p. It is easy to see that f_i is a signed total Roman k-dominating function of $K_{p,p}$ for $1 \leq i \leq p$ and $\{f_1, f_2, \ldots, f_p\}$ is a signed total Roman k-dominating family on $K_{p,p}$. Hence $d^k_{stR}(K_{p,p}) \geq p$ and thus $d^k_{stR}(K_{p,p}) = p$ in this case.

Case 2. Assume that $p - k$ is even. If $k = 1$ and $p = 3$, then Theorem 7 and Proposition 5 imply that $d^1_{stR}(K_{3,3}) \leq 3/2$ and thus $d^1_{stR}(K_{3,3}) = 1$.

Let now $p + k \geq 6$. Define $f_1(a_i) = f_1(b_i) = 1$ for $1 \leq i \leq \frac{p+k-6}{2}$, $f_1(a_i) = f_1(b_i) = -1$ for $\frac{p+k-4}{2} \leq i \leq p-2$ and $f_1(a_{p-1}) = f_1(a_p) = f_1(b_{p-1}) = f_1(b_p) = 2$. For $2 \leq j \leq p$ and $1 \leq i \leq p$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo p. It is easy to see that f_i is a signed total Roman k-dominating function of $K_{p,p}$ for $1 \leq i \leq p$ and $\{f_1, f_2, \ldots, f_p\}$ is a signed total Roman k-dominating family on $K_{p,p}$. Hence $d^k_{stR}(K_{p,p}) \geq p$ and thus $d^k_{stR}(K_{p,p}) = p$. ■

Examples 8 and 9 demonstrate that Theorem 6 is sharp. If $k \geq 3$, then Example 8 and Proposition 5 show that Theorem 7 is sharp too.

Corollary 10. *If K_n is the complete graph of order n such that $n \geq k + 2$, then*

$$d^k_{stR}(K_n) \leq \frac{k}{k+2} n.$$

Proof. It follows from Theorem 7 and Proposition 3 that

$$(k + 2) d^k_{stR}(K_n) = \gamma^k_{stR}(K_n) \cdot d^k_{stR}(K_n) \leq kn.$$

This yields the desired bound immediately. ■

Example 11. If $n \geq 6$ is an even integer, then $d^2_{stR}(K_n) = \frac{n}{2}$.

Proof. Corollary 10 implies $d^2_{stR}(K_n) \le \frac{n}{2}$. Let $n = 2p$ for an integer $p \ge 3$, and let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of K_n. For the opposite inequality we distinguish two cases.

Case 1. Assume that $p = 2t+1$ for an integer $t \ge 1$ and thus $n = 4t+2$. Define the function f_1 by $f_1(x_i) = -1$ for $1 \le i \le 2t$, $f_1(x_i) = 1$ for $2t+1 \le i \le 4t$ and $f_1(x_{4t+1}) = f_1(x_{4t+2}) = 2$. For $2 \le j \le 2t+1$ and $1 \le i \le 4t+2$ define $f_j(x_i) = f_1(x_{i+2j-2})$, where the indices are taken modulo n. It is easy to see that f_i is a signed total Roman 2-dominating function of K_n for $1 \le i \le 2t+1$ and $\{f_1, f_2, \ldots, f_{2t+1}\}$ is a signed total Roman 2-dominating family on K_n. Hence $d^2_{stR}(K_n) \ge 2t+1 = \frac{n}{2}$ and thus $d^2_{stR}(K_n) = \frac{n}{2}$ in this case.

