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A DEGREE CONDITION IMPLYING ORE-TYPE CONDITION FOR EVEN $[2, b]$-FACTORS IN GRAPHS

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Abstract

For a graph G and even integers $b \geqslant a \geqslant 2$, a spanning subgraph F of G such that $a \leqslant \deg_F(x) \leqslant b$ and $\deg_F(x)$ is even for all $x \in V(F)$ is called an even $[a, b]$-factor of G. In this paper, we show that a 2-edge-connected graph G of order n has an even $[2, b]$-factor if $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\left\{\frac{2n}{2+b}, 3\right\}$ for any nonadjacent vertices x and y of G. Moreover, we show that for $b \geqslant 3a$ and $a > 2$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies the condition $\deg_G(x) + \deg_G(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, but has no even $[a, b]$-factors. In particular, the infinite family of graphs gives a counterexample to the conjecture of Matsuda on the existence of an even $[a, b]$-factor.

Keywords: $[a, b]$-factor, even factor, 2-edge-connected, minimum degree.

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1. Introduction

In this paper, we consider only finite undirected graphs with no loops and no multiple edges. For a graph G, we let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex x of G, $\deg_G(x)$ denotes the *degree* of x in G. We let $\delta(G)$ denote the *minimum degree* of G. For two integers a and b with $1 \leqslant a \leqslant b$, a spanning subgraph F of G such that $a \leqslant \deg_F(x) \leqslant b$ for all $x \in V(F)$ is called an $[a,b]$*-factor* of G. A $[k,k]$-factor is usually called a k*-factor*. An $[a,b]$-factor F is said to be a *parity* $[a,b]$*-factor* if $\deg_F(x) \equiv a \equiv b \pmod 2$ for all $x \in V(F)$. In particular, a parity $[a,b]$-factor is an *even* $[a,b]$*-factor* if $a \equiv b \equiv 0 \pmod 2$.

We first introduce some known results on degree conditions for the existence of an even $[2,b]$-factor.

Theorem 1 (Kouider and Vestergaard [1])**.** *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $\delta(G) \geqslant \max\{\frac{2n}{2+b}, 3\}$, then G has an even $[2,b]$-factor.*

Theorem 2 (Matsuda [4])**.** *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $\deg_G(x) + \deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$ for any nonadjacent vertices x and y of G, then G has an even $[2,b]$-factor.*

In this paper, we prove the following theorem, which implies Theorems 1 and 2.

Theorem 3. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\left\{\frac{2n}{2+b}, 3\right\} \tag{1}$$

for any nonadjacent vertices x and y of G, then G has an even $[2,b]$-factor.

Let x and y be nonadjacent vertices of G. Then $\delta(G) \geqslant \max\{\frac{2n}{2+b}, 3\}$ implies $\deg_G(x)+\deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$, and $\deg_G(x)+\deg_G(y) \geqslant \max\{\frac{4n}{2+b}, 5\}$ implies $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\{\frac{2n}{2+b}, 3\}$. Hence Theorem 3 implies Theorems 1 and 2.

Additionally, we here show that Theorem 3 is stronger than Theorem 2. In order to show that, we construct an infinite family of graphs as follows: For a positive integer t and an even integer $b \geqslant 4$, we define the graph G_0 obtained from $\frac{b}{2}$ cliques $K_t^1, K_t^2, \ldots, K_t^{\frac{b}{2}}$ of order t and one vertex v_0 by joining a vertex v_0 to two vertices of K_t^i for each $1 \leqslant i \leqslant \frac{b}{2}$ (a clique means a complete graph), and let $\mathcal{G}_0 = \{G_0(b,t) \mid t \in \mathbb{Z}^+, b \in 2\mathbb{Z}^+, t > \frac{b^2+b-6}{b-2}\}$. For each $G_0 \in \mathcal{G}_0$, it is easily seen that G_0 is 2-edge-connected, and that the order of G_0 is $n = \frac{b}{2}t + 1$. By the definition of G_0, we have $n > \frac{b^3+b^2-4b-4}{2(b-2)}$. Hence it follows that if $b \geqslant 4$, then

$$\deg_{G_0}(x) + \deg_{G_0}(v_0) = |V(K_t^1)| - 1 + b = t - 1 + b = \frac{2n-2}{b} - 1 + b < \frac{4n}{2+b},$$

and

$$\max\{\deg_{G_0}(x), \deg_{G_0}(v_0)\} = |V(K_t^1)| - 1 = t - 1 = \frac{2n-2}{b} - 1 > \frac{2n}{2+b}$$

for any vertex $x \in \left(\bigcup_{1 \leqslant i \leqslant \frac{b}{2}} V(K_t^i)\right) \setminus N_{G_0}(v_0)$. Thus Theorem 3 guarantees the existence of an even $[2, b]$-factor in G_0, but Theorem 2 does not. Consequently, Theorem 3 is stronger than Theorem 2.

