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PACKING COLORING OF SOME UNDIRECTED AND ORIENTED CORONAE GRAPHS

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two distinct vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$. For a given integer $p \geq 1$, the p-corona of a graph G is the graph obtained from G by adding p degree-one neighbors to every vertex of G. In this paper, we determine the packing chromatic number of p-coronae of paths and cycles for every $p \geq 1$.

Moreover, by considering digraphs and the (weak) directed distance between vertices, we get a natural extension of the notion of packing coloring to digraphs. We then determine the packing chromatic number of orientations of p-coronae of paths and cycles.

Keywords: packing coloring, packing chromatic number, corona graph, path, cycle.

2010 Mathematics Subject Classification: 05C15, 05C70, 05C05.

1. Introduction

All the graphs we considered are simple and loopless. For an undirected graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The *distance* $d_G(u, v)$, or simply $d(u, v)$, between vertices u and v in G is the length (number

of edges) of a shortest path joining u and v. The *diameter* of G is the maximum distance between two vertices of G. We denote by P_n the path of order n and by C_n, $n \geq 3$, the cycle of order n.

A *packing k-coloring* of G is a mapping $\pi : V(G) \to \{1, \ldots, k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d(u, v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k-coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$. A packing coloring of G is *optimal* if it uses exactly $\chi_\rho(G)$ colors.

Packing coloring has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [12, 13] under the name *broadcast coloring* and has been studied by several authors in recent years. Several papers deal with the packing chromatic number of certain classes of graphs such as trees [3, 4, 13, 16, 17], lattices [4, 5, 9, 10, 14, 18], Cartesian products [4, 9, 16], distance graphs [6, 7, 19] or hypercubes [13, 20, 21]. Complexity issues of the packing coloring problem were adressed in [1, 2, 3, 8, 11, 13].

The following proposition, which states that having packing chromatic number at most k is a hereditary property, will be useful in the sequel.

Proposition 1 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [13])**.** *If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.*

Fiala and Golovach [8] proved that determining the packing chromatic number is an NP-hard problem for trees. Determining the packing chromatic number of special subclasses of trees is thus an interesting problem. The exact value of the packing chromatic number of trees with diameter at most 4 was given in [13]. In the same paper, it was proved that $\chi_\rho(T_n) \leq (n + 7)/4$ for every tree T_n of order $n \neq 4, 8$, and this bound is tight, while $\chi_\rho(T_n) \leq 3$ if $n = 4$ and $\chi_\rho(T_n) \leq 4$ if $n = 8$, these two bounds being also tight.

The packing chromatic numbers of paths and cycles have been determined by Goddard *et al.*

Theorem 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [13])**.**

- $\chi_\rho(P_n) = 2$ *if* $n \in \{2, 3\}$,
- $\chi_\rho(P_n) = 3$ *if* $n \geq 4$,
- $\chi_\rho(C_n) = 3$ *if* $n = 3$ *or* $n \equiv 0 \pmod 4$,
- $\chi_\rho(C_n) = 4$ *if* $n \geq 5$ *and* $n \equiv 1, 2, 3 \pmod 4$.

The *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbor to every vertex of G. We call such a degree-one neighbor a *pendant vertex* or a *pendant neighbor*. More generally, for a given integer $p \geq 1$,

the *p-corona* $G \odot pK_1$ of a graph G is the graph obtained from G by adding p pendant neighbors to every vertex of G.

A *caterpillar of length* $\ell \geq 1$ is a tree whose set of internal vertices (vertices with degree at least 2) induces a path of length $\ell - 1$, called the *central path*. Sloper proved the following result.

Theorem 3 (Sloper [17])**.** *Let CT_ℓ be a caterpillar of length ℓ. Then $\chi_\rho(CT_\ell) \leq 6$ if $\ell \leq 34$, and $\chi_\rho(CT_\ell) \leq 7$ otherwise. Moreover, these two bounds are tight.*

Since every p-corona of a path is a caterpillar, we get that for every integer $p \geq 1$, $\chi_\rho(P_n \odot pK_1) \leq 6$ if $n \leq 34$ and $\chi_\rho(P_n \odot pK_1) \leq 7$ otherwise.

By considering digraphs instead of undirected graphs, and using the (*weak*) *directed distance* between vertices — defined as the number of arcs in a shortest directed path linking these vertices, in either direction — we get a natural extension of packing colorings to digraphs. In this paper, we will consider *orientations* of some undirected graphs, obtained by giving to each edge of such a graph one of its two possible orientations. The so-obtained *oriented graphs* are thus digraphs having no pair of opposite arcs.

In this paper, we determine the packing chromatic number of (simple) coronae of paths and cycles (Section 2) and of p-coronae (for $p \geq 2$) of paths and cycles (Section 3). In Section 4, we consider the oriented version of packing colorings and determine the packing chromatic number of oriented paths, oriented cycles and oriented p-coronae of paths and cycles. Some of the presented results for undirected graphs were obtained by the first author in [15].

2. Coronae of Undirected Paths and Cycles

We study in this section coronae of paths and cycles. William and Roy proved in [22] that for every $n \geq 8$, $\chi_\rho(P_n \odot K_1) \leq 5$. We complete their result in the following theorem which gives the exact value of the packing chromatic number of the corona of any path. Note that any corona $P_n \odot K_1$ is a caterpillar of length n.

