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THE DEGREE-DIAMETER PROBLEM FOR OUTERPLANAR GRAPHS

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Abstract

For positive integers Δ and D we define $n_{\Delta,D}$ to be the largest number of vertices in an outerplanar graph of given maximum degree Δ and diameter D. We prove that $n_{\Delta,D} = \Delta^{\frac{D}{2}} + O\left(\Delta^{\frac{D}{2}-1}\right)$ if D is even, and $n_{\Delta,D} = 3\Delta^{\frac{D-1}{2}} + O\left(\Delta^{\frac{D-1}{2}-1}\right)$ if D is odd. We then extend our result to maximal outerplanar graphs by showing that the maximum number of vertices in a maximal outerplanar graph of maximum degree Δ and diameter D asymptotically equals $n_{\Delta,D}$.

Keywords: outerplanar, diameter, degree, degree-diameter problem, distance, separator theorem.

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1. Introduction

The degree-diameter problem is one of the best studied problems in graph theory. The problem is to find or bound the largest possible number of vertices in a graph of given maximum degree and diameter. It has been studied for various families of graphs as for example bipartite graphs [8, 16], vertex-transitive graphs [2], Cayley graphs [1], Abelian Cayley graphs [15] and claw-free graphs [5].

In this paper we study the degree-diameter problem for a subclass of planar graphs, for outerplanar graphs. First we present known results on the degree-diameter problem for planar graphs. Let $p_{\Delta,D}$ be the largest number of vertices in a planar graph of given maximum degree and diameter.

The degree-diameter problem for maximal planar graphs was considered by Seyffarth [13], who proved that for $\Delta \geq 8$ the number of vertices in a maximal planar graph of diameter two is at most $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ vertices and that there exist maximal planar graphs of diameter two having exactly $\left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$. Hell and Seyffarth [9] extended this result to all planar graphs and showed that $p_{\Delta,2} = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$ if $\Delta \geq 8$. Yang, Lin and Dai [17] determined $p_{\Delta,2}$ for the remaining values of Δ, i.e., for $\Delta \leq 7$.

Fellows, Hell and Seyffarth [7] proved that for diameter three we have the bounds $\left\lfloor \frac{9}{2}\Delta \right\rfloor - 3 \leq p_{\Delta,3} \leq 8\Delta + 12$ and for general diameter, $p_{\Delta,D} \leq (6D + 3)\left(2\Delta^{\lfloor \frac{D}{2} \rfloor} + 1\right)$ if $\Delta \geq 4$. Tishchenko [14] improved the bound and showed that $p_{\Delta,D} = \frac{3\Delta}{2}\frac{(\Delta-1)^{\frac{D}{2}-1}}{\Delta-2} + 1$ if D is even and $\Delta \geq 6(6D+1)$. Nevo, Pineda-Villavicencio and Wood [11] generalised Tishchenko's result to all the diameters and all the surfaces. From [11] we have $p_{\Delta,D} = O\left(\Delta^{\lfloor \frac{D}{2} \rfloor}\right)$ for $D \geq 2$. The degree-diameter problem for maximal planar bipartite graphs was studied by Dalfó, Huemer and Salas [3].

In this paper we focus on outerplanar graphs, an important subclass of planar graphs. There are only few results on distances in outerplanar graphs. A classical result by Proskurowski [12] states that the centre of a maximal outerplanar graph is isomorphic to one of seven graphs, and Farley and Proskurowski [6] gave linear algorithms to compute the diameter and the centre of an outerplanar graph. Recently, eccentric sequences of maximal outerplanar graphs have been characterized (see [4]).

For positive integers Δ and D we denote by $n_{\Delta,D}$ the largest number of vertices in an outerplanar graph of given maximum degree Δ and diameter D. In Section 2, we prove that $n_{\Delta,D} = \Delta^{\frac{D}{2}} + O\left(\Delta^{\frac{D}{2}-1}\right)$ if D is even, and $n_{\Delta,D} = 3\Delta^{\frac{D-1}{2}} + O\left(\Delta^{\frac{D-1}{2}-1}\right)$ if D is odd. One of our main tools is a separator theorem for outerplanar graphs, which shows that every outerplanar graph has two vertices whose removal renders the graph disconnected such that no component contains more than approximately two thirds of all vertices. In Section 3 we consider maximal outerplanar graphs and show that $n_{\Delta,D}$ gives asymptotically also the maximum number of vertices in a maximal outerplanar graph of maximum degree Δ and diameter D. The main tool is a lemma that shows that every tree is a spanning tree of a maximum outerplanar graph whose maximum degree is only slightly higher.