In order to prove Theorem 3, we actually prove the following two theorems, which are obtained from Theorem 3 by dividing it into two cases on the order n of a graph G.

Theorem 4. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $n \geqslant b+3$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b} \tag{2}$$

for any nonadjacent vertices x and y of G, then G has an even $[2, b]$-factor.

Theorem 5. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n. If $n \leqslant b+2$ and*

$$\max\{\deg_G(x), \deg_G(y)\} \geqslant 3 \tag{3}$$

for any nonadjacent vertices x and y of G, then G has an even $[2, b]$-factor.

Combining these, we can obtain Theorem 3.

In the rest of this section, we discuss extending "an even $[2, b]$-factor" in Theorem 3 to "an even $[a, b]$-factor" briefly. In 2005, Matsuda [4] posed the following conjecture as a natural generalization of Theorem 2.

Conjecture 6 (Matsuda [4])**.** *Let $2 \leqslant a \leqslant b$ be even integers, and let G be a 2-edge-connected graph of order $n \geqslant 2a + b + \frac{a^2-3a}{b} - 2$. If $\delta(G) \geqslant a$ and $\deg_G(x) + \deg_G(y) \geqslant \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, then G has an even $[a, b]$-factor.*

In 2004, Kouider and Vestergaard constructed an infinite family of k-connected graphs G^* of order n with $\delta(G^*) \geqslant \frac{an}{a+b}$ having no even $[a, b]$-factors such that $b > 3a^2$, $k \leqslant a - 1$ and k is odd (see Example 3 in [2]). If n is sufficiently large and $k \geqslant 3$, then the graph G^* satisfies the hypothesis of Conjecture 6. Thus G^* is a kind of counterexamples in the case where $b > 3a^2$. Nevertheless, Conjecture 6 was open when $b \leqslant 3a^2$.

In this paper, we also prove that Conjecture 6 does not hold even when $3a \leqslant b \leqslant 3a^2$. Furthermore, we prove that the similar degree condition to (1) (i.e., $\max\{\deg_G(x), \deg_G(y)\} > \frac{an}{a+b}$) does not guarantee the existence of an even $[a, b]$-factor even when the difference of a and b is not so large.

Proposition 7. *Let $4 \leqslant a \leqslant b$ be even integers. Then the following assertions hold:*

(i) *For $b \geqslant 3a$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies $\deg_G(x) + \deg_G(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G, but has no $[a,b]$-factors.*

(ii) *For $b > a$, there exists an infinite family of 2-edge-connected graphs G of order n with $\delta(G) \geqslant a$ such that G satisfies $\max\{\deg_G(x), \deg_G(y)\} > \frac{an}{a+b}$ for any nonadjacent vertices x and y of G, but has no $[a,b]$-factors.*

Although Conjecture 6 is not true in the case where $b \geqslant 3a$ by Proposition 7, the case where $3a > b \geqslant a$ is still open.

The organization of the paper is as follows. In Section 2, Proposition 7 is described in detail. We introduce preliminaries used in our proofs of Theorems 4 and 5 in Section 3, and we show the sharpness of Theorems 4 and 5 in Section 4. In Section 5, we prove Theorems 4 and 5.

2. Construction of Graphs Without Even $[a,b]$-Factors

In this section, we mention in more detail on Proposition 7. We here construct an infinite family, which gives a new counterexample to the conjecture of Matsuda.