Theorem 4. *The packing chromatic number of the corona graph $P_n \odot K_1$ is given by*

$$\chi_\rho(P_n \odot K_1) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \in \{2, 3\}, \\ 4 & \text{if } 4 \leq n \leq 9, \\ 5 & \text{if } n \geq 10. \end{cases}$$

Proof. We obviously have $\chi_\rho(P_1 \odot K_1) = \chi_\rho(P_2) = 2$. Optimal packing colorings of $P_n \odot K_1$ are given in Figure 1 for every n, $2 \leq n \leq 9$. Since $P_2 \odot K_1 = P_4$, we have $\chi_\rho(P_2 \odot K_1) = 3$ by Theorem 2. Observe that the packing 3-coloring

of $P_3 \odot K_1$ depicted in Figure 1 is unique, up to the exchange of colors 2 and 3. Hence, if $P_4 \odot K_1$ would be packing 3-colorable, one of these packing 3-colorings of $P_3 \odot K_1$ would appear on the left or right hand side of $P_4 \odot K_1$. But in that case, the fourth vertex of the central path of $P_4 \odot K_1$ could not be colored. Hence $\chi_\rho(P_4 \odot K_1) = 4$. Finally, since $P_2 \odot K_1$ is a subgraph of $P_3 \odot K_1$ and $P_4 \odot K_1$ is a subgraph of $P_n \odot K_1$ for every n, $5 \leq n \leq 9$, all the packing colorings given in Figure 1 are optimal by Proposition 1.

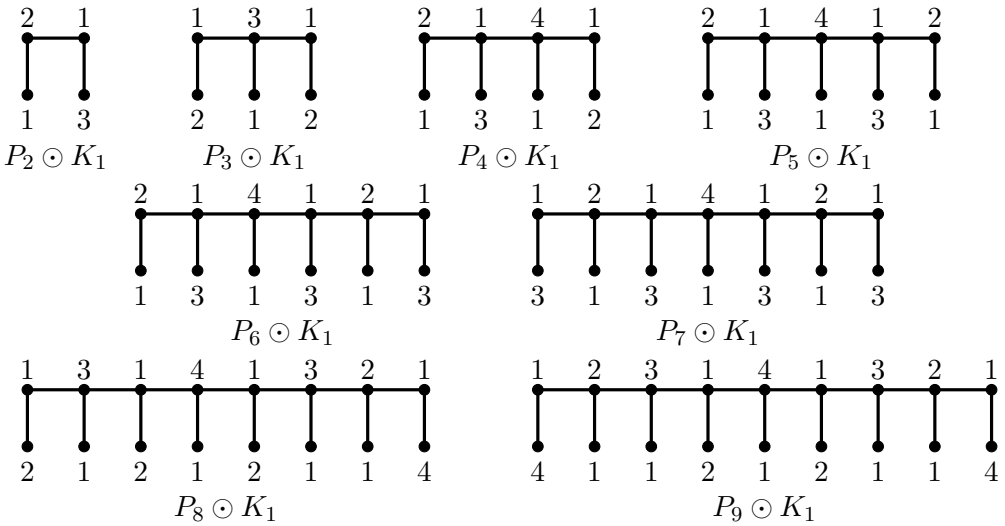

Figure 1. Optimal packing colorings of $P_n \odot K_1$, $2 \leq n \leq 9$.

Let us now consider $P_n \odot K_1$ with $n \geq 10$. Let $x_1 x_2 \cdots x_n$ denote the central path of $P_n \odot K_1$ and y_i denote the pendant neighbor of x_i for every i, $1 \leq i \leq n$. Let π be the 4-periodic 5-coloring of $P_n \odot K_1$ defined as follows (see Figure 2).

$$\pi(x_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 2, \\ 2 & \text{if } i \equiv 2 \pmod 4, \\ 3 & \text{if } i \equiv 0 \pmod 4, \end{cases}$$

$$\pi(y_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2, \\ 4 & \text{if } i \equiv 1 \pmod 4, \\ 5 & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

It is not difficult to check that π is indeed a packing 5-coloring of $P_n \odot K_1$ and, therefore, $\chi_\rho(P_n \odot K_1) \leq 5$ for every $n \geq 10$.

To finish the proof, it is enough to prove that $\chi_\rho(P_{10} \odot K_1) \geq 5$, thanks to Proposition 1. By computer exhaustive search, we get that the largest packing 4-colorable corona of path is $P_9 \odot K_1$, which admits two distinct packing 4-colorings:

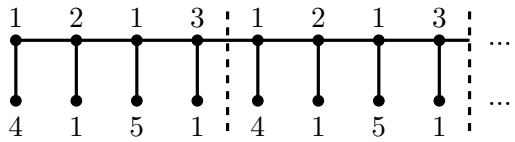

Figure 2. Periodic packing coloring of $P_n \odot K_1$, $n \geq 8$.

one is given in Figure 1, the other one is obtained by coloring the middle pendant vertex by 2 instead of 1. ■

In [23], William, Roy and Rajasingh proved that $\chi_\rho(C_n \odot K_1) \leq 5$ for every even $n \geq 6$. We complete their result as follows.

Theorem 5. *The packing chromatic number of the corona graph $C_n \odot K_1$ is given by*

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

Proof. Optimal packing 4-colorings of $C_3 \odot K_1$ and $C_4 \odot K_1$ are given in Figure 3. We claim indeed that these two coronae graphs cannot be packing 3-colored. If there would exist such colorings then color 1 would necessarily be used for the cycle and its two neighbors on the cycle would get colors 2 and 3. But then, it would not be possible to color the pendant neighbor of the vertex with color 1.

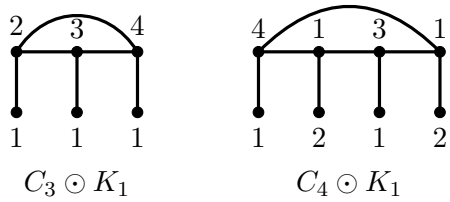

Figure 3. Optimal packing colorings of $C_3 \odot K_1$ and $C_4 \odot K_1$.

