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EVERY 8-TRACEABLE ORIENTED GRAPH IS TRACEABLE

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Abstract

A digraph of order n is k-traceable if $n \geq k$ and each of its induced subdigraphs of order k is traceable. It is known that if $2 \leq k \leq 6$, every k-traceable oriented graph is traceable but for $k = 7$ and for each $k \geq 9$, there exist k-traceable oriented graphs that are nontraceable. We show that every 8-traceable oriented graph is traceable.

Keywords: oriented graph, traceable, hypotraceable, k-traceable, generalized tournament.

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1. Introduction and Background

A digraph is *traceable* if it contains a path that visits every vertex and *hamiltonian* if it contains a cycle that visits every vertex. A digraph is *k-traceable* if it has at least k vertices and each of its induced subdigraphs of order k is traceable. Obviously, an oriented graph is 2-traceable if and only if it is a nontrivial tournament. Thus k-traceable oriented graphs may be regarded as generalized tournaments. It is well-known that every tournament is traceable. The following theorem shows that k-traceable oriented graphs retain this property for small values of k.

Theorem 1 [3]**.** *For $k = 2, 3, 4, 5, 6$, every k-traceable oriented graph is traceable.*

However, not all k-traceable oriented graphs are traceable. In particular, we know the following.

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Theorem 2 [2, 6]**.**

(1) *For $k = 7$ and for every $k \geq 9$, there exist k-traceable oriented graphs of order $k+1$ that are nontraceable.*

(2) *There exist nontraceable k-traceable oriented graphs of order $k+2$ for infinitely many k.*

The following traceability conjecture, called the TC, is considered in [1, 3, 4, 5, 7, 9].

Conjecture 1 (TC)**.** *For $k \geq 2$, every k-traceable oriented graph of order at least $2k-1$ is traceable.*

The next two results were established by means of exhaustive computer searches.

Theorem 3 [7]**.** *All 8-traceable oriented graphs of order 9, 10 and 11 are traceable.*

Theorem 3 was used in [1] to prove, by means of an iterative procedure, that the TC holds for $k = 8$. In fact, the following slightly stronger result was proved.

Theorem 4 [1]**.** *Every 8-traceable oriented graph of order at least 14 is traceable.*

In this paper we prove analytically that all 8-traceable oriented graphs of order 12 and 13 are also traceable. Hence we conclude that every 8-traceable oriented graph is traceable.

2. Preliminaries and Auxiliary Results

The set of vertices and the set of arcs of a digraph D are denoted by $V(D)$ and $A(D)$, respectively, and the order of D is denoted $n(D)$. If D is a digraph and $X \subset V(D)$, then $\langle X \rangle$ denotes the subdigraph induced by X in D. If $v \in V(D)$, we denote the sets of *out-neighbours* and *in-neighbours* of v by $N^+(v)$ and $N^-(v)$ and the cardinalities of these sets by $d^+(v)$ and $d^-(v)$, respectively. For undefined concepts we refer the reader to [8].

A digraph is *strong* (or *strongly connected*) if for every pair of vertices x, y in D there is a path from x to y. A maximal strong subdigraph of a digraph D is called a *strong component* of D. We say that a strong component is *trivial* if has only one vertex. If D is a digraph with h strong components, then its strong components have an acyclic ordering $D_1, D_2, \ldots, D_h$ such that if there is an arc from D_i to D_j, then $i \leq j$. If D is k-traceable for some $k \geq 2$, this acyclic ordering is unique since there is at least one arc from D_i to D_{i+1} for $i = 1, 2, \ldots, h-1$.

Throughout this paper we label the strong components of a k-traceable digraph in accordance with this acyclic ordering.

A digraph D is *hypotraceable* if $n(D) \geq 3$ and D is nontraceable but the removal of any vertex leaves a traceable digraph. Thus for $k \geq 2$ a k-traceable digraph of order $k+1$ is hypotraceable if and only if it is nontraceable.

The following immediate consequence of Theorem 3 plays an important role in the proof of our main result.