Construction of the family $\mathcal{G}^*$. For an integer t and even integers a and b such that $t \geqslant a+2$ and $b \geqslant a \geqslant 4$, we construct a graph $G^*(a,b,t)$ as follows: Recall that a clique means a complete graph. Let C^0, C_t^1, C_t^2 be three disjoint cliques of order 2, t and t, respectively. Let $V(C^0) = \{x, y\}$, and let $u_1, u_2, \ldots, u_{a-1}$ (resp., $v_1, v_2, \ldots, v_{a-1}$) be distinct $a-1$ vertices of C_t^1 (resp., C_t^2). We define the graph $G^*(a,b,t)$ obtained from C^0, C_t^1 and C_t^2 by adding xv_1, xu_i, yu_1 and yv_i for $2 \leqslant i \leqslant a-1$ (see Figure 1), and let $\mathcal{G}^* = \{G^*(a,b,t) \mid t \geqslant a+2, b \geqslant a \geqslant 4\}$.

For each $G^* \in \mathcal{G}^*$, it is easy to check the following:

(i) $\delta(G^*) = a$ $(= \deg_{G^*}(x) = \deg_{G^*}(y))$,

(ii) the order of G^* is $n \geqslant 2a + b + \frac{a^2-3a}{b} - 2$ if t is large enough,

(iii) G^* is 2-edge-connected from $a \geqslant 4$.

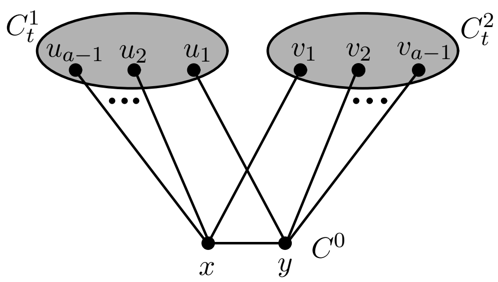

Figure 1. The graph $G^*(a,b,t)$.

Lemma 8. *Let $4 \leqslant a \leqslant b$ be even integers. Then the following assertions hold:*

(i) *For $b \geqslant 3a$, every graph $G^* \in \mathcal{G}^*$ satisfies $\deg_{G^*}(x) + \deg_{G^*}(y) > \frac{2an}{a+b}$ for any nonadjacent vertices x and y of G^*.*

(ii) *For $b > a$, every graph $G^* \in \mathcal{G}^*$ satisfies $\max\{\deg_{G^*}(x), \deg_{G^*}(y)\} > \frac{an}{a+b}$ for any nonadjacent vertices x and y of G^*.*

Proof. (i) Let $G^* \in \mathcal{G}^*$. By the construction of G^*, the following two facts hold:

(F1) Vertices having the minimum degree are only x and y, and $G^*[\{x, y\}]$ is a clique;

(F2) Vertices having the second smallest degree belong to $V(C_t^1) \setminus N_{G^*}(V(C^0))$ or to $V(C_t^2) \setminus N_{G^*}(V(C^0))$, each of which is nonadjacent to x and y.

In view of (F1) and (F2), it suffices to check the degree condition only for two vertices $w \in V(C_t^1) \setminus N_{G^*}(V(C^0))$ and $z \in V(C^0)$. By $b \geqslant 3a$ and $a \geqslant 4$, we obtain

$$\deg_{G^*}(w) + \deg_{G^*}(z) = t - 1 + a \geqslant t + 3 = \frac{n}{2} + 2 > \frac{2an}{a+b}.$$

(ii) Let $G^* \in \mathcal{G}^*$. Similarly to the proof of (i), it suffices to check the degree condition only for two vertices $w \in V(C_t^1) \setminus N_{G^*}(V(C^0))$ and $z \in V(C^0)$. By $b > a \geqslant 4$, we get

$$\max\{\deg_{G^*}(w), \deg_{G^*}(z)\} = t - 1 = \frac{n}{2} - 2 > \frac{an}{a+b}.$$

■

Lemma 9. *Every graph $G^* \in \mathcal{G}^*$ has no even $[a, b]$-factors.*

Proof. Suppose that $G^* \in \mathcal{G}^*$ has an even $[a, b]$-factor F. Since $\deg_{G^*}(x) = a = \deg_{G^*}(y)$, we obtain $\deg_F(x) = a = \deg_F(y)$. Also, since $|V(C_t^1) \cap N_F(V(C^0))| = a - 1$ is odd, $F[V(C_t^1)]$ is a graph having odd number of vertices with odd degree. This is a contradiction. ■

By Lemmas 8, 9 and the construction of $\mathcal{G}^*$, Proposition 7 can be proved if t is large enough.

3. Preliminaries

In this section, we give notation and lemmas used in our proofs of Theorems 4 and 5.