The notation we use is as follows. If G is a graph then we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively. The distance between two vertices u and v of G is the minimum number of edges on a path from u to v, we denote it by $d_G(u,v)$. If G is connected, i.e., if there is a path between any two vertices of G, then the diameter $D(G)$ is the largest of the distances between all pairs of vertices of G. If v is a vertex of G and $i \in \mathbb{N}$, then $N_{\leq i}(v)$ $(N_i(v))$ denotes the set of all vertices of distance at most i (of distance exactly i) from v. Similarly we define $N_{<i}(v)$ and $N_{>i}(v)$. The degree of a vertex v, denoted by $\deg_G(v)$, is the number of vertices to which v is jointed by an edge. The maximum degree $\Delta(G)$ is the largest of the vertex degrees of G. If the graph is understood we sometimes drop the subscript or argument G. A component of a graph G is a maximal connected subgraph, and a part of G is a disjoint union of some, but not all components of G. For positive integers m, n, the complete graph on n vertices and the complete bipartite graph whose partite sets have m and n vertices are denoted by K_n and $K_{m,n}$, respectively.

A graph is planar if it can be embedded in the plane such that no two edges cross. A planar graph is outerplanar if it can be embedded such that all vertices are on the boundary of the outer face, and an outerplanar graph is maximal outerplanar if adding any edge results in a graph that is not outerplanar. If G is an outerplanar graph, then by adding suitable edges we obtain a (not necessarily unique) maximal outerplanar supergraph G' of G on the same vertex set; we call such a supergraph an extension of G. If G is a maximal outerplanar graph, then G has a unique Hamilton cycle whose edges form the boundary of the outer face of G. We call this cycle the outer cycle of G. An edge of a maximal outerplanar graph G that is not on its outer cycle is called a chord of G. The outerdistance $\mathrm{od}(u,v)$ between two vertices u and v of a maximal outerplanar graph is their distance on the outer cycle.

2. Outerplanar Graphs

In this section we asymptotically determine $n_{\Delta,D}$, the maximum number of vertices in an outerplanar graph of maximum degree Δ and diameter D. First we prove two lemmas which will be used in the proof of our main result, Theorem 3. The first one, Lemma 1, is a separator theorem for outerplanar graphs. It is a strengthening of a well-known separator theorem for planar graphs by Lipton and

Tarjan [10] which states that a planar graph of order n always has a cutset of not more than $2\sqrt{2n}$ vertices whose removal disconnects the graph such that none of its remaining components has more than $\frac{2}{3}n$ vertices. In Lemma 1 we show that an outerplanar graph has a cutset with the same property, but of cardinality two. The second lemma gives a bound on the number of vertices of an outerplanar graph whose distance from two vertices is at most a given value.

Let G be an outerplanar graph, G' an extension of G. Then every chord uv of G' yields a cutset $\{u,v\}$ of G', whose removal leaves two components G'_1 and G'_2 of $G'-\{u,v\}$. The subgraphs G_1 and G_2 of $G-\{u,v\}$ induced by $V(G'_1)$ and $V(G'_2)$, respectively, are disjoint, nonempty unions of components of $G-\{u,v\}$. We refer to them as the two parts of $G-\{u,v\}$ with respect to G'.

Lemma 1. *Let G be an outerplanar graph with n vertices, $n \geq 4$, and G' an extension of G. Then there exists a chord uv of G' such that each of the two parts of $G-\{u,v\}$ with respect to G' has at least $\frac{n}{3}-1$ vertices.*

Proof. Let G be an outerplanar graph with n vertices, $n \geq 4$ and G' an extension of G. Let $v_0v_1\cdots v_{n-1}$ be the outer cycle of G'. It suffices to show that there exists a chord v_iv_j of G' with

$$\text{od}(v_i,v_j) \geq \frac{n}{3}, \tag{1}$$

since then $G'-\{v_i,v_j\}$ has exactly two components, each corresponding to a part of $G-\{v_i,v_j\}$, the smallest of which has $\text{od}(v_i,v_j)$ vertices, where $\text{od}(v_i,v_j) \geq \frac{n}{3}-1$.