Lemma 5. *If D is an 8-traceable oriented graph of order $n \geq 12$, then D is also 9-, 10- and 11-traceable.*

Lemma 5 implies that if there exists a nontraceable 8-traceable oriented graph of order 12, then it would be hypotraceable. We observe the following.

Observation 6. *A hypotraceable digraph does not have a vertex with in-degree or out-degree equal to 1.*

We shall also use the following properties of hypotraceable digraphs.

Lemma 7. *Let D be be hypotraceable digraph with strong components $D_1, \ldots, D_h$. Then the following hold.*

(1) *If D_t is a trivial strong component, then t is either 1 or h.*

(2) *D_1 and D_h are nonhamiltonian.*

Proof. (1) Let $D_t = \{w\}$ for some $t \in \{2, \ldots, h-1\}$. Now let $x \in D_1$ and $z \in D_h$. Then $D - z$ and $D - x$ have Hamilton paths P and Q, respectively. Note that P has a subpath with vertex set $\bigcup_{i=1}^{t} V(D_i)$ ending at w, and Q has a subpath with vertex set $\bigcup_{i=t}^{h} V(D_i)$ starting at w. Hence the concatenation of P and Q is a Hamilton path of D.

(2) Suppose to the contrary that D_1 has a Hamilton cycle $v_1 \cdots v_\ell v_1$. Then $D - v_1$ is traceable. Hence $D - V(D_1)$ has a Hamilton path P whose initial vertex has an in-neighbour v_i in D_1. But then $v_{i+1}v_{i+2} \cdots v_\ell v_1 \cdots v_i P$ is a Hamilton path of D. ■

We shall also use the following result of Grötschel and Wakabayashi [10].

Lemma 8 [10]**.** *Every nontrivial strong component of a hypotraceble digraph has order at least 5.*

The proof of our main theorem relies heavily on results proved in the papers [1, 3, 5, 7] and [9]. In the sequel, results extracted from these papers are stated without proof.

Lemma 9 [3, 5]**.** *Let $2 \leq k \leq n$ and let D be a k-traceable oriented graph of order n. Then the following hold.*

(1) *$d(v) \geq n-k+1$ for every $v \in V(D)$.*
(2) *$|N^+(x) \cup N^+(y)| \geq n-k+1$ and $|N^-(x) \cup N^-(y)| \geq n-k+1$ for every pair of distinct nonadjacent vertices $x, y \in V(D)$.*

Lemma 10 [3, 9]**.** *Let $2 \leq k \leq n$ and let D be a nontraceable k-traceable oriented graph of order n. Suppose x and y are distinct nonadjacent vertices in D and let*

$$S \in \left\{N^+(x),\ N^-(x),\ N^+(x) \cup N^+(y),\ N^-(x) \cup N^-(y)\right\}.$$

If $|S| = n-k+1$, then $\langle S \rangle$ is nontraceable.

We shall often use the following corollary of Lemma 10 in combination with Lemma 5.

Corollary 11 [3]**.** *Let $k \geq 2$ and suppose D is a nontraceable oriented graph of order n that is j-traceable for $j = k, k+1, \ldots, n-1$. If $x \in V(D)$ such that $d^+(x) \geq 2$ (or $d^-(x) \geq 2$) and P is a path in $D - x$ that contains all the out-neighbours (or all the in-neighbours) of x, then $n(P) \geq n-k+2$.*

We shall also use the next result, which is easily derived from Corollary 11.

Corollary 12. *Suppose $x \in V(D)$ such that $d^+(x) \geq 2$ (or $d^-(x) \geq 2$) and C is a t-cycle in $D-x$ such that $N^+(x)$ (or $N^-(x)$) is contained in C. If i consecutive vertices of C are not out-neighbours (or in-neighbours) of x, then $i \leq t-n+k-2$.*

Another consequence of Corollary 11 is the following.