Our notation is standard possibly except the following. Let G be a graph. For a vertex x of G, $N_G(x)$ denotes the set of vertices adjacent to x in G; $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N_G(A)$ denote the union of $N_G(x)$ as x ranges over A. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, $e_G(A, B)$ denotes the number of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the subgraph of G induced by A is denoted by $G[A]$, and $G-A$ denotes the subgraph $G[V(G)-A]$. A vertex set A is called *independent* if $G[A]$ has no edges.

In our proofs of Theorems 4 and 5, we depend on the following lemma, which is a special case of the parity (g,f)-factor theorem of Lovász [3] (for this necessary and sufficient criterion, an alternative proof was given by Tutte [5]).

Lemma 10 (Lovász [3]). *Let $b \geqslant 2$ be an even integer, and let G be a graph. Then G has an even $[2,b]$-factor if and only if*

$$\begin{aligned}\theta_G(S,T) &:= b|S| + \sum_{y\in T}(\deg_{G-S}(y)-2) - h_G(S,T)\\ &= b|S| + \sum_{y\in T}(\deg_G(y)-2) - e_G(S,T) - h_G(S,T) \geqslant 0\end{aligned}$$

for all disjoint subsets S and T of $V(G)$, where $h_G(S,T)$ is the number of components C of $G-S-T$ such that $e_G(V(C),T) \equiv 1 \pmod 2$, and such a component C is briefly called an odd component of $G-S-T$.

In addition to the above lemma, we use the following two lemmas in our proofs. Since they are well-known, we omit the proofs (see [4] in detail).

Lemma 11. *Let G be a graph, and let S and T be disjoint subsets of $V(G)$. Then the following assertion hold:*

$$\theta_G(S,T) \equiv 0 \pmod 2.$$

Lemma 12. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph. Let S and T be disjoint subsets of $V(G)$ for which $\theta_G(S,T) \leqslant -2$. Then the following assertions holds:*

(i) $2|T| \geqslant b|S| + 2$,
(ii) $|T| \geqslant 2$.

For a graph G satisfying the hypothesis in Theorem 3, we show the following lemma.

Lemma 13. *Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order n such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant \max\{\frac{2n}{2+b}, 3\}$ for any nonadjacent vertices x and y of G. Assume that there exist disjoint subsets S and T of $V(G)$ satisfying $\theta_G(S,T) \leqslant -2$. Choose such subsets S and T so that $|T|$ is as small as possible. Then the following assertions hold:*

(i) *T is an independent set of G,*
(ii) $\sum_{y\in T}\deg_G(y) \geqslant 3|T| - 1$.

Proof. To prove (i), let $T' = T - \{v\}$ for any $v \in T$. Then $T' \neq \emptyset$ by Lemma 12(ii). By the choice of T and Lemma 11, we have $\theta_G(S, T') \geqslant 0$ and $\theta_G(S, T) \leqslant -2$. Thus, by subtracting these inequalities, $2 \leqslant \theta_G(S, T') - \theta_G(S, T) \leqslant -\deg_{G-S}(v) + 2 + h_G(S, T) - h_G(S, T')$, which implies $\deg_{G-S}(v) \leqslant h_G(S, T) - h_G(S, T')$. This inequality together with $e_G(v, V(G) - S - T) \geqslant h_G(S, T) - h_G(S, T')$ yields $\deg_{G[T]}(v) = \deg_{G-S}(v) - e_G(\{v\}, V(G) - S - T) \leqslant 0$, which means that T is an independent subset of $V(G)$. Thus, (i) holds.

Suppose that there exist two vertices $x, y \in T$ satisfying $\deg_G(x) = \deg_G(y) = 2$. Then, in the case where $n \geqslant b + 3$, by (i) and the condition of Theorem 4, we have $2 = \max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}$, which contradicts $n \geqslant b + 3$. In the case where $n \leqslant b + 2$, by (i) and the condition of Theorem 5, we have $2 = \max\{\deg_G(x), \deg_G(y)\} \geqslant 3$, a contradiction. In either case, we obtain a contradiction. Hence T has at most one vertex t with $\deg_G(t) = 2$. Consequently, we have $\sum_{y \in T} \deg_G(y) \geqslant 3(|T| - 1) + 2 = 3|T| - 1$. Thus, (ii) holds. ■

4. Sharpness of Theorems 4 and 5

In this section, we discuss the sharpness of Theorems 4 and 5. In Theorems 4 and 5, the degree conditions (2) and (3) are best possible. Moreover, the hypothesis "2-edge-connected" cannot be dropped. For Theorem 4, the lower bound of the order (i.e., "$b + 3$") is sharp. Although our result is a generalization of Theorem 2, the examples in [4] are applicable to Theorems 4 and 5 as they stand. Here we include them for the convenience of the reader.