Indeed, let v_iv_j be a chord of G' for which $\text{od}(v_i,v_j)$ is maximum. Without loss of generality we may assume that $i=0$ and $\text{od}(v_0,v_j)=j$. We prove that $j \geq \frac{n}{3}$. Suppose to the contrary that $j < \frac{n}{3}$. Since except for the outer face, every face of a maximal outerplanar graph is a triangle, there exists a vertex v_k, where $j+1 \leq k \leq n-1$, which is adjacent to both v_0 and v_j in G'. It is easy to see that $k \neq j+1$, since otherwise v_0v_{j+1} is a chord with $\text{od}(v_0,v_{j+1}) = j+1 > \text{od}(v_0,v_j)$, a contradiction to the choice of v_iv_j. Similarly $k \neq n-1$. Hence v_jv_k and v_kv_0 are chords of G'.

If now $j+1 \leq k \leq \frac{2n}{3}$, then v_0v_k is a chord of G' with $\text{od}(v_0,v_k) = \min\{k, n-k\} > j$, a contradiction to the maximality of $\text{od}(v_i,v_j)$. On the other hand, if $\frac{2n}{3} < k \leq n-1$, then $k-j > \frac{n}{3}$ and $\text{od}(v_j,v_k) = \min\{k-j, n-k+j\} > \min\{\frac{n}{3}, k\} > j$, again a contradiction to the maximality of $\text{od}(v_i,v_j)$. Hence $j \geq \frac{n}{3}$, and so (1) follows, completing the proof of Lemma 1. ■

In the following lemma we make use of the well-known fact that a graph is outerplanar if and only if it contains no minor isomorphic to $K_{2,3}$ or K_4.

Lemma 2. *Let G be an outerplanar graph of maximum degree Δ and let u,v be two vertices of G. Let $k \in \mathbb{N}$ with $k \geq 2$.*

(i) *Then*
$$|N_{\le k}(u) \cap N_{\le k}(v)| \le 4\Delta^{k-1} + O\big(\Delta^{k-2}\big).$$

(ii) *If furthermore uv is a chord of some extension G' of G, then for the two parts G_1 and G_2 of $G - \{u, v\}$ with respect to G' we have*
$$|(N_k(u) \cap N_k(v)) \cap V(G_i)| \le \Delta^{k-1} + O\big(\Delta^{k-2}\big)$$
for $i = 1, 2$.

Proof. Let X be the set of all vertices x with $d(u,x) = d(v,x) \le k$. Then

$$N_{\le k}(u) \cap N_{\le k}(v) \subseteq N_{\le k-1}(u) \cup N_{\le k-1}(v) \cup X. \tag{2}$$

For $1 \le i \le k-1$ the number of vertices at distance i from u (from v) is at most $\Delta(\Delta-1)^{i-1}$. Hence

$$|N_{\le k-1}(u)| \le 1 + \sum_{i=1}^{k-1} \Delta(\Delta-1)^{i-1}, \quad |N_{\le k-1}(v)| \le 1 + \sum_{i=1}^{k-1} \Delta(\Delta-1)^{i-1}. \tag{3}$$

In order to bound $|X|$ define the set X^* to be the set of all vertices $x \in X$ which have no neighbour y with $d(u,y) = d(v,y) = d(u,x) - 1$. Then for every $x \in X$, there exists a vertex $x^* \in X^*$ with $d(x, x^*) \le k-1$. To see this consider shortest paths from x to u and v, respectively, which have a maximum number of vertices in common. Then the last vertex belonging to both paths is in X^*, distinct from u and v, and thus at distance at most $k-1$ from x. Therefore

$$|X| \le |X^*| \left(1 + \sum_{i=1}^{k-1} \Delta(\Delta-1)^{i-1}\right). \tag{4}$$