Lemma 13. *Suppose D is an oriented graph of order $n \geq 2k-4$ with at most two strong components and D is j-traceable for $j = k, k+1, \ldots, n-1$. If D contains a t-cycle for some t such that $k-4 \leq t \leq n-k$, then D is traceable.*

Proof. Suppose D is nontraceable and D contains a t-cycle C, with $k-4 \leq t \leq n-k$. Then $k \leq n-t < n-1$, so D is $(n-t)$-traceable by our assumption. Hence $D - V(C)$ is traceable. Let $P = v_1 \cdots v_{n-t}$ be a Hamilton path of $D - V(C)$. Since D is nontraceable, $D - V(C)$ is nonhamitonian, so $v_{n-t} \notin N^-(v_1)$. We may assume that v_1 is not a source. (If it is, then v_{n-t} is not a sink and we consider it instead.) If v_1 has an in-neighbour on C, then D is traceable, so we may assume that $N^-(v_1) \subseteq V(P)$. But then $N^-(v_1)$ is contained in a path of order at most $n-t-3 \leq n-k+1$ (since $t \geq k-4$). This contradicts Corollary 11. ■

Lemma 14 [9]**.** *Let $k \geq 2$ and suppose D is a k-traceable oriented graph of order n. Then any nontrivial strong component of D that is nonhamiltonian has at least $n-k+5$ vertices.*

Lemma 15 [5]**.** *Let $k \geq 2$ and suppose D is a k-traceable oriented graph of order $n \geq 2k-3$. If D is nontraceable, then D has a nonhamiltonian strong component.*

Lemma 16 [4]. *If D is a k-traceable oriented graph with a strong component X such that $g(X) \geq 6$, then the order of D is at most $2k-4$ if k is odd, and at most $2k-5$ if k is even.*

Corollary 17. *If D is an 8-traceable oriented graph of order at least 12, then the girth of every nontrivial strong component of D is at most 5.*

We also need the following result, which is a special case of Lemma 17 of [1].

Lemma 18. *Let D be a k-traceable oriented graph of order $2k-3$ consisting of three strong components D_1, D_2, D_3 such that D_1 and D_3 are singletons. Then D is traceable if there is a vertex v in D_2 such that $d^-_{D_2}(v) < k-1$ and $d^+_{D_2}(v) < k-1$.*

3. The Main Result

Theorem 19. *Every 8-traceable oriented graph of order 12 is traceable.*

Proof. Suppose, to the contrary, that D is a nontraceable 8-traceable oriented graph of order 12. By Lemma 5, D is also 9-, 10- and 11-traceable and thus hypotraceable.

We note that Corollary 11 implies the following.

Claim 1. *If v is a vertex in $V(D)$, then any path in $D-v$ that contains all the in-neighbours (or all the out-neigbhours) of v has order at least 6.*

We distinguish three cases.

Case (i) *D is strong.* First, suppose D contains a 3-cycle C. Since D is 9-traceable, $D-V(C)$ has a Hamilton path, say $P = v_1 \cdots v_9$. First observe that since D is nontraceable, $D-V(C)$ is nonhamiltonian, and hence $v_9v_1 \notin A(D)$. Also, no vertex in C is an in-neighbour of the initial vertex of any Hamilton path in $D-V(C)$ and similarly, no vertex in C is an out-neighbour of the terminal vertex of any Hamilton path of $D-V(C)$. Hence $N^-(v_1) \subseteq \{v_3, \ldots, v_8\}$. Since D is hypotraceable, it follows from Observation 6 that $d^-(v_1) \geq 2$. Hence, by Corollary 11, $v_3, v_8 \in N^-(v_1)$. Similarly, $v_2, v_7 \in N^+(v_9)$. But now $Q = v_9v_2v_3v_4v_5 v_6v_7v_8v_1$, $R = v_4v_5v_6v_7v_8v_9v_2v_3v_1$ and $S = v_9v_7v_8v_1v_2v_3v_4v_5v_6$ are also Hamilton paths in $D-V(C)$.