Case 2. Assume that $p = 2t$ for an integer $t \ge 2$ and thus $n = 4t$. Define the functions f_1 by $f_1(x_i) = -1$ for $1 \le i \le 2t$, $f_1(x_i) = 1$ for $2t+1 \le i \le 4t-4$ and $f_1(x_i) = 2$ for $4t-3 \le i \le 4t$. For $2 \le j \le 2t$ and $1 \le i \le 4t$ define $f_j(x_i) = f_1(x_{i+2j-2})$, where the indices are taken modulo n. It is easy to see that f_i is a signed total Roman 2-dominating function of K_n for $1 \le i \le 2t$ and $\{f_1, f_2, \ldots, f_{2t}\}$ is a signed total Roman 2-dominating family on K_n. Hence $d^2_{stR}(K_n) \ge 2t = \frac{n}{2}$ and thus $d^2_{stR}(K_n) = \frac{n}{2}$. ■

Example 12. If $n \ge 5$, then $d^{n-2}_{stR}(K_n) = n-2$.

Proof. Corollary 10 implies $d^{n-2}_{stR}(K_n) \le n-2$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of K_n. Define the function f_1 by $f_1(x_1) = -1$, $f_1(x_2) = f_2(x_3) = 2$ and $f_1(x_i) = 1$ for $4 \le i \le n$. For $2 \le j \le n-2$ and $1 \le i \le n-2$ define $f_j(x_i) = f_1(x_{i+j-1})$, where the indices are taken modulo $n-2$ and $f_j(x_{n-1}) = f_j(x_n) = 1$. It is easy to see that f_i is a signed total Roman $(n-2)$-dominating function of K_n for $1 \le i \le n-2$ and $\{f_1, f_2, \ldots, f_{n-2}\}$ is a signed total Roman $(n-2)$-dominating family on K_n. Hence $d^{n-2}_{stR}(K_n) \ge n-2$ and thus $d^{n-2}_{stR}(K_n) = n-2$. ■

These are further examples showing the sharpness of Theorem 7 and Corollary 10. For some regular graphs we will improve the upper bound given in Theorem 6.

Theorem 13. *Let G be a δ-regular graph of order n with $\delta \ge \max\{2, k\}$ such that $n = p\delta + r$ with integers $p \ge 1$ and $1 \le r \le \delta - 1$ and $kr = t\delta + s$ with integers $t \ge 0$ and $1 \le s \le \delta - 1$. Then $d^k_{stR}(G) \le \delta - 1$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STRkD family on G such that $d = d^k_{stR}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) = \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(G)} k = kn.$$

Proposition 4 implies

$$\begin{aligned}\omega(f_i) \;\ge\; \gamma^k_{stR}(G) \ge \left\lceil \frac{kn}{\delta} \right\rceil = \left\lceil \frac{kp\delta + kr}{\delta} \right\rceil = kp + \left\lceil \frac{kr}{\delta} \right\rceil \\ = \; kp + \left\lceil \frac{t\delta + s}{\delta} \right\rceil = kp + t + 1,\end{aligned}$$

for each $i \in \{1, 2, \ldots, d\}$. If we suppose to the contrary that $d = \delta$, then the above inequality chains lead to the contradiction

$$\begin{aligned}kn \ge \sum_{i=1}^{d} \omega(f_i) \ge d(kp + t + 1) = \delta(kp + t + 1) = kp\delta + \delta t + \delta \\ = \; kp\delta + kr - s + \delta > kp\delta + kr = k(p\delta + r) = kn.\end{aligned}$$

Thus $d \le \delta - 1$, and the proof is complete. ■

Examples 8 and 9 demonstrate that Theorem 13 is not valid in general.

Theorem 14. *If G is a graph of order n with $\delta(G) \ge k$, then*

$$\gamma^k_{stR}(G) + d^k_{stR}(G) \le n + k.$$

Proof. If $d^k_{sR}(G) \le k$, then Proposition 1 implies $\gamma^k_{stR}(G) + d^k_{stR}(G) \le n + k$ immediately. Let now $d^k_{stR}(G) \ge k$. It follows from Theorem 7 that

$$\gamma^k_{stR}(G) + d^k_{stR}(G) \le \frac{kn}{d^k_{stR}(G)} + d^k_{stR}(G).$$

According to Theorem 6, we have $k \le d^k_{stR}(G) \le n$. Using these bounds, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, we obtain

$$\gamma^k_{stR}(G) + d^k_{stR}(G) \le \frac{kn}{d^k_{stR}(G)} + d^k_{stR}(G) \le \max\{n + k, k + n\} = n + k,$$

and the desired bound is proved. ■

Proposition 5 and Example 8 demonstrate that Theorem 14 is sharp for $k \ge 3$. For $\delta(G) \ge k + 2$, we will improve the bound in Theorem 14 slightly.