We claim that

$$|X^*| \le 2. \tag{5}$$

Suppose to the contrary that X^* contains three vertices, x, y and z say. Let $P(u,x)$, $P(u,y)$ and $P(u,z)$ be shortest paths from u to x, y and z, and let $P(v,x)$, $P(v,y)$ and $P(v,z)$ be shortest paths from v to x, y and z, respectively. Then the sets $U_0 := [V(P(u,x)) \cup V(P(u,y)) \cup V(P(u,z))] - \{x,y,z\}$ and $V_0 := [V(P(v,x)) \cup V(P(v,y)) \cup V(P(v,z))] - \{x,y,z\}$ are disjoint since U_0 contains only vertices that are closer to u than to v, and V_0 contains only vertices that are closer to v than to u. Both sets induce connected graphs. Hence contracting U_0 to a single vertex u_0 and V_0 to a single vertex v_0 yields a graph in which vertices u_0, v_0, x, y, z induce a graph containing $K_{2,3}$. Hence G has a $K_{2,3}$-minor, contradiction to the fact that G is outerplanar. This proves (5).

Combining (2), (3), (4) and (5), we obtain

$$\begin{aligned}|N_{\le k}(u) \cap N_{\le k}(v)| &\le |N_{\le k-1}(u)| + |N_{\le k-1}(v)| + |X| \\ &\le 4 + 4\sum_{i=1}^{k-1} \Delta(\Delta-1)^{i-1} = 4\Delta^{k-1} + O\big(\Delta^{k-2}\big),\end{aligned}$$

and part (i) of the lemma follows.

We prove part (ii) only for $i = 1$, the proof for $i = 2$ is identical. It suffices to show that $|X^* \cap V(G_1)| \le 1$ since every vertex in $(N_k(u) \cap N_k(v)) \cap V(G_1)$ is at distance at most $k-1$ from some vertex in $X^* \cap V(G_1)$. Suppose to the contrary that $X^* \cap V(G_1)$ contains two vertices, x and y say. Let H be the outerplanar graph obtained from G by adding all edges of the subgraph of G' induced by $V(G_2) \cup \{u, v\}$ that are not in G. Then there exists a vertex $z \in V(G_2)$ that is adjacent to u and to v in H. As in (i), we now conclude that H has a $K_{2,3}$-minor, a contradiction to H being outerplanar. Hence $|X^* \cap V(G_1)| \le 1$ and part (ii) follows. ■

We now present our main result, an asymptotically sharp upper bound on the number of vertices in outerplanar graphs of given diameter and maximum degree.

Theorem 3. *Let G be an outerplanar graph of maximum degree Δ and let $k \in \mathbb{N}$.*
(i) *If $D(G) = 2k$, then $|V(G)| \le \Delta^k + O(\Delta^{k-1})$.*
(ii) *If $D(G) = 2k+1$, then $|V(G)| \le 3\Delta^k + O(\Delta^{k-1})$.*

Proof. Let G' be an extension of G. By Lemma 1, G' contains a chord uv, such that $G - \{u, v\}$ consists of two disjoint parts G_1 and G_2 with respect to G', and each part has at least $\frac{n}{3} - 1$ vertices. We partition the vertex set of G as follows:

$$\begin{aligned} A &= N_{<k}(u) \cup N_{<k}(v), \\ X_i &= N_k(u) \cap N_k(v) \cap V(G_i) \quad i = 1, 2, \\ U_i &= N_k(u) \cap N_{>k}(v) \cap V(G_i) \quad i = 1, 2, \\ V_i &= N_{>k}(u) \cap N_k(v) \cap V(G_i) \quad i = 1, 2, \\ Z &= N_{>k}(u) \cap N_{>k}(v). \end{aligned}$$

Since the number of vertices at distance i from u (from v) is at most $\Delta(\Delta-1)^{i-1} = \Delta^i + O(\Delta^{i-1})$, we have

$$|A| = O\big(\Delta^{k-1}\big), \tag{6}$$

$$|U_1 \cup U_2| \le \Delta^k + O\big(\Delta^{k-1}\big), \tag{7}$$

$$|V_1 \cup V_2| \le \Delta^k + O\big(\Delta^{k-1}\big). \tag{8}$$

From part (i) of Lemma 2 we obtain

$$|X_1 \cup X_2| = O\big(\Delta^{k-1}\big). \tag{9}$$

Let the diameter of G be even, namely $D(G) = 2k$. We distinguish the following cases.