Let $\mathcal{P}$ be the set of all Hamilton paths in $D-V(C)$. By considering the paths P and Q, we note that the initial vertex of any path in $\mathcal{P}$ is also the terminal vertex of some path in $\mathcal{P}$ and vice versa. Also, by considering the paths P and R, we observe that the fourth vertex of any path in $\mathcal{P}$ is an initial vertex of some path in $\mathcal{P}$. Hence v_4, the initial vertex of R, is the terminal vertex of some path in $\mathcal{P}$ and v_6, the terminal vertex of S, is an initial vertex of some path in $\mathcal{P}$. Also v_7, the fourth vertex of R, is the initial vertex of a path in $\mathcal{P}$ and the terminal vertex of another. Hence no vertex in C has a neighbour in $\{v_1, v_4, v_6, v_7, v_9\}$. But now $\langle\{v_1, v_4, v_6, v_7, v_9\} \cup V(C)\rangle$ is a nontraceable subdigraph of D with 8 vertices, contradicting that D is 8-traceable. Hence D does not contain a 3-cycle.

By Corollary 17, $g(G) < 6$ and by Lemma 13, D does not contain a 4-cycle. Hence $g(D) = 5$. Now let X be a 5-cycle in D.

By Claim 1, every vertex in $D - V(X)$ has at least one out-neighbour in $D - V(X)$. Hence $D - V(X)$ contains a cycle.

If $D - V(X)$ contains a 7-cycle, D is obviously traceable.

Now suppose Y is a 6-cycle in $D - V(X)$ and let v be the vertex of $D - (V(X) \cup V(Y))$. Then by Claim 1, v has an in- and an out-neighbour on Y. Hence, since D is nontraceable, v has no neighbour on X. But then by Corollary 11 every vertex in Y is an in-neighbour and an out-neighbour of v contradicting the fact that D is an oriented graph.

Hence we may assume that $D - V(X)$ contains no 6-cycle or 7-cycle. Now suppose Y is a 5-cycle in $D - V(X)$ and let $v_1, v_2 \in V(D) - (V(X) \cup V(Y))$. By Lemma 9(1), $\delta(D) \geq 5$, so we may assume without loss of generality that $d^+(v_1) \geq 3$ and $d^-(v_1) \geq 2$. If v_1, v_2 are nonadjacent, then it follows from Claim 1 that each of v_1 and v_2 has an in-neighbour as well as an out-neighbour on each of X and Y. Thus we may further assume that v_1 has two out-neighbours and an in-neighbour on X. But since D has no 3-, 4- or 6-cycle, this is not possible. Hence we may assume that $v_1v_2 \in A(D)$. But then, by Claim 1, v_1 has an in-neighbour in X and v_2 has an out-neighbour in Y. Hence D is traceable.

Case (ii) *D has exactly two strong components.* By Lemma 7(2) we may assume that neither D_1 nor D_2 is hamiltonian. Hence by Lemma 14 we may assume without loss of generality that D_1 is a singleton x and D_2 is nonhamiltonian, and $n(D_2) = 11$.

Now suppose D_2 contains a 3-cycle Z. Then since D is 9-traceable, $D - V(Z)$ has a 9-path $P = xv_1v_2 \cdots v_8$ and $N^+(v_8) \subseteq V(P)$. By Claim 1, $v_1, v_6 \in N^+(v_8)$, so $\langle V(P)\rangle$ contains an 8-cycle Y. By Observation 6, every vertex in Z has an out-neighbour on Y. Hence, since D is nontraceable, no vertex in Z is in $N^+(x)$. Thus by Lemma 9(1), x has at least five out-neighbours on Y and since D is nontraceable, the predecessor of each out-neighbour of x on Y is not an in-neighbour of any vertex in Z. Now let $z \in V(Z)$. Then z has at most three in-neighbours on Y and thus at most four in-neighbours in D_2, so $|N^-(z) \cup N^-(x)| \leq 4$, contradicting Lemma 9(2). Hence D has no 3-cycle. By Corollary 17, $g(G) < 6$ and by Lemma 13, D does not contain a 4-cycle. Hence $g(D) = 5$.

Let Z be an induced 5-cycle in D_2. By Corollary 11, every vertex in $D - V(Z)$ has an out-neighbour in $D - V(Z)$, so $D - V(Z)$ contains a cycle Y.