Example 1. The degree condition (2) is best possible in the sense that we cannot replace $\frac{2n}{2+b}$ with $\frac{2n-2}{2+b}$ (noting that $\frac{2n-1}{2+b}$ cannot be an integer, and thus an integer $a > \frac{2n}{2+b}$ if and only if $a > \frac{2n-1}{2+b}$). To check it, we construct an infinite family of 2-edge-connected graphs G_1 of order sufficiently large n without even $[2, b]$-factors such that the degree condition of G_1 is a little smaller than $\frac{2n}{2+b}$ as follows: For a positive integer t and an even integer $b \geqslant 2$, let K_{2t} (resp., $(bt+1)K_1$) be a clique of order $2t$ (resp., $bt+1$ cliques of order 1). We define the graph $G_1(b, t)$ obtained by joining K_{2t} and $(bt + 1)K_1$, and let $\mathcal{G}_1 = \{G_1(b, t) \mid t \in \mathbb{Z}^+, b \geqslant 2 \text{ is even}\}$. For each $G_1 \in \mathcal{G}_1$, the order of G_1 is $n = (2 + b)t + 1$ and G_1 is 2-edge-connected. Also, it follows that

$$\frac{2n}{2+b} > \max\{\deg_{G_1}(x), \deg_{G_1}(y)\} = 2t = \frac{2n}{2+b} - \frac{2}{2+b} > \frac{2n}{2+b} - 1$$

for any nonadjacent vertices $x, y \in V((bt + 1)K_1)$. However, G_1 has no $[2, b]$-factors as $b|V(K_{2t})| < 2|V((bt + 1)K_1)|$.

Example 2. The condition "2-edge-connected" in Theorem 4 cannot be deleted for $b \geqslant 6$. To check it, we construct an infinite family of connected graphs G_2

of order sufficiently large n without even $[2,b]$-factors such that G_2 satisfies the condition $\max\{\deg_{G_2}(x), \deg_{G_2}(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices $x, y \in V(G_2)$, but is not 2-edge-connected as follows: For a positive integer t and an even integer $b \geqslant 6$, we define the graph $G_2(t)$ obtained from two cliques K_t^1, K_t^2 and one vertex v_0 by joining a vertex v_0 to a vertex of K_t^1 and to a vertex of K_t^2, and let $\mathcal{G}_2 = \{G_2(t) \mid t \in \mathbb{Z}^+\}$. For each $G_2 \in \mathcal{G}_2$, G_2 is not 2-edge-connected. Also, the order of G_2 is $n = 2t+1$, and it follows that $\max\{\deg_{G_2}(u), \deg_{G_2}(v_0)\} = \deg_{G_2}(u) = t-1 = \frac{n-3}{2} \geqslant \frac{2n}{2+b}$ for any vertex $u \in (V(K_t^1) \setminus N_{G_2}(v_0)) \cup (V(K_t^2) \setminus N_{G_2}(v_0))$ for $b \geqslant 6$. However, G_2 has no even $[2,b]$-factors. In fact, putting $S = \emptyset$ and $T = \{v_0\}$ in Lemma 10, we can check that both K_t^1 and K_t^2 are odd components of $G - S - T$, and thus $\theta_{G_2}(\emptyset, \{v_0\}) = \deg_{G_2}(v_0) - 2 - 2 = -2 < 0$.