Case 1. $Z \neq \emptyset$. Let $x \in Z$. Without loss of generality we assume that $x \in V(G_1)$. Then every vertex $y \in V(G_2)$ satisfies $d(y,u) \leq k-1$ or $d(y,v) \leq k-1$ since otherwise we would obtain $d(x,y) = \min\{d(x,u) + d(u,y), d(x,v) + d(v,y)\} > 2k$. This implies that $V(G_2) \subset A$ and thus $|V(G_2)| = O(\Delta^{k-1})$ by (6). Since it follows from Lemma 1 that $|V(G_2)| \geq \frac{1}{3}(|V(G_1)| + |V(G_2)| + 2) - 1$ and thus $|V(G_1)| \leq 2|V(G_2)|+1$, we have $|V(G)| = |V(G_1)|+|V(G_2)|+2 = O(\Delta^{k-1})$.

Case 2. $Z = \emptyset$.

Case 2A. At most one of the sets U_1, U_2, V_1, V_2 is nonempty. Without loss of generality we can assume that U_1 is the set which is nonempty. Then $U_2 = \emptyset$ and $V_2 = \emptyset$, which means that $V(G_2) \subset A \cup X_2$. Therefore $|V(G_2)| = O(\Delta^{k-1})$ by (6) and (9), and as in Case 1 we conclude that $|V(G)| = O(\Delta^{k-1})$.

Case 2B. At least two of the sets U_1, U_2, V_1, V_2 are nonempty. Note that at most one of the sets U_1 and V_2 can be nonempty because if there exists $u_1 \in U_1$ and $v_2 \in V_2$, then we obtain the contradiction $d(u_1, v_2) = \min\{d(u_1, u) + d(u, v_2), d(u_1, v) + d(v, v_2)\} > 2k$. By the same argument at most one of the sets V_1 and U_2 can be nonempty. Therefore it is enough to consider the following two subcases:

Case 2B1. $U_1 \neq \emptyset$, $V_1 \neq \emptyset$, $U_2 = \emptyset$, $V_2 = \emptyset$. In this case $V(G_2) \subset A \cup X_2$ and as in Case 2A we conclude that $|V(G)| = O(\Delta^{k-1})$.

Case 2B2. $U_1 \neq \emptyset$, $U_2 \neq \emptyset$, $V_1 = \emptyset$, $V_2 = \emptyset$. Then $V(G) = A \cup X_1 \cup X_2 \cup U_1 \cup U_2$ and by (6), (7) and (9) we get

$$|V(G)| = |A| + |X_1 \cup X_2| + |U_1 \cup U_2| \leq \Delta^k + O\big(\Delta^{k-1}\big).$$

(ii) Let the diameter of G be odd, namely $D(G) = 2k+1$. Note that either $Z \subseteq V(G_1)$ or $Z \subseteq V(G_2)$; otherwise if there are $x_1, x_2 \in Z$ with $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$, we obtain the contradiction $d(x_1, x_2) = \min\{d(x_1, u) + d(u, x_2), d(x_1, v) + d(v, x_2)\} \geq 2(k+1)$. Without loss of generality we assume that $Z \subset V(G_1)$. We distinguish two cases.

Case 1. For all vertices $x \in Z$, we have $d(x,u) = d(x,v) = k+1$. From part (ii) of Lemma 2 it follows that $|Z| \leq \Delta^k + O(\Delta^{k-1})$. In conjunction with (6), (7), (8), and (9) we get

$$|V(G)| = |A| + |X_1 \cup X_2| + |U_1 \cup U_2| + |V_1 \cup V_2| + |Z| \leq 3\Delta^k + O\big(\Delta^{k-1}\big),$$

as desired.