Suppose Y is a 6-cycle. By Claim 1, every vertex in Z has an out-neighbour in Y and hence every vertex in Z is the initial vertex of a Hamilton path in D_2. Therefore no vertex on Z is an out-neighbour of x. By Corollary 12, every vertex of Y is an out-neighbour of x. But, since D_2 is strong, at least one vertex on Y has an out-neighbour on X and hence D is traceable.

Therefore we may assume that Y is a 5-cycle. Hence neither $D_2 - V(Z)$ nor $D_2 - V(Y)$ has a 6-cycle. Let v be the vertex in $V(D_2) - (V(Z) \cup V(Y))$. If $v \notin N^+(x)$, then by Lemma 9(2), $d^-(v) \geq 5$, so v has at least three in-neighbours in at least one of Z and Y, say Y. However, Claim 1 implies that v also has an out-neighbour in Y, and then $D_2 - V(Z)$ has a 6-cycle. This contradiction shows that $xv \in A(D)$. But v has an out-neighbour in Y and every vertex in Y has at least one out-neighbour in X, so D is traceable.

Case (iii) *D has at least three strong components.* By Lemma 7(2) and Lemma 14, each of D_1 and D_h is either a singleton or has order at least 9. Since $n = 12$ it follows from Lemma 7(1) that D_1 and D_h are both singletons. By Lemma 7(1) and Lemma 8, either $h = 4$ and $n(D_2) = n(D_3) = 5$, or $h = 3$ and $n(D_2) = 10$.

Suppose the former. Let $D_1 = x$ and let S be a set of vertices in D such that $|S| = 8$ and $\{x \cup V(D_2)\} \subset S$. Since D is 8-traceable, $\langle S\rangle$ has a Hamilton path with subpath $xv_1 \cdots v_5$ where $v_i \in V(D_2)$, $i = 1, \ldots, 5$. Note that $\langle D - \{x, v_1, v_2, v_3\}\rangle$ has order 8 and therefore has a Hamilton path Q with initial arc v_4v_5. But then $xv_1v_2v_3Q$ is a Hamilton path in D — a contradiction.

Now suppose $h = 3$ and $n(D_2) = 10$. Let x and z be the vertices in D_1 and D_3 respectively. Note that if $xz \in A(D)$, then $D - xz$ is also a nontraceable 8-traceable oriented graph of order 12. Hence we may assume that $xz \notin A(D)$.

Claim 2. *Let $v \in V(D_2)$. If v is not a neighbour of x, then $d^-(v) \geq 6$. Similarly, if v is not a neighbour of z, then $d^+(v) \geq 6$.*

Proof. Suppose $v \notin N(x)$. Since $d^-(x) = 0$, Lemma 9(2) implies that $d^-(v) \geq 5$. Suppose $d^-(v) = 5$. Let $B = V(D_2) - \{v \cup N^-(v)\}$. Then $|B| = 4$. Since D is 8-traceable, the oriented graph $\langle V(D) - B\rangle$ has a Hamilton path $xu_1u_2u_3u_4u_5vz$ where $N^-(v) = \{u_1, \ldots, u_5\}$. Also $\langle B \cup \{x, u_1, v, z\}\rangle$ has a Hamilton path Q and, since u_1 is the only in-neighbour of v in this oriented graph, $u_1v \in A(Q)$. But then the path obtained by replacing the arc u_1v with the path $u_1u_2u_3u_4u_5v$ is a Hamilton path of D. This proves that $d^-(v) \geq 6$ and, similarly, $d^+(v) \geq 6$ if $v \notin N(z)$. This proves Claim 2. $\square$

Claim 3. *D_2 is nonhamiltonian.*

Proof. Suppose to the contrary that $v_1v_2 \cdots v_{10}v_1$ is a Hamilton cycle in D_2. Let $v_i \notin N^+(x)$ for some $i \in \{1, \ldots, 10\}$. Then by Claim 2, $v_i \in N^-(z)$ and hence, since D is nontraceable, $v_{i+1} \notin N^+(x)$. Since $D - x$ is traceable, we may assume, without loss of generality, that $v_1 \in N^-(z)$. Hence it follows by induction that $v_i \notin N^+(x)$ for $i = 2, \ldots, 10$, contradicting that $d^+(x) \geq 5$. This proves Claim 3. $\square$