Example 3. The lower bound of order $n \geqslant b+3$ in Theorem 4 is sharp for $b \geqslant 4$. To check it, we construct an infinite family of 2-edge-connected graphs G_3 of order $n = b+2$ without even $[2,b]$-factors such that G_3 satisfies the condition $\max\{\deg_{G_3}(x), \deg_{G_3}(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices x and y of G_3 as follows: For an even integer $b \geqslant 4$, we define the graph $G_3(b)$ obtained from two vertices v_1, v_2 and a path P_b of order b by joining each v_i to two endvertices of P_b, and let $\mathcal{G}_3 = \{G_3(b) \mid b \geqslant 4 \text{ is even}\}$. For each $G_3 \in \mathcal{G}_3$, G_3 is 2-edge-connected. Also, the order of G_3 is $n = b+2$, and it follows that $\max\{\deg_{G_3}(v_1), \deg_{G_3}(v_2)\} = 2 = \frac{2n}{2+b}$. However, it is clear that G_3 has no $[2,b]$-factors. Note that G_3 also shows that the degree condition (3) in Theorem 5 is best possible in the sense that we cannot replace 3 with 2.

5. Proof of Theorems 4 and 5

In this section, we prove Theorems 4 and 5. Suppose that a graph G satisfies the hypothesis of Theorems 4 or 5. By Lemmas 10 and 11, it suffices to show that there exist no disjoint subsets S and T of $V(G)$ for which

$$\theta_G(S,T) \leqslant -2. \tag{4}$$

5.1. Proof of Theorem 4

Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order $n \geqslant b+3$ such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}$ for any nonadjacent vertices x and y of G. By way of contradiction, suppose that G does not have an even $[2,b]$-factor. Then by Lemmas 10 and 11, there exist disjoint subsets S and T of G satisfying (4). We choose such S and T so that $|T|$ is as small as possible.

Let $t_1, t_2, \ldots, t_{|T|}$ be the vertices of T. Note that $|T| \geqslant 2$ by Lemma 12(ii). Without loss of generality, we may assume that $\deg_G(t_1) \leqslant \deg_G(t_2) \leqslant \cdots \leqslant$

$\deg_G(t_{|T|})$. By Lemma 13(i), $T = \{t_1, t_2, \ldots, t_{|T|}\}$ is an independent set of G. Consequently, by the condition of Theorem 4, we have

$$\max\{\deg_G(t_1), \deg_G(t_i)\} = \deg_G(t_i) \geqslant \frac{2n}{2+b}$$

for each $2 \leqslant i \leqslant |T|$. By this inequality, we obtain

$$\sum_{y \in T} \deg_G(y) = \sum_{y \in T \setminus \{t_1\}} \deg_G(y) + \deg_G(t_1) \geqslant (|T| - 1)\frac{2n}{2+b} + \deg_G(t_1). \tag{5}$$

We divide the proof into two cases on the cardinality of $|T|$.

Case 1. $|T| \geqslant b + 1$.

Claim 14. $|S| \leqslant \frac{2n}{2+b} - 1$.

Proof. Suppose that $|S| > \frac{2n}{2+b} - 1$, i.e., $2n-(2+b)|S| < 2+b$. Since the both sides of this inequality are even, $2n - (2+b)|S| \leqslant b$ holds. By $n \geqslant |S|+|T|+h_G(S,T)$, this implies

$$\begin{aligned} 2|T| - b|S| &\leqslant 2(n - |S| - h_G(S,T)) - b|S| \\ &= 2n - (2+b)|S| - 2h_G(S,T) \leqslant b - 2h_G(S,T). \end{aligned}$$

Thus, it follows from (4) and $2|T| - b|S| \leqslant b - 2h_G(S,T)$ that

$$\begin{aligned} \sum_{y \in T} \deg_{G-S}(y) &\leqslant 2|T| - b|S| + h_G(S,T) - 2 \\ &\leqslant b - 2h_G(S,T) + h_G(S,T) - 2 \leqslant b - 2. \end{aligned}$$

Since $|T| \geqslant b + 1$, there exist at least two vertices x and y of T such that $\deg_{G-S}(x) = \deg_{G-S}(y) = 0$. Therefore by the condition of Theorem 4, we have

$$|S| \geqslant \max\{\deg_G(x), \deg_G(y)\} \geqslant \frac{2n}{2+b}. \tag{6}$$

On the other hand, by Lemma 12(i) and $n \geqslant |S| + |T| + h_G(S,T)$, we have $2(n-|S|-h_G(S,T)) \geqslant 2|T| \geqslant b|S|+2$, which implies $|S| \leqslant \frac{2(n-h_G(S,T)-1))}{2+b} < \frac{2n}{2+b}$. This contradicts (6). □