Theorem 15. If G is a graph of order n with $\delta(G) \ge k + 2$, then

$$\gamma^k_{stR}(G) + d^k_{stR}(G) \le n + k - 1.$$

Proof. If $d^k_{sR}(G) \le k$, then Proposition 2 implies $\gamma^k_{stR}(G) + d^k_{stR}(G) \le n + k - 1$. Let now $d^k_{stR}(G) \ge k + 1$. It follows from Theorem 7 that

$$\gamma^k_{stR}(G) + d^k_{stR}(G) \le \frac{kn}{d^k_{stR}(G)} + d^k_{stR}(G).$$

According to Theorem 6, we have $k + 1 \le d^k_{stR}(G) \le n - 1$. Using these bounds, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k+1 \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n - 1$, we obtain

$$\begin{aligned}\gamma^k_{stR}(G) + d^k_{stR}(G) &\le \frac{kn}{d^k_{stR}(G)} + d^k_{stR}(G)\\ &\le \max\left\{\frac{kn}{k+1} + k + 1, \frac{kn}{n-1} + n - 1\right\} < n + k,\end{aligned}$$

and this leads to the desired bound. ■

The special case $k = 1$ of Theorems 6, 7, 13 and 14 can be found in [7].

3. Nordhaus-Gaddum Type Results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their current classical paper [3], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the signed total Roman k-domatic number.

Theorem 16. *If G is a graph of order n such that $\delta(G), \delta(\overline{G}) \ge k$, then*

$$d^k_{stR}(G) + d^k_{stR}(\overline{G}) \le n - 1.$$

Furthermore, if $d^k_{stR}(G) + d^k_{stR}(\overline{G}) = n - 1$, then G is regular.

Proof. It follows from Theorem 6 that

$$d^k_{stR}(G) + d^k_{stR}(\overline{G}) \le \delta(G) + \delta(\overline{G}) = \delta(G) + (n - \Delta(G) - 1) \le n - 1.$$

If G is not regular, then $\Delta(G) - \delta(G) \ge 1$, and hence the above inequality chain implies the better bound $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \le n - 2$. ■

In the special case $k = 1$, we have proved the following theorem in [7].

Theorem 17. *If G is a graph of order n such that $\delta(G), \delta(\overline{G}) \ge 1$, then*

$$d_{stR}(G) + d_{stR}(\overline{G}) \le n - 1,$$

with equality if and only if $G = C_4$.

As a supplement to Theorem 17, we prove the next result.

Theorem 18. *Let $k \geq 2$ be an integer. Then there is only a finite number of graphs G with $\delta(G), \delta(\overline{G}) \geq k$ such that*

$$d^k_{stR}(G) + d^k_{stR}(\overline{G}) = n(G) - 1.$$

Proof. Let $n(G) = n$, $\delta(G) = \delta$ and $\delta(\overline{G}) = \overline{\delta}$. The strategy of our proof is as follows. For a fixed integer $k \geq 2$, we show that $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$ or $n \leq 2k^3 + 5k^2 - 5k + 1$. Together with Theorem 16 this implies the desired result.

If G is not regular, then it follows from Theorem 16 that $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$. Assume now that G is δ-regular. Then $\overline{G}$ is $\overline{\delta}$-regular such that $\delta + \overline{\delta} + 1 = n$. Assume, without loss of generality, that $\overline{\delta} \leq \delta$. We distinguish three cases.