Case 2. There exists $x \in Z$, such that $d(x, u) = k + 2$ or $d(x, v) = k + 2$. We can assume that $d(x, u) = k + 2$. Then U_2 is empty, since otherwise for $u_2 \in U_2$, we would obtain $d(x, u_2) \geq 2k+2$. Therefore, we have $V(G_2) \subset A \cup X_2 \cup V_2$, which implies that $|V(G_2)| \leq \Delta^k + O(\Delta^{k-1})$ by (6), (9), and (8). As in Case 1, Lemma 1 implies that $|V(G_2)| \leq 2|V(G_1)| + 1$ and thus $|V(G_2)| \leq 2\Delta^k + O(\Delta^{k-1})$. In total we obtain that $|V(G)| \leq 3\Delta^k + O(\Delta^{k-1})$, as desired. ■

The following example shows that the bounds given in Theorem 3 are best possible.

Example 4. For given $a, b, c \in \mathbb{N}$ with $a, b \geq 2$ let $T_{a,b,c}$ be the tree rooted at a vertex r of degree a, all vertices at distance less than c from r have degree b, and all vertices at distance exactly c from r are leaves.

Given integers Δ and k with $\Delta \geq 3$ and $k \geq 1$, we construct large outerplanar graphs of maximum degree Δ and of even diameter $2k$ and odd diameter $2k+1$. For even diameter define $G_{\Delta,2k}$ to be the tree $T_{\Delta,\Delta,k}$. For odd diameter define $G_{\Delta,2k+1}$ to be the graph obtained from three disjoint copies of the tree $T_{\Delta-2,\Delta,k}$ by adding three edges joining the roots of these three graphs to each other.

It is easy to see that the graphs $G_{\Delta,2k}$ and $G_{\Delta,2k+1}$ are outerplanar, that their maximum degree is Δ, and their diameter is $2k$ and $2k+1$, respectively. Moreover, for large Δ,

$$|V(G_{\Delta,2k})| = \Delta^k + O\big(\Delta^{k-1}\big), \quad |V(G_{\Delta,2k+1})| = 3\Delta^k + O\big(\Delta^{k-1}\big).$$

From Theorem 3 and Example 4, we obtain the following result.

Corollary 5. *For any $D \geq 2$ and large Δ, we have*
(i) $n_{\Delta,D} = \Delta^{\frac{D}{2}} + O\left(\Delta^{\frac{D}{2}-1}\right)$ *if D is even,*
(ii) $n_{\Delta,D} = 3\Delta^{\frac{D-1}{2}} + O\left(\Delta^{\frac{D-1}{2}-1}\right)$ *if D is odd.*

3. Maximal Outerplanar Graphs

It is natural to ask if the bounds given in Theorem 3 can be improved for maximal outerplanar graphs. We show below that for large enough Δ this is not the case. We achieve this by showing that we can add edges to any given tree to obtain a maximal outerplanar graph with only slightly greater maximum degree than the original tree, and then applying this result to the graphs $G_{\Delta,2k}$ and $G_{\Delta,2k+1}$. Towards this goal we prove the following proposition.

Proposition 6. *Let T be a tree of order at least three. Then there exists a cycle C on the same vertex set as T such that $T \cup C$ has an outerplanar embedding with the following properties.*

(i) *C forms the boundary of the outer face,*
(ii) *no internal vertex of T is incident with more than one edge in $E(C) - E(T)$,*
(iii) *every bounded face of $T \cup C$ has exactly one edge of $E(C) - E(T)$ on its boundary.*

Proof. We prove the statement by induction on n, the order of T. If T is a star, then let u be its centre and $v_1, v_2, \ldots, v_{n-1}$ be its leaves. Define the cycle C by $E(C) = \{uv_1, v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}u\}$. Then it is easy to see that (i), (ii) and (iii) are satisfied. This proves the statement in particular for $n = 3$.