By (4), (5), Claim 14, $e_G(S,T) \leqslant |S||T|$, $h_G(S,T) \leqslant n - |S| - |T|$ and $b + 1 - |T| \leqslant 0$ (by the assumption of Case 1), we obtain

$$\begin{aligned} -2 &\geqslant \theta_G(S,T) \\ &\geqslant b|S| + (|T| - 1) \cdot \frac{2n}{2+b} + \deg_G(t_1) - |S||T| - 2|T| - (n - |S| - |T|) \end{aligned}$$

$$
\begin{aligned}
&= (b+1-|T|)|S| + \left(\frac{2n}{2+b}-1\right)|T| + \deg_G(t_1) - \frac{2n}{2+b} - n \\
&\geqslant (b+1-|T|)\left(\frac{2n}{2+b}-1\right) + \left(\frac{2n}{2+b}-1\right)|T| + \deg_G(t_1) - \frac{2n}{2+b} - n \\
&= b\left(\frac{2n}{2+b}-1\right) + \deg_G(t_1) - 1 - n,
\end{aligned}
$$

which implies $\deg_G(t_1) \leqslant \frac{(2-b)n}{2+b} + b - 1$. If $b \geqslant 4$, then by $n > b+2$,

$$
\deg_G(t_1) \leqslant \frac{n}{2+b}(2-b) + b - 1 < 2 - b + b - 1 = 1,
$$

i.e., $\deg_G(t_1) = 0$, which means that t_1 is an isolated vertex. If $b = 2$, then $\deg_G(t_1) \leqslant 1$ holds. In either case, we get a contradiction because G is 2-edge-connected.

Case 2. $|T| \leqslant b$. By Lemma 12(i), we have $|S| < \frac{2|T|}{b} \leqslant 2$, which means that $|S| = 0$ or $|S| = 1$.

Let h_1 (resp., h_2) be the number of odd components C of $G-S-T$ such that $e_G(V(C),T) = 1$ (the number of odd components C of $G-S-T$ such that $e_G(V(C),T) \neq 1$, i.e., $e_G(V(C),T) \geqslant 3$). Then $h_G(S,T) = h_1 + h_2$.

Claim 15. $|S| = 1$.

Proof. Suppose that $|S| = 0$, i.e., $S = \emptyset$. Since G is 2-edge-connected, we obtain $h_1 = 0$. Then $h_G(\emptyset, T) = h_2$ holds. Hence it follows from (4) and $\sum_{y\in T} \deg_G(y) \geqslant 3h_2$ that

$$
\begin{aligned}
-2 \geqslant \theta_G(\emptyset, T) &= \sum_{y\in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\
&\geqslant 3h_2 - 2|T| - h_2 = 2h_2 - 2|T|,
\end{aligned}
$$

implying $|T| \geqslant h_2 + 1$. By this inequality, (4) and Lemma 13(ii), we have

$$
\begin{aligned}
-2 \geqslant \theta_G(\emptyset, T) &= \sum_{y\in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\
&\geqslant (3|T|-1) - 2|T| - h_2 = |T| - h_2 - 1 \geqslant 0.
\end{aligned}
$$

This is a contradiction. □

Since $\sum_{y\in T} \deg_{G-S}(y) \geqslant h_1 + 3h_2$ and $h_G(S,T) = h_1 + h_2$, it follows from Claim 15 and (4) that

$$
-2 \geqslant \theta_G(S,T) \geqslant b + (h_1 + 3h_2) - 2|T| - (h_1 + h_2) = 2h_2 - 2|T| + b,
$$

that is,

$$
|T| \geqslant h_2 + \frac{b+2}{2}. \tag{7}
$$

Claim 16. $h_1 \geqslant \frac{b+4}{2}$.

Proof. By (4), (7), Lemma 13(ii), Claim 15, $e_G(S,T) \leqslant |S||T| \leqslant b$ and $h_G(S,T) = h_1 + h_2$,we obtain

$$\begin{aligned} -2 \geqslant \theta_G(S,T) &\geqslant b + (3|T| - 1) - b - 2|T| - (h_1 + h_2) \\ &\geqslant |T| - h_1 - h_2 - 1 \geqslant \frac{b+2}{2} - h_1 - 1, \end{aligned}$$