Let us now consider $C_n \odot K_1$ with $n \geq 5$. Figure 4 describes 5-colorings of $C_5 \odot K_1$, $C_6 \odot K_1$ and $C_7 \odot K_1$. Figure 3 describes "almost 4-periodic" packing 5-colorings of $C_n \odot K_1$, $n \geq 8$, according to the value of n mod 4 (the leftmost pattern of length 4 can be repeated any number of times). It is not difficult to check that all these colorings are indeed packing 5-colorings and, therefore, $\chi_\rho(C_n \odot K_1) \leq 5$ for every $n \geq 5$.

It remains to prove that $\chi_\rho(C_n \odot K_1) \geq 5$ for every $n \geq 5$. Assume to the contrary that there exists a packing 4-coloring of $C_5 \odot K_1$. By "unfolding" this coloring and considering it as a pattern of a 5-periodic coloring for coronae of

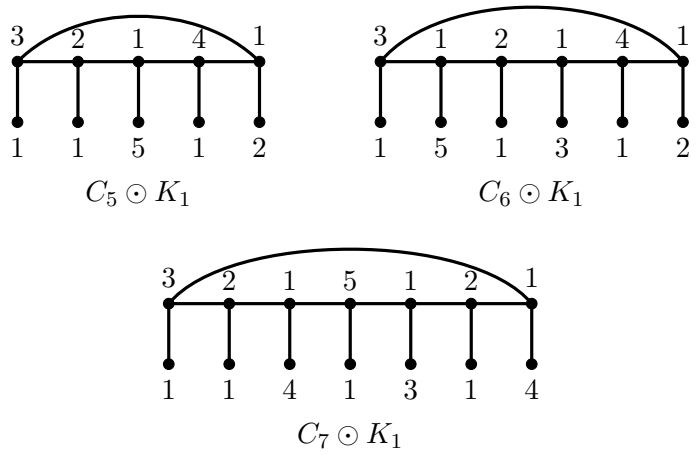

Figure 4. Optimal packing colorings of $C_5 \odot K_1$, $C_6 \odot K_1$ and $C_7 \odot K_1$.

paths we obtain a packing 4-coloring of every corona graph $P_n \odot K_1$, $n \geq 5$, in contradiction with Theorem 4. The same argument proves that there is no packing 4-coloring of $C_n \odot K_1$ for every $n \geq 6$. This completes the proof. ■

3. p-Coronae of Undirected Paths and Cycles

As observed in the introduction, we know, by Theorem 3, that for every integer $p \geq 1$, $\chi_\rho(P_n \odot pK_1) \leq 6$ if $n \leq 34$ and $\chi_\rho(P_n \odot pK_1) \leq 7$ otherwise.

When considering p-coronae of paths or cycles, the following proposition is useful.

Proposition 6. *Let $P_n = x_1 \cdots x_n$, $n \geq 2$, be a path and $P_n \odot pK_1$, $p \geq 1$, be a p-corona of P_n. Any packing coloring π of $P_n \odot pK_1$ with $\pi(x_i) = 1$ for some vertex x_i must use at least $p+3$ colors if $2 \leq i \leq n-1$, or at least $p+2$ colors if $i \in \{1, n\}$.*

Similarly, if $C_n \odot pK_1$, $p \geq 3$, is a p-corona of $C_n = y_1 \cdots y_n$, then any packing coloring π' of $C_n \odot pK_1$ with $\pi'(y_i) = 1$ for some vertex y_i must use at least $p+3$ colors.

Proof. To see that, simply note that if $\pi(x_i) = 1$ then no two neighbors of x_i can receive the same color. Since the degree of x_i is $p+2$ if $2 \leq i \leq n-1$, or $p+1$ if $i \in \{1, n\}$, the claim follows. The proof if similar for $C_n \odot pK_1$. ■

In order to describe packing colorings of p-coronae of paths and cycles, we will use the following notation in the rest of this paper. Observe first that whenever a

vertex of the path, or the cycle, in any such graph is colored with a color distinct from 1, all the pendant vertices attached to this vertex can be colored 1. Hence, it is necessary to give the colors of the pendant vertices only when the color of their neighbor is 1. In that case, these colors will be given within parenthesis, following the color 1. Such a sequence of colors, called a *pattern*, can thus unambigously describe a packing coloring of a (p-)corona of a given path. For instance, the colorings of $P_4 \odot K_1$ and $P_5 \odot K_1$ given in the previous section (see Figure 1) will be denoted by 21(3)41(2) and 21(3)41(3)2, respectively. For packing colorings of (p-)coronae of cycles, we will put the whole sequence of colors in brackets in order to emphasize the fact that the pattern is circular. For instance, the colorings of $C_5 \odot K_1$ and $C_6 \odot K_1$ given in the previous section (see Figure 4) will be denoted by [321(5)41(2)] and [31(5)21(3)41(2)], respectively.

Let u and v be two words on the alphabet of colors, such that $[u]$ is a circular pattern. We will say that the pattern v is *compatible with* $[u]$ if $[uv]$ is a circular pattern.

The value of the packing chromatic number of p-coronae of paths $P_n \odot pK_1$ with $p \geq 4$ is given by the following theorem.