Now assume that T is not a star. Let v_1 be an end of a longest path in T and let u be its unique neighbour. Then all but one neighbour of u are leaves, denote these by $v_2, v_3, \ldots, v_t$. Let $T' = T - \{v_1, v_2, \ldots, v_t\}$. By our induction hypothesis there exists a cycle C' on $V(T')$ such that $T' \cup C'$ has an outerplanar embedding satisfying (i), (ii) and (iii). Since u is a leaf of T', at least one of its two neighbours on C', say u', is not a neighbour of u in T. Then replacing the edge uu' of C' with the path $u, v_1, v_2, \ldots, v_t, u'$ and adding the edges uv_i for $i = 2, 3, \ldots, t$ on the inside of C' yields a cycle C. It is easy to check that C satisfies (i), (ii) and (iii). ■

Lemma 7. *Let T be a tree of order at least three. Then T is a spanning subgraph of a maximal outerplanar graph G with the property that*

$$\deg_G(v) \leq \begin{cases} \deg_T(v) + 5 & \textit{if } v \textit{ is an internal vertex of } T, \\ 6 & \textit{if } v \textit{ is a leaf of } T. \end{cases}$$

Proof. Let T be a tree. Let C be a cycle on $V(T)$ as in Proposition 6. By property (ii) of Proposition 6, we have

$$\deg_{T\cup C}(v) \leq \begin{cases} \deg_T(v) + 1 & \text{if } v \text{ is an internal vertex of } T, \\ 3 & \text{if } v \text{ is a leaf of } T. \end{cases} \tag{10}$$

Now choose an internal vertex r and root T at r. If f is a bounded face of $T \cup C$, then we say that a vertex v on its boundary is a good vertex of f if $d_T(v, r) \leq d_T(x, r)$ for all vertices x on the boundary of f. A vertex on the boundary of f that is not good is said to be a bad vertex of f. Since f contains exactly one edge not in T, it follows that the vertices on the boundary of f induce a path in T, and among those there is a unique vertex closest to r. Hence every bounded face of $T \cup C$ has exactly one good vertex. We claim that

(11) every vertex is a bad vertex of at most two bounded faces.

Indeed, let v be an arbitrary vertex of $T \cup C$. If $v = r$, then clearly v is a good vertex of every face that has v on its boundary. If $v \neq r$, then there exists exactly one neighbour w of v in T which is closer to r, while all other neighbours of v are further from r. It follows that every face that contains v as a bad vertex has the edge vw on its boundary. Since there are at most two such faces, (11) follows.

We now triangulate the bounded faces of $T \cup C$. Let f be a bounded face of length greater than three, let v_0 be its unique good vertex, and let $v_0, v_1, v_2, \ldots, v_t$ be the boundary vertices of f in clockwise order. Then adding the edges of the path $v_1 v_t v_2 v_{t-1} v_3 v_{t-2} \cdots v_{\frac{t+4}{2}} v_{\frac{t}{2}}$ if t is even, and $v_1 v_t v_2 v_{t-1} v_3 v_{t-2} \cdots v_{\frac{t-1}{2}} v_{\frac{t+3}{2}}$ if t is odd yields a triangulation of f. Clearly, every bad vertex of f is incident with at most two new edges, while the good vertex is not incident with any new edge. Performing this triangulation for every face f of $T \cup C$ of length greater than three yields a maximal outerplanar graph in which every vertex of $T \cup C$ is incident with at most four new edges by (11), so $\deg_G(v) \le \deg_{T\cup C}(v) + 4$. In conjunction with (10) it follows that $\Delta(G) \le \Delta(T) + 5$, as desired. ■

Corollary 8. *Let $k \in \mathbb{N}$ be fixed. For large Δ there exist maximal outerplanar graphs of maximum degree at most Δ, diameter at most $2k$ and $2k+1$, and order $\Delta^k + O(\Delta^{k-1})$ and $3\Delta^k + O(\Delta^{k-1})$, respectively.*

Proof. We may assume that $\Delta \ge 8$. We first construct maximal outerplanar graphs of even diameter. Let $G_{\Delta-5,2k}$ be the tree constructed in Example 1. By Lemma 7 the tree T is contained in a maximum outerplanar graph $H_{\Delta,2k}$ on the same vertex set with maximum degree Δ. Since $H_{\Delta,2k}$ has $G_{\Delta-5,2k}$ as a spanning subgraph, its diameter at most $2k$, and its order is bounded by

$$|V(H_{\Delta,2k})| = |V(G_{\Delta-5,2k})| = (\Delta-5)^k + O\big(\Delta^{k-1}\big) = \Delta^k + O\big(\Delta^{k-1}\big).$$