Case 1. Assume that $\overline{\delta} = \delta$. Then $n = 2\delta + 1$ and $k = t\delta + s$ with integers $t \geq 0$ and $1 \leq s \leq \delta - 1$. If $\delta = k$, then $n = 2k + 1 \leq 2k^3 + 5k^2 - 5k + 1$. If $\delta \geq k + 1$, then $t = 0$ and $s = k \leq \delta - 1$, and Theorem 13 implies that $d^k_{stR}(G) \leq \delta - 1$. Applying now Theorem 6, we conclude that

$$d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq \delta - 1 + \overline{\delta} \leq n - 2.$$

Case 2. Assume that $\overline{\delta} = \delta - 1$. Then $n = 2\overline{\delta} + 2$. If $\overline{\delta} = 2$, then $n = 6 \leq 3k \leq 2k^3 + 5k^2 - 5k + 1$. If $\overline{\delta} \geq 3$, then let $2k = t\overline{\delta} + s$ with integers $t \geq 0$ and $0 \leq s \leq \overline{\delta} - 1$. Since $\overline{\delta} \geq k$, we observe that $0 \leq t \leq 2$. If $t = 2$, then $2k = 2\overline{\delta}$ and therefore $n = \delta + \overline{\delta} + 1 = 2k + 2 \leq 2k^3 + 5k^2 - 5k + 1$. If $t = 1$, then $2k = \overline{\delta} + s$. If $s = 0$, then $\overline{\delta} = 2k$ and thus $n = \delta + \overline{\delta} + 1 = 4k + 2 \leq 2k^3 + 5k^2 - 5k + 1$. If $s \neq 0$, then accoring to Theorem 13, $d^k_{stR}(\overline{G}) \leq \overline{\delta} - 1$ and Theorem 6 leads to $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$. If $t = 0$, then $2k = s \leq \overline{\delta} - 1$, and again Theorems 6 and 13 yield $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$.

Case 3. Assume that $\overline{\delta} \leq \delta - 2$. Then $n = \delta + \overline{\delta} + 1 = \delta + r$ with $1 \leq r = \overline{\delta} + 1 \leq \delta - 1$. Let now $kr = k(\overline{\delta} + 1) = t\delta + s$ with integers $t \geq 0$ and $0 \leq s \leq \delta - 1$. If $s \neq 0$, then we deduce from Theorem 13 that $d^k_{stR}(G) \leq \delta - 1$, and Theorem 6 yields to $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$. If $s = 0$, then the condition $\overline{\delta} \leq \delta - 2$ shows that

$$k(\overline{\delta} + 1) = t\delta \text{ with } 1 \leq t \leq k - 1 \tag{1}$$

and thus

$$\delta = \frac{k(\overline{\delta} + 1)}{t}. \tag{2}$$

Let now

$$n = p\overline{\delta} + r \text{ with integers } p \geq 1 \text{ and } 0 \leq r \leq \overline{\delta} - 1 \tag{3}$$

and when $r \neq 0$

$$kr = a\overline{\delta} + b \text{ with integers } a \geq 0 \text{ and } 0 \leq b \leq \overline{\delta} - 1. \tag{4}$$

If $b, r \neq 0$, then we conclude from Theorem 13 that $d^k_{stR}(\overline{G}) \leq \overline{\delta} - 1$, and Theorem 6 implies that $d^k_{stR}(G) + d^k_{stR}(\overline{G}) \leq n - 2$. Now let $r \neq 0$ and $b = 0$. Then (3) and (4) yield to

$$kr = a\overline{\delta} \ \text{ with } \ 1 \leq a \leq k - 1$$

and thus

$$\overline{\delta} = \frac{kr}{a}. \tag{5}$$

Using (2) and (3), we obtain

$$\frac{k(\overline{\delta} + 1)}{t} + \overline{\delta} + 1 = \delta + \overline{\delta} + 1 = n = p\overline{\delta} + \frac{a\overline{\delta}}{k}$$