Theorem 7. *Let $P_n \odot pK_1$, $p \geq 4$, be a p-corona of the path P_n. Then we have*

$$\chi_\rho(P_n \odot pK_1) = \begin{cases} 2 & \textit{if } n = 1, \\ 3 & \textit{if } n = 2, \\ 4 & \textit{if } n \in \{3, 4\}, \\ 5 & \textit{if } 5 \leq n \leq 8, \\ 6 & \textit{if } 9 \leq n \leq 34, \\ 7 & \textit{otherwise.} \end{cases}$$

Proof. If $n \leq 8$, optimal packing colorings of $P_n \odot pK_1$ are given by the patterns 2, 23, 234, 2342, 23425, 234253, 2342532 and 23425324, respectively.

Note that 23425324 is the longest pattern on five colors which do not use color 1 and, moreover, none of the patterns 123425324 or 234253241 can be used for coloring $P_9 \odot 4K_1$ (the pendant neighbors of vertices with color 1 cannot be colored). Therefore, $\chi_\rho(P_9 \odot pK_1) \geq 6$. In [17], Sloper exhibited the following pattern of length 34, which uses colors 2 to 6, and proved that no such pattern of greater length exists:

23425 62342 53264 23524 62352 43265 2342.

As before, this pattern cannot be extended by adding color 1 to the left or to the right, so that $\chi_\rho(P_{35} \odot pK_1) \geq 7$. Sloper also gave the circular pattern

[23425 62342 57],

of length 12, that uses colors 2 to 7, which can be used when $n \geq 35$. By Proposition 6, all these colorings are optimal. ∎

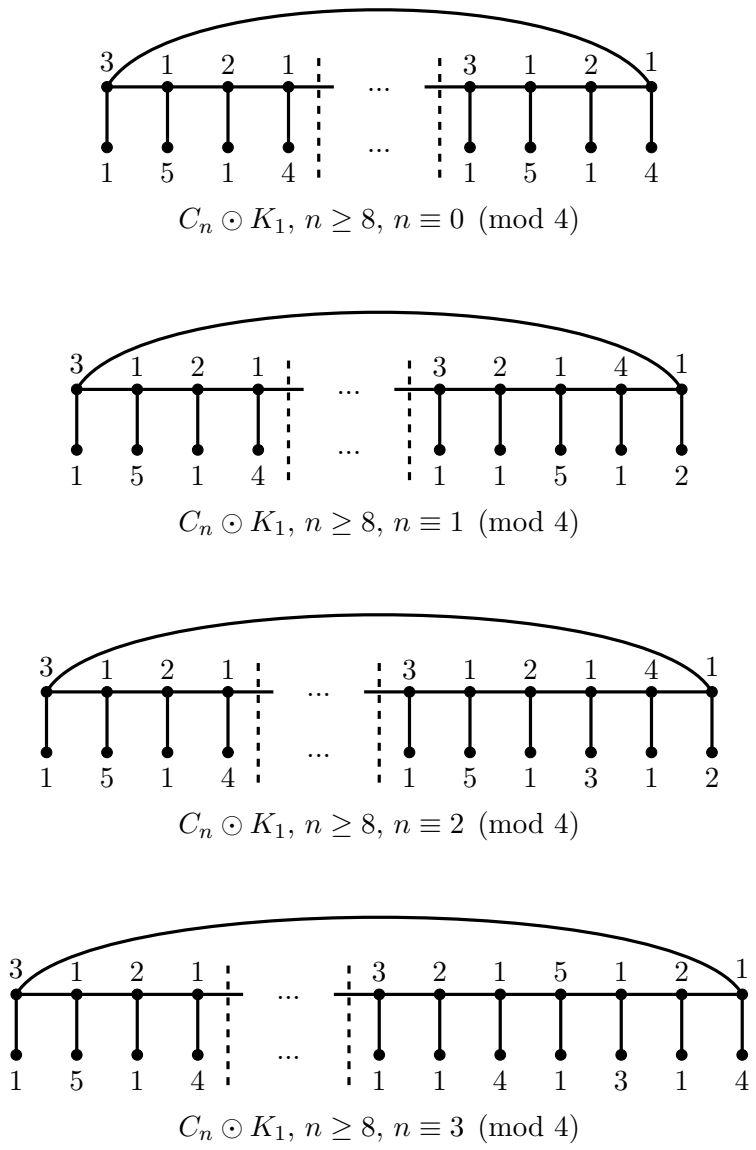

Figure 5. Optimal packing colorings of $C_n \odot K_1$, $n \geq 8$.

The value of the packing chromatic number of p-coronae of paths $P_n \odot pK_1$, when $p \in \{2, 3\}$, is given by the next two results. We will see that the maximum value of the packing chromatic number of such graphs is 6, slightly better than the bound given in Theorem 7. This is due to the fact that the number of pendant vertices is now bounded by 3, which allows us to use color 1 for coloring the vertices of the path P_n.

Theorem 8. *Let $P_n \odot 2K_1$ be the 2-corona of the path P_n. Then we have*

$$\chi_\rho(P_n \odot 2K_1) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } 5 \leq n \leq 11, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. To see that $\chi_\rho(P_n \odot 2K_1) \leq 6$ for every n, it is enough to use the following circular pattern of length 12:

$$[1(36)2432\ 56234\ 25].$$

Since $P_m \odot pK_1$ is a subgraph of $P_n \odot pK_1$ for all $m \leq n$, every packing ℓ-coloring of $P_n \odot pK_1$ induces a packing ℓ-coloring of $P_m \odot pK_1$. Therefore, it suffices to construct optimal packing colorings of $P_1 \odot 2K_1$, $P_2 \odot 2K_1$, $P_4 \odot 2K_1$ and $P_{11} \odot 2K_1$, to get that all the claimed values are upper bounds. This can be done by using the patterns 2, 23, 2342 and 1(35)243251(23)4231(25), respectively.