Since D is hypotraceable, we may assume that $P = v_1 v_2 \cdots v_{10} z$ is a Hamilton path in $D - x$. It follows from Claim 2 that every vertex in D_2 is adjacent to at least one of x and z. In particular, $v_1 \notin N^+(x)$, so $v_1 \in N^-(z)$. Let v_ℓ be the last in-neighbour of v_1 on P. By Claim 3, $\ell \neq 10$ and hence it follows from Claim 1 that $\ell \in \{8, 9\}$. Let $C = v_1 \cdots v_\ell v_1$.

First we show that $xv_{\ell+1} \in A(D)$. Suppose to the contrary that $xv_{\ell+1} \notin A(D)$. Then it follows from Claim 2 that $v_{\ell+1}$ has at least six in-neighbours on C. If v_j is any in-neighbour of $v_{\ell+1}$ on C, then $v_{j+1} \notin N^+(x)$, since otherwise $xv_{j+1} \cdots v_\ell v_1 \cdots v_j v_{\ell+1} \cdots v_{10} z$ is a Hamilton path of D. Hence there are at least six vertices on C that are not out-neighbours of x. But this contradicts the fact that x has at least five out-neighbours in D_2. Hence $xv_{\ell+1} \in A(D)$.

Since D_2 is strong and $\ell \in \{8, 9\}$, v_{10} has at least one out-neighbour on C. By Claim 3, v_1 is not an out-neighbour of v_{10}. Also, v_2 is not an out-neighbour of v_{10} otherwise $xv_{\ell+1} \cdots v_{10} v_2 v_3 \cdots v_\ell v_1 z$ is a Hamilton path in D. Thus the out-neighbours of v_{10} in D_2 lie on the path $Q = v_3 v_4 \cdots v_8$. Note that by Claim 2, there is at most one vertex on $v_3 \cdots v_\ell$ that is not an in-neighbour of v_1. Let v_j and v_{j+r}, $r \geq 0$ be the first and last out-neighbours of v_{10} on Q. Since either v_{j+r} or v_{j+r+1} is an in-neighbour of v_1, it follows that $v_j v_{j+1} \cdots v_{j+r} v_1 z$ or $v_j v_{j+1} \cdots v_{j+r} v_{j+r+1} v_1 z$ is a path in $D - v_{10}$ that contains all the out-neighbours of v_{10}. Hence it follows from Claim 1 that $r \geq 2$ and hence v_{10} has at least two nonconsecutive out-neighbours on Q.

Hence v_7 and v_8 cannot be the only out-neighbours of v_{10} and we may therefore assume that v_j is an out-neighbour of v_{10} for some $j \in \{3, \ldots, 6\}$. Then $v_{j-1} z \notin A(D)$ otherwise $xv_{\ell+1} \cdots v_{10} v_j \cdots v_\ell v_1 \cdots v_{j-1} z$ is a Hamilton path in D. Hence by Claim 2, v_{j-1} has at least six out-neighbours in D_2. Since $v_{j-1} \in \{v_2, v_3, v_4, v_5\}$, v_{j-1} has at most three out-neigbhours on the path $v_1 \cdots v_{j-3}$ and therefore at least two out-neighbours on the path $v_{j+1} \cdots v_{10}$.

Now suppose $\ell = 8$. Then by Claim 2, $N^-(v_1) = \{v_3, \ldots, v_8\}$. In this case v_{j-1} has at least one out-neighbour v_i on the path $v_{j+1} \cdots v_9$, for $j \in \{3, \ldots, 6\}$. Then v_{i-1} lies on the path $v_j \cdots v_8$, for $j \in \{3, \ldots, 6\}$ and hence $v_{i-1} v_1 \in A(D)$. But now $v_{10} v_j v_{j+1} \cdots v_{i-1} v_1 v_2 \cdots v_{j-1} v_i \cdots v_{10}$ is a Hamilton cycle in D_2, contradicting Claim 3.