and thus

$$p\overline{\delta} = \overline{\delta}\left(\frac{k}{t} + 1 - \frac{a}{k}\right) + \frac{k}{t} + 1 \leq \overline{\delta}\left(\frac{k}{t} + 2 - \frac{a}{k}\right) + 1 \leq \overline{\delta}\left(k + 2 - \frac{a}{k}\right) + 1$$

and so $p \leq k + 2$. Combining (2) and (5), we obtain

$$\delta = \frac{k}{t}\left(\frac{kr}{a} + 1\right)$$

and so

$$n = \delta + \overline{\delta} + 1 = \frac{k}{t}\left(\frac{kr}{a} + 1\right) + \frac{kr}{a} + 1. \tag{6}$$

According to (3) and (5), we have

$$n = p\overline{\delta} + r = \frac{pkr}{a} + r. \tag{7}$$

Combining (6) and (7), we find that

$$r\left(\frac{pk}{a} + 1\right) = \frac{kr}{a}\left(\frac{k}{t} + 1\right) + \frac{k}{t} + 1$$

and therefore

$$1 + \frac{k}{t} = r\left(\frac{pk}{a} + 1 - \frac{k}{a} - \frac{k^2}{at}\right). \tag{8}$$

This equality shows that

$$\frac{pk}{a} + 1 - \frac{k}{a} - \frac{k^2}{at} > 0$$

and hence

$$\frac{pk}{a} + 1 - \frac{k}{a} - \frac{k^2}{at} \geq \frac{1}{at}.$$

Using this and (8), we obtain

$$1+\frac{k}{t}\geq\frac{r}{at}$$

and thus

$$r\leq a(t+k)\leq(k-1)(2k-1)\,. \tag{9}$$

In view of (5), it follows that

$$\overline{\delta}=\frac{kr}{a}\leq\frac{ka(t+k)}{a}=k(t+k)\leq k(2k-1)\,.$$

Applying $p\leq k+2$, (3), (9) and the last inequality, we arrive at the desired bound

$$n=p\overline{\delta}+r\leq(k+2)k(2k-1)+(k-1)(2k-1)=2k^3+5k^2-5k+1.$$

It remains the case that $r=0$ and thus $n=p\overline{\delta}$ with an integer $p\geq 2$. Since $n=\delta+\overline{\delta}+1$, we deduce that

$$\delta=(p-1)\overline{\delta}-1.$$

Using this identity and (1), we obtain

$$k(\overline{\delta}+1)=t\delta=t(p-1)\overline{\delta}-t \tag{10}$$

and thus

$$k+t=\overline{\delta}(t(p-1)-k)\,.$$

It follows that $t(p-1)-k\geq 1$ and so $k+t=\overline{\delta}(t(p-1)-k)\geq\overline{\delta}$ and therefore $\overline{\delta}\leq k+t\leq 2k-1$. Furthermore, (10) leads to

$$k(\overline{\delta}+1)=t((p-1)\overline{\delta}-1)\geq(p-1)\overline{\delta}-1$$

and consequently,

$$p-1\leq\frac{k\overline{\delta}+k+1}{\overline{\delta}}=k+\frac{k+1}{\overline{\delta}}\leq k+\frac{k+1}{2}\leq 2k.$$

Using $\overline{\delta}\leq 2k-1$, we finally arrive at

$$n=p\overline{\delta}\leq(2k+1)(2k-1)\leq 2k^3+5k^2-5k+1.$$

This completes the proof. ■

Conjecture 19. *Let $k\geq 2$ be an integer. If G is a graph of order n such that $\delta(G),\delta(\overline{G})\geq k$, then*

$$d^k_{stR}(G)+d^k_{stR}(\overline{G})\leq n-2.$$

If $n \geq 5$ is an integer, then Examples 9 and 12 show that

$$d_{stR}^{n-2}(K_{n,n}) + d_{stR}^{n-2}(\overline{K_{n,n}}) = 2n - 2 = n(K_{n,n}) - 2.$$

Thus Conjecture 19 would be tight for $k \geq 3$.

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