which implies $h_1 \geqslant \frac{b+4}{2}$, as desired. □

For each $1 \leqslant i \leqslant h_1$, let C'_i be the odd components of $G - S - T$ such that $e_G(V(C'_i, T) = 1$. Without loss of generality, we may assume that $|C'_1| \leqslant |C'_2| \leqslant \cdots \leqslant |C'_{h_1}|$. Note that there exist at least two components C'_1 and C'_2 by Claim 16. For two vertices $u_1 \in V(C'_1)$ and $u_2 \in V(C'_2)$, it follows from the definition of C'_i, Claim 15 and the condition of Theorem 4 that

$$\begin{aligned} \frac{2n}{2+b} &\leqslant \max\{\deg_G(u_1), \deg_G(u_2)\} \\ &\leqslant \max\{|C'_1| - 1 + e_G(u_1, S \cup T), |C'_2| - 1 + e_G(u_2, S \cup T)\} \\ &\leqslant \max\{|C'_1| + 1, |C'_2| + 1\} = |C'_2| + 1, \end{aligned}$$

that is, $|C'_2| \geqslant \frac{2n}{2+b} - 1$. Hence, we have

$$\sum_{i=1}^{h_1} |C'_i| \geqslant |C'_1| + (h_1 - 1)\left(\frac{2n}{2+b} - 1\right).$$

It follows from this inequality, (7) and Claim 16 that

$$\begin{aligned} n &\geqslant |S| + |T| + |C'_1| + (h_1 - 1)\left(\frac{2n}{2+b} - 1\right) \\ &\geqslant 1 + h_2 + \frac{b+2}{2} + |C'_1| + \frac{b+2}{2}\left(\frac{2n}{2+b} - 1\right) > n, \end{aligned}$$

which is a contradiction. Consequently, this completes the proof of Theorem 4.

5.2. Proof of Theorem 5

Let $b \geqslant 2$ be an even integer, and let G be a 2-edge-connected graph of order $n \leqslant b + 2$ such that $\max\{\deg_G(x), \deg_G(y)\} \geqslant 3$ for any nonadjacent vertices x and y of G. By way of contradiction, suppose that G does not have an even $[2, b]$-factor. Then by Lemmas 10 and 11, there exist disjoint subsets S and T of G satisfying (4). We choose such S and T so that $|T|$ is as small as possible.

By Lemma 12(i), $|T| \geqslant \frac{b|S|}{2} + 1$. If $|S| \geqslant 2$, then we obtain $n \geqslant |S| + |T| \geqslant |S| + \left(\frac{b|S|}{2} + 1\right) \geqslant b + 3$, which contradicts that $n \leqslant b + 2$. Hence we have that $|S| = 0$ or $|S| = 1$.

Claim 17. $|S| = 1$.

Proof. Suppose that $|S| = 0$, i.e., $S = \emptyset$. Since G is 2-edge-connected, all of the odd components C of $G - T$ satisfy $e_G(V(C), T) \geqslant 3$. By (4),

$$\begin{aligned} -2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(x) - 2|T| - h_G(\emptyset, T) \\ &\geqslant 3h_G(\emptyset, T) - 2|T| - h_G(\emptyset, T) = 2h_G(\emptyset, T) - 2|T|, \end{aligned}$$

implying

$$|T| \geqslant h_G(\emptyset, T) + 1. \tag{8}$$

Then it follows from (4), (8) and Lemma 13(ii) that

$$\begin{aligned} -2 \geqslant \theta_G(\emptyset, T) &= \sum_{y \in T} \deg_G(y) - 2|T| - h_G(\emptyset, T) \\ &\geqslant (3|T| - 1) - 2|T| - h_G(\emptyset, T) = |T| - h_G(\emptyset, T) - 1 \geqslant 0. \end{aligned}$$

This is a contradiction. □

By (4), Lemma 13(ii), Claim 17 and $e_G(S, T) \leqslant |T|$, we have

$$\begin{aligned} h_G(S, T) &\geqslant b + \sum_{y \in T} \deg_G(y) - e_G(S, T) - 2|T| + 2 \\ &\geqslant b + (3|T| - 1) - |T| - 2|T| + 2 = b + 1. \end{aligned}$$

Therefore by the above inequality and Lemma 12(ii), we obtain $n \geqslant |S| + |T| + h_G(S, T) \geqslant 1+2+(b+1) \geqslant b+4$, which contradicts the assumption that $n \leqslant b+2$. This completes the proof of Theorem 5.

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