Finally suppose $\ell = 9$. If v_{10} is an out-neighbour of v_{j-1}, then $v_{j-1} v_{10} v_j \cdots v_9$ $v_1 \cdots v_{j-1}$ is a Hamilton cycle in D_2. Hence $v_{j-1} v_{10} \notin A(D)$. It therefore follows that v_{j-1} has at least two out-neighbours on the path $v_{j+1} \cdots v_9$. By Claim 2, at most one of the vertices on the path $v_j \cdots v_8$ is not an in-neighbour of v_1. Hence v_{j-1} has an out-neighbour v_i on $v_{j+1} \cdots v_9$ such that v_{i-1} is an in-neighbour of v_1. In this case $v_{10} v_j v_{j+1} \cdots v_{i-1} v_1 v_2 \cdots v_{j-1} v_i v_{i+1} \cdots v_{10}$ is a Hamilton cycle in D, again contradicting Claim 3. ■

Theorem 20. *Every* 8*-traceable oriented graph of order* 13 *is traceable.*

Proof. Suppose D is a nontraceable 8-traceable oriented graph of order 13. Then by Lemma 5 and Theorem 19, D is also 9-, 10-, 11- and 12-traceable.

By Lemmas 14 and 15, D has a nonhamiltonian strong component of order at least 10. Hence it follows from Lemma 8 and Lemma 7(1) that we only need to consider the following three cases.

Case (i) *D has three strong components.* In this case we can assume that $D_1 = x$ and $D_3 = z$ and $n(D_2) = 11$. Let $P = v_1v_2 \cdots v_{11}z$ be a Hamilton path in $D - x$. By Lemma 18 we may assume that either $d^-_{D_2}(v) \geq 7$ or $d^+_{D_2}(v) \geq 7$ for every $v \in V(D_2)$. If $v \notin N(x)$, then it follows from Lemma 9(2) that $d^-(v) \geq 6$. Then $d^+(v) \leq 12 - 6 = 6$ and hence $d^-(v) \geq 7$. Similarly $d^+(v) \geq 7$ for every vertex $v \notin N^-_{D_2}(z)$. Hence, $d^-_{D_2}(v_1) \geq 7$. Let v_ℓ be the last in-neighbour of v_1 on P. Since D_2 is nonhamiltonian, $\ell \in \{9, 10\}$. Let $C = v_1 \cdots v_\ell v_1$. By Lemma 9(1), x has at least four out-neighbours on C. If v_j is any out-neighbour of x on C, v_{j-1} is not in $N^-(v_{\ell+1})$, since otherwise $xv_j \cdots v_\ell v_1 \cdots v_{j-1}v_\ell v_{\ell+1} \cdots v_{11}z$ is a Hamilton path of D. Hence there are at least four vertices on C that are not in-neighbours of $v_{\ell+1}$. Since $v_{\ell+1}$ is not an in-neighbour of itself, $v_{\ell+1}$ has at most six in-neighbours in D_2 and therefore by Lemma 9(2) we may assume that $xv_{\ell+1} \in A(D)$ and that $N^+_{D_2}(v_{\ell+1}) \geq 7$.

Suppose $\ell = 9$. If $v_j \in N^+_C(v_{10})$, then v_{j-1} is not an in-neighbour of v_{11} otherwise $xv_{10}v_j \cdots v_9v_1 \cdots v_{j-1}v_{11}z$ is a Hamilton path of D. Hence there are at least six vertices on C that are not in-neighbours of v_{11} and therefore $d^-_{D_2}(v_{11}) < 7$ and hence by Lemma 18, $N^+_C(v_{11}) \geq 7$. But for each out-neighbour v_j of v_{11} in D_2, v_{j-1} is not an in-neighbour of z since otherwise $xv_{10}v_{11}v_j \cdots v_9v_1 \cdots v_{j-1}z$ is a Hamilton path of D. Hence $|N^-_{D_2}(z)| \leq 4$, contradicting Lemma 9(1).