To finish the proof, we need to show that all these bounds are tight. This is obvious for $n = 1$ and this is a direct consequence of Proposition 6, for $2 \leq n \leq 4$, since it implies that we cannot use color 1 on the vertices of the path, so that no packing coloring using less colors than stated in the theorem can exist in those cases. For $n = 5$, Proposition 6 again implies that we cannot use color 1 for the vertices of P_5 in a packing 4-coloring and it is easily checked that no such pattern exists (the longest one is 2342). Finally, we have to check that there exists no packing 5-coloring of $P_{12} \odot 2K_1$. We did it by means of a computer program. ■

Theorem 9. *Let $P_n \odot 3K_1$ be the 3-corona of the path P_n. Then we have*

$$\chi_\rho(P_n \odot 3K_1) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } 5 \leq n \leq 8, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. To see that $\chi_\rho(P_n \odot 3K_1) \leq 6$ for every n, it is enough to consider the following circular pattern of length 14:

$$[1(234)5234\ 26325\ 4326].$$

As before, it suffices to construct optimal packing colorings of $P_1 \odot 3K_1$, $P_2 \odot 3K_1$, $P_4 \odot 3K_1$ and $P_8 \odot 3K_1$, to get that all the claimed values are upper bounds. This can be done by using the patterns 2, 23, 2342 and 23425324, respectively.

To finish the proof, we need to show that all these bounds are tight. This is obvious for $n = 1$ and this is a direct consequence of Proposition 6, for $n \in \{2, 3, 5, 9\}$, since it implies that we cannot use color 1 on the vertices of the path. It is then not difficult to check that the longest such patterns are the ones given above, and the result follows. ■

We now turn to p-coronae of cycles $C_n \odot pK_1$. When $p \geq 4$, we have the following (note the particular case when $n = 11$).

Theorem 10. *Let $C_n \odot pK_1$, $p \geq 4$, be a p-corona of the cycle C_n. Then we have*

$$\chi_\rho(C_n \odot pK_1) = \begin{cases} 4 & \textit{if } n = 3, \\ 5 & \textit{if } n = 4, \\ 6 & \textit{if } n \in \{5, 6\}, \\ 8 & \textit{if } n = 11, \\ 7 & \textit{otherwise.} \end{cases}$$

Proof. Note first that by Proposition 6, since $p \geq 4$, color 1 cannot be used on the vertices of C_n in any packing coloring of $C_n \odot pK_1$ using at most 6 colors.

Packing colorings of $C_n \odot pK_1$, for $3 \leq n \leq 6$, are given by the following circular patterns:

$$[234] \quad [2345] \quad [23456] \quad [234256].$$

It is not difficult to check that these packing colorings are optimal.

On the other hand, a packing 8-coloring of $C_{11} \odot pK_1$ is given by the following circular pattern:

$$[23425324678].$$

Let us show that no packing 7-coloring of $C_{11} \odot pK_1$ can exist. If color 1 is not used then, due to the length of the cycle, color 2 can be used at most three times, colors 3 and 4 at most twice each, and colors 5, 6 and 7 at most once each. Hence, at most 10 vertices of the cycle can be colored. Now, if color 1 is used on the cycle, then the pendant vertices must be colored 2, 3, 4 and 5, as otherwise the packing coloring cannot be extended far enough. The coloring is then "forced" around the color 1 as $\ldots 43271(2345)6234 \ldots$. It is then easy to check that this pattern cannot be extended to a packing 7-coloring of $C_{11} \odot pK_1$ (the smallest extension has length 14 and is given by [43271(2345)623425362]).

Packing 7-colorings of $C_n \odot pK_1$, for $7 \leq n \leq 15$, $n \neq 11$, are given by the following circular patterns:

$$\begin{aligned} n = 7: &\quad [2342567]; \\ n = 8: &\quad [23425367]; \\ n = 9: &\quad [234253267]; \end{aligned}$$

$$\begin{aligned} n = 10: &\quad [2342532467]; \\ n = 12: &\quad [234253246257]; \\ n = 13: &\quad [2342532462357]; \\ n = 14: &\quad [23425362432576]; \\ n = 15: &\quad [234253264235276]. \end{aligned}$$

Moreover, all the above circular patterns for $n \geq 9$ are compatible with the circular pattern [23425367] of length 8. Hence, if $n \geq 16$, $n = 8q + r$ with $0 \leq r \leq 7$, $r \neq 3$, a packing 7-coloring of $C_n \odot pK_1$ can be obtained by combining $q - 1$ patterns of length 8 followed by a pattern of length $q + r$ (if $r = 0$, we thus have q occurrences of the pattern of length 8).

Finally, for $n = 8q + 3$, $q \geq 2$, a packing 7-coloring of $C_n \odot pK_1$ can be obtained by combining $q - 2$ patterns of length 8 followed by the circular pattern [2342532462352432657] of length 19, which is also compatible with [23425367]. This concludes the proof. ■

We now consider the remaining cases, that is $p \in \{2, 3\}$. For $p = 2$, we have the following (note the particular case when $n = 9$).