We now construct maximal outerplanar graphs of odd diameter. Let T_1, T_2 and T_3 be three copies of the rooted tree $T_{\Delta-8,\Delta-6,k}$, and let r_1, r_2 and r_3 be their respective roots. By Lemma 7 each tree T_i is contained in a maximum outerplanar graph F_i in which r_i has degree at most $\Delta - 3$, and all other vertices have degree at most $\Delta - 1$. For $i = 1, 2, 3$ let s_i be the vertex following r_i on the outer cycle of F_i in clockwise order. Let $H_{\Delta,2k+1}$ be obtained from $F_1 \cup F_2 \cup F_3$ by joining r_i to r_{i+1} and s_{i+1} for $i = 1, 2, 3$, with subscripts taken modulo 3. It is easy to see that $H_{\Delta,2k+1}$ is maximum outerplanar and has maximum degree at most Δ. Since $H_{\Delta,2k+1}$ has $G_{\Delta-6,2k+1}$ as a spanning subgraph, its diameter is at most $2k+1$, and its order is bounded by

$$|V(H_{\Delta,2k+1})| = |V(G_{\Delta-6,2k+1})| = 3(\Delta-6)^k + O\big(\Delta^{k-1}\big) = 3\Delta^k + O\big(\Delta^{k-1}\big).$$

In both cases the order of the graphs $H_{\Delta,2k}$ and $H_{\Delta,2k+1}$ asymptotically attains the upper bound in Theorem 3. ■

4. Conclusion

Corollaries 5 and 8 determine asymptotically the number of vertices in an outerplanar or maximum outerplanar graph of given diameter as the maximum degree gets large. It would be interesting to determine good estimates for fixed values of Δ. This suggests the following problem.

Problem 9. *For fixed Δ with $\Delta \geq 3$ find upper and lower bounds on $n_{\Delta,D}$.*

Our results in Corollary 5 determine the leading coefficient of $n_{\Delta,D}$. Can this result be strengthened by determining or bounding also the second coefficient?

Problem 10. *Find numbers $b, b', c, c' \in \mathbb{R}$, such that $n_{\Delta,D} \leq \Delta^{\frac{D}{2}} + b\Delta^{\frac{D}{2}-1} + O\left(\Delta^{\frac{D}{2}-2}\right)$, $n_{\Delta,D} \geq \Delta^{\frac{D}{2}} + b'\Delta^{\frac{D}{2}-1} + O\left(\Delta^{\frac{D}{2}-2}\right)$ if D is even, and $n_{\Delta,D} \leq 3\Delta^{\frac{D-1}{2}} + c\Delta^{\frac{D-1}{2}-1} + O\left(\Delta^{\frac{D-1}{2}-2}\right)$, $n_{\Delta,D} \geq 3\Delta^{\frac{D-1}{2}} + c'\Delta^{\frac{D-1}{2}-1} + O\left(\Delta^{\frac{D-1}{2}-2}\right)$ if D is odd.*

Does there exist numbers $b, c \in \mathbb{R}$, such that $n_{\Delta,D} = \Delta^{\frac{D}{2}} + b\Delta^{\frac{D}{2}-1} + O\left(\Delta^{\frac{D}{2}-2}\right)$ if D is even, and $n_{\Delta,D} = 3\Delta^{\frac{D-1}{2}} + c\Delta^{\frac{D-1}{2}-1} + O\left(\Delta^{\frac{D-1}{2}-2}\right)$ if D is odd?

We can ask similar questions for maximal ouerplanar graphs. Find upper and lower bounds on the largest number of vertices of a maximal outerplanar graph for given D and large Δ of the form $\Delta^{\frac{D}{2}} + b\Delta^{\frac{D}{2}-1} + O\left(\Delta^{\frac{D}{2}-2}\right)$ if D is even, and $3\Delta^{\frac{D-1}{2}} + c\Delta^{\frac{D-1}{2}-1} + O\left(\Delta^{\frac{D-1}{2}-2}\right)$ if D is odd.

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