Now suppose $\ell = 10$. Then v_{11} has at least seven out-neighbours in D_2. If v_j is an out-neighbour of v_{11}, then v_{j-1} is not an in-neighbour of z, since otherwise $xv_{11}v_j \cdots v_{10}v_1 \cdots v_{j-1}z$ is a Hamilton path of D. Again we have that $|N^-_{D_2}(z)| \leq 4$, which contradicts Lemma 9(2). This proves Case (i).

In Cases (ii) and (iii) it follows from Lemma 12 that D does not contain a 4-cycle or a 5-cycle and so by Corollary 17, D has a 3-cycle. Hence, for the remainder of the proof let $W = w_1w_2w_3w_1$ be a 3-cycle in D. Since D is 10-traceable, $D - V(W)$ has a Hamilton path $P = v_1 \cdots v_{10}$.

Case (ii) *D has two strong components.* In this case we may assume one strong component is trivial and the other has order 12. Hence, either v_1 is a source or v_{10} is a sink. Assume the former. Since D is nontraceable, $N^+(v_{10}) \subseteq \{v_2, \ldots, v_8\}$. It therefore follows from Corollary 11 that $v_2, v_8 \in N^+(v_{10})$. Now let C denote the 9-cycle $v_2v_3 \cdots v_{10}v_2$. From Observation 6, every vertex on W has an out-neighbour on C. Suppose w_1 is an out-neighbour of v_1. Then $v_1w_1w_2w_3$ followed by an out-neighbour of w_3 on C and the remainder of the vertices of C is a Hamilton path of D. Hence, $N^+(v_1) \subseteq C$ and by Lemma 9(1),

$|N_C^+(v_1)| \geq 6$. Now for each out-neighbour of v_1 on C, its predecessor on C is not an in-neighbour of any vertex of W, since otherwise D is traceable. But then $|N^-(w_1) \cup N^-(v_1)| \leq 4$, contradicting Lemma 9(2).

Case (iii) *D is strong.* Note that $D - V(W)$ is nonhamiltonian, since otherwise D would be traceable. Hence $v_{10} \notin N^-(v_1)$ and therefore by Corollary 11, $v_3, v_9 \in N^-(v_1)$ and $v_2, v_8 \in N^+(v_{10})$. But now $Q = v_{10}v_2v_3v_4v_5v_6v_7v_8v_9v_1$, $R = v_4v_5v_6v_7v_8v_9v_{10}v_2\ v_3v_1$ and $S = v_{10}v_8v_9v_1v_2v_3v_4v_5v_6v_7$ are also Hamilton paths in $D - V(W)$.

Let $\mathcal{P}$ be the set of all Hamilton paths in $D - V(W)$. By considering the paths P and Q, we note that for any path in $\mathcal{P}$ with initial vertex v_i and terminal vertex v_j, there is a path in $\mathcal{P}$ with initial vertex v_j and terminal vertex v_i. Hence, since $D - V(W)$ is nonhamiltonian, v_1 and v_{10} are nonadjacent.

Applying this observation to the paths R and S it follows that v_1 and v_4 are nonadjacent and v_7 and v_{10} are nonadjacent.

By considering the paths P and R, we observe that for any path $P \in \mathcal{P}$, there is a path in $\mathcal{P}$ whose initial vertex is the fourth vertex of P and whose terminal vertex is the first vertex of P. Hence by our previous observation, the first and fourth vertices of any path in $\mathcal{P}$ are nonadjacent.

Similarly, by considering the paths P and S, we observe that for any path in $P \in \mathcal{P}$, there is a path in $\mathcal{P}$ whose initial vertex is the terminal vertex of P and whose terminal vertex is the seventh vertex of P. Hence the seventh and terminal vertices of any path in $\mathcal{P}$ are nonadjacent.

By applying these observations to the paths Q and R it follows that v_4 and v_{10} are nonadjacent, v_1 and v_7 are nonadjacent, and v_4 and v_7 are nonadjacent.

Thus $\{v_1, v_4, v_7, v_{10}\}$ is an independent set and therefore, for any $w_i \in V(W)$, the set $\{w_i, v_1, v_4, v_7, v_{10}\}$ is an independent set of order 5 contradicting the fact that D is 8-traceable. ■

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