Theorem 11. *Let $C_n \odot 2K_1$ be the 2-corona of the cycle C_n. Then we have*

$$\chi_\rho(C_n \odot 2K_1) = \begin{cases} 4 & \textit{if } n = 3, \\ 5 & \textit{if } n = 4, \\ 7 & \textit{if } n = 9, \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. The packing colorings of $C_n \odot 2K_1$, for $n \leq 13$, $n \neq 9$ are given by the following circular patterns:

$$\begin{aligned} n = 3: &\quad [234]; \\ n = 4: &\quad [2345]; \\ n = 5: &\quad [23456]; \\ n = 6: &\quad [234256]; \\ n = 7: &\quad [1(23)423526]; \\ n = 8: &\quad [1(24)3251(24)326]; \\ n = 10: &\quad [1(23)41(23)523421(35)6]; \\ n = 11: &\quad [1(23)4231(25)624325]; \\ n = 12: &\quad [1(23)41(23)521(26)423526]; \\ n = 13: &\quad [1(23)41(23)5231(26)423526]. \end{aligned}$$

It is not difficult to check that these colorings are optimal for $n \leq 6$. For $n \geq 7$, any packing 5-coloring of $C_n \odot 2K_1$ would induce a packing 5-coloring of $P_{12} \odot 2K_1$, in contradiction with Theorem 8.

We now consider the case $n \geq 14$. Similarly, no packing 5-coloring of $C_n \odot 2K_1$ can exist in this case. All the patterns given above for $n \geq 8$ are compatible with the circular pattern [1(23)423526] of length 7. Moreover, the pattern 423524326 of length 9 is also compatible with the same pattern [1(23)423526]. This allows us to construct a packing 6-coloring of any 2-corona $C_n \odot 2K_1$ with $n \geq 14$. If $n = 7q + r$, with $q \geq 2$ and $0 \leq r < 7$, the coloring is obtained by repeating $q - 1$ times the pattern u of length 7 and adding the compatible pattern of length $7 + r$ (note that since the pattern u is a circular pattern, it is compatible with itself).

The last case to consider is the case $n = 9$. A packing 7-coloring of $C_9 \odot 2K_1$ is given by the circular pattern

$$[1(24)3251(24)3267].$$

It is then tedious but not difficult to check that $C_9 \odot 2K_1$ does not admit any packing 6-coloring. (The main idea is that in such a case, each of the colors 4, 5 and 6 can be used only once on the vertices of C_9 while the color 3 can be used at most twice and the color 2 at most three times, so that color 1 has to be used on some vertex of C_9; but in that case, the colors assigned to the pendant neighbors of this vertex forces the color 1 to be used again on the cycle, leading eventually to a contradiction.) ■

Finally, for $p = 3$, we have the following.

Theorem 12. *Let $C_n \odot 3K_1$ be the 3-corona of the cycle C_n. Then we have*

$$\chi_\rho(C_n \odot 3K_1) = \begin{cases} 4 & \text{if } n = 3, \\ 5 & \text{if } n = 4, \\ 7 & \text{if } n \in \{7, \ldots, 13, 15, \ldots, 22, 24, \ldots, 27, 30, \ldots, 36, 39, 40, 41\} \\ & \qquad \cup \{45, 47, \ldots, 50, 53, 54, 55, 59, 62, 63, 64, 68, 77, 78, 91\}, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 10 and Proposition 1, we know that $\chi_\rho(C_n \odot 3K_1) \leq 7$ for every $n \geq 3$, $n \neq 11$. Packing colorings of $C_3 \odot 3K_1$, $C_4 \odot 3K_1$, $C_5 \odot 3K_1$ and $C_6 \odot 3K_1$ are given by the following circular patterns:

$$[234], \quad [2345], \quad [23456], \quad [234256],$$

whose optimality is easy to check.

Table 1 gives, as circular patterns, packing 6-colorings of $C_n \odot 3K_1$ for every $n \in \{14, 23, 29, 38, 44, 46, 61, 67, 69, 73, 76, 82, 92\}$ (pendant neighbors of vertices colored 1 are always assigned colors 2, 3 and 4). Since all these patterns begin with $152342\ldots$ and end with $\ldots 524326$, they are all pairwise compatible. Therefore, by repeating the pattern of length 14 a certain number of times, and adding one of the patterns of Table 1, we can produce a packing 6-coloring of $C_n \odot 3K_1$ in all the following cases, according to the value of n mod 14.

n	circular pattern
14	[1523426325 4326]
23	[1523426324 5236423524 326]
29	[1523426324 5236423524 623524326]
38	[1523426324 5236243251 6234253246 23524326]
44	[1523426324 5236243251 6234253264 2352462352 4326]
46	[1523426324 5236423524 3261523426 3245236423 524326]
61	[1523426324 5236243251 6234253246 2352432615 2342632452 3642352432 6]
67	[1523426324 5236243251 6234253246 2352432615 2342632452 3642352462 3524326]
69	[1523426324 5236423524 3261523426 3245236423 5243261523 4263245236 423524326]
73	[1523426324 5236243251 6234253264 2352462352 4326152342 6324523642 3524623524 326]
76	[1523426324 5236243251 6234253246 2352432615 2342632452 3624325162 3425324623 524326]
82	[1523426324 5236243251 6234253246 2352432615 2342632452 3624325162 3425326423 5246235243 26]
92	[1523426324 5236423524 3261523426 3245236423 5243261523 4263245236 4235243261 5234263245 2364235243 26]

Table 1. Circular patterns for the proof of Theorem 12.

- $n = 14q$, $n \geq 14$,
- $n = 14q + 1$, $n \geq 29$ (by repeating $q - 2$ times the pattern of length 14 and adding the pattern of length 29),
- $n = 14q + 2$, $n \geq 44$ (by repeating $q - 3$ times the pattern of length 14 and adding the pattern of length 44),
- $n = 14q + 3$, $n \geq 73$ (by repeating $q - 5$ times the pattern of length 14 and adding the pattern of length 73),
- $n = 14q + 4$, $n \geq 46$ (by repeating $q - 3$ times the pattern of length 14 and adding the pattern of length 46),
- $n = 14q + 5$, $n \geq 61$ (by repeating $q - 4$ times the pattern of length 14 and adding the pattern of length 61),
- $n = 14q + 6$, $n \geq 76$ (by repeating $q - 5$ times the pattern of length 14 and adding the pattern of length 76),
- $n = 14q + 7$, $n \geq 105$ (by repeating $q - 7$ times the pattern of length 14 and adding the patterns of length 44 and 61),
- $n = 14q + 8$, $n \geq 92$ (by repeating $q - 6$ times the pattern of length 14 and adding the pattern of length 92),
- $n = 14q + 9$, $n \geq 23$ (by repeating $q - 1$ times the pattern of length 14 and adding the pattern of length 23),

- $n = 14q + 10$, $n \geq 38$ (by repeating $q - 2$ times the pattern of length 14 and adding the pattern of length 38),
- $n = 14q + 11$, $n \geq 67$ (by repeating $q - 4$ times the pattern of length 14 and adding the pattern of length 67),
- $n = 14q + 12$, $n \geq 82$ (by repeating $q - 5$ times the pattern of length 14 and adding the pattern of length 82),
- $n = 14q + 13$, $n \geq 69$ (by repeating $q - 4$ times the pattern of length 14 and adding the pattern of length 69).

It is now easy to check that the remaining values of n, for which a packing 6-coloring cannot be produced in this way, are exactly those given in the statement of the theorem. The fact that, for each of these values, $\chi_\rho(C_n \odot 3K_1) = 7$ has been checked by means of a computer program. ■

4. Oriented Paths, Oriented Cycles and Their p-Coronae

In this section, we extend the notion of packing colorings to digraphs and study the case of oriented graphs whose underlying undirected graph is a path, a cycle, or a p-corona of a path or a cycle.

Let $\overrightarrow{D}$ be a digraph, with vertex set $V(\overrightarrow{D})$ and arc set $E(\overrightarrow{D})$. A *directed path* of length k in $\overrightarrow{D}$ is a sequence $u_0 \cdots u_k$ of vertices of $V(\overrightarrow{D})$ such that for every i, $0 \leq i \leq k-1$, $u_i u_{i+1}$ is an arc in $E(\overrightarrow{D})$. The *weak directed distance* between two vertices u and v in $\overrightarrow{D}$, denoted $d_{\overrightarrow{D}}(u, v)$, is the smallest length (number of arcs) of a directed path in $\overrightarrow{D}$ going either from u to v or from v to u.

A *packing k-coloring* of a digraph $\overrightarrow{D}$ is a mapping $\pi : V(\overrightarrow{D}) \to \{1, \ldots, k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d_{\overrightarrow{D}}(u, v) > i$. The *packing chromatic number* $\chi_\rho(\overrightarrow{D})$ of $\overrightarrow{D}$ is then the smallest k such that $\overrightarrow{D}$ admits a packing k-coloring.

A digraph $\overrightarrow{O}$ with no pair of opposite arcs, that is $uv \in E(\overrightarrow{O})$ implies $vu \notin E(\overrightarrow{O})$, is called an *oriented graph*. If G is an undirected graph, an *orientation* of G is any oriented graph $\overrightarrow{G}$ obtained by giving to each edge of G one of its two possible orientations.

By definition, if $\overrightarrow{G}$ is any orientation of an undirected graph G then, for any two vertices u and v in G, $d_{\overrightarrow{G}}(u, v) \geq d_G(u, v)$. Therefore, every packing coloring of G is a packing coloring of $\overrightarrow{G}$. Hence, we have the following.

Proposition 13. *For every orientation $\overrightarrow{G}$ of an undirected graph G, $\chi_\rho(\overrightarrow{G}) \leq \chi_\rho(G)$.*

Note also that Proposition 1 is still valid for oriented graphs.

Proposition 14. *If $\overrightarrow{H}$ is a subgraph of $\overrightarrow{G}$, then $\chi_\rho(\overrightarrow{H}) \leq \chi_\rho(\overrightarrow{G})$.*

The characterization of oriented graphs with packing chromatic number 2 is given by the following result.

Proposition 15. *For every orientation $\overrightarrow{G}$ of an undirected graph G, $\chi_\rho(\overrightarrow{G}) = 2$ if and only if* (i) *G is bipartite and* (ii) *one part of the bipartition of G contains only sources or sinks in $\overrightarrow{G}$.*

Proof. Clearly, $\chi_\rho(\overrightarrow{G}) > 2$ whenever G is not bipartite. Assume thus that G is bipartite. Since color 1 cannot be used for the central vertex of any directed path of length 2, we get that $\chi_\rho(\overrightarrow{G}) = 2$ if and only if all the vertices from one of the two parts are sources or sinks in $\overrightarrow{G}$. ■

We now determine the packing chromatic number of orientations of paths, cycles, and coronae of paths and cycles.

For oriented paths, we have the following.

Theorem 16. *Let $\overrightarrow{P_n}$ be any orientation of the path $P_n = x_1 \cdots x_n$. Then, for every $n \geq 2$, $2 \leq \chi_\rho(\overrightarrow{P_n}) \leq 3$. Moreover, $\chi_\rho(\overrightarrow{P_n}) = 2$ if and only if one part of the bipartition of P_n contains only sources or sinks in $\overrightarrow{P_n}$.*

Proof. Since adjacent vertices cannot receive the same color, we clearly have $\chi_\rho(\overrightarrow{P_n}) \geq 2$ for all $n \geq 2$. By Theorem 2, we know that $\chi_\rho(P_n) \leq 3$ for every $n \geq 2$ and thus, by Proposition 13, we get that $\chi_\rho(\overrightarrow{P_n}) \leq 3$ for every $n \geq 2$.

The last claim directly follows from Proposition 15. ■

For oriented cycles, we have the following.

Theorem 17. *Let $\overrightarrow{C_n}$ be any orientation of the cycle $C_n = x_0 \cdots x_{n-1}x_0$. Then, for every $n \geq 3$, $2 \leq \chi_\rho(\overrightarrow{C_n}) \leq 4$. Moreover,*

(1) *$\chi_\rho(\overrightarrow{C_n}) = 2$ if and only if C_n is bipartite (that is, n is even) and one part of the bipartition contains only sources or sinks in $\overrightarrow{C_n}$.*
(2) *$\chi_\rho(\overrightarrow{C_n}) = 4$ if and only if $\overrightarrow{C_n}$ is a directed cycle (all arcs have the same direction), $n \geq 5$ and $n \not\equiv 0 \pmod 4$.*

Proof. Since adjacent vertices cannot receive the same color, we clearly have $\chi_\rho(\overrightarrow{C_n}) \geq 2$ for all $n \geq 3$. By Theorem 2, we know that $\chi_\rho(C_n) \leq 4$ for every $n \geq 3$ and thus, by Proposition 13, we get that $\chi_\rho(\overrightarrow{C_n}) \leq 4$ for every $n \geq 3$.

Claim (1) directly follows from Proposition 15.

Let us now consider Claim (2). By Theorem 2, we know that $\chi_\rho(C_n) = 4$ if and only if $n \geq 5$ and $n \not\equiv 0 \pmod 4$. By Proposition 13, we get that $\chi_\rho(\overrightarrow{C_n}) \leq 3$

in all other cases. Thus suppose that $n \geq 5$ and $n \not\equiv 0 \pmod 4$. If $\overrightarrow{C_n}$ is a directed cycle, with all arcs having the same direction, then $d_{\overrightarrow{C_n}}(x_i, x_j) = d_{C_n}(x_i x_j)$ for every $0 \leq i, j \leq n-1$ and thus $\chi_\rho(\overrightarrow{C_n}) = 4$. If $\overrightarrow{C_n}$ is not a directed cycle, it contains a source vertex, say x_0 without loss of generality. We will prove that, in this case, $\overrightarrow{C_n}$ admits a packing 3-coloring.

We consider three cases according to the value of (n mod 4). In each case, a packing 3-coloring of $\overrightarrow{C_n}$ is given by a pattern, starting at vertex x_0 (recall that x_0 is a source or a sink).

- If $n \equiv 1 \pmod 4$, we use the pattern 1231 | 2131 | ... | 2131 | 2.
- If $n \equiv 2 \pmod 4$, we use the pattern 1 | 2131 | ... | 2131 | 2.
- If $n \equiv 3 \pmod 4$, we use the pattern 13 | 1213 | ... | 1213 | 2.

This completes the proof. ■

For orientations of p-coronae of paths, we have the following.

Theorem 18. *Let $\overrightarrow{G}$ be any orientation of a p-corona $P_n \odot pK_1$, with $p \geq 1$ and $P_n = x_1 \cdots x_n$. Then, for every $n \geq 1$, $2 \leq \chi_\rho(\overrightarrow{G}) \leq 3$. Moreover, $\chi_\rho(\overrightarrow{G}) = 2$ if and only if one part of the bipartition of $P_n \odot pK_1$ contains only sources or sinks in $\overrightarrow{G}$.*

Proof. Since a packing coloring is a proper coloring, we clearly have $\chi_\rho(\overrightarrow{G}) \geq 2$ for every orientation $\overrightarrow{G}$ of $P_n \odot pK_1$, $n, p \geq 1$.

We first consider the case $p = 1$. For any orientation $\overrightarrow{G}$ of $P_1 \odot K_1$, the coloring given by the pattern 1(2) is clearly a packing 2-coloring of $\overrightarrow{G}$. Assume now that $n \geq 2$ and let $\overrightarrow{G}$ be any orientation of $P_n \odot K_1$. Let $z_1, \ldots, z_n$ denote the pendant vertices associated with $x_1, \ldots, x_n$, respectively. We will construct inductively a packing 3-coloring π of $\overrightarrow{G}$. We first set $\pi(x_1) := 1$ and $\pi(z_1) := 2$. Assume now that all the vertices $x_1, z_1, \ldots, x_i, z_i$, $1 \leq i \leq n-1$ have been colored in such a way that $\pi(x_i) = 1$ if and only if i is odd and $\pi(z_i) = 1$ if and only if i is even. Then, use the following rule.

- If $\pi(x_i) = 1$ then set $\pi(x_{i+1}) := 5 - \pi(z_i)$ if $z_i x_i x_{i+1}$ is a directed path (in either direction) and $\pi(x_{i+1}) := \pi(z_i)$ otherwise. In both cases, set $\pi(z_{i+1}) := 1$.
- If $\pi(x_i) \neq 1$ then set $\pi(z_{i+1}) := 5 - \pi(x_i)$ if $x_i x_{i+1} z_{i+1}$ is a directed path (in either direction) and $\pi(z_{i+1}) := \pi(x_i)$ otherwise. In both cases, set $\pi(x_{i+1}) := 1$.

The coloring π thus obtained (see Figure 6(a) for an example) has the following property:

(P) every vertex with color 1 is such that all its in-neighbors have the same color $\alpha \in \{2, 3\}$ and all its out-neighbors have the same color $5 - \alpha \in \{2, 3\}$.

The coloring π is thus a packing 3-coloring of $\overrightarrow{G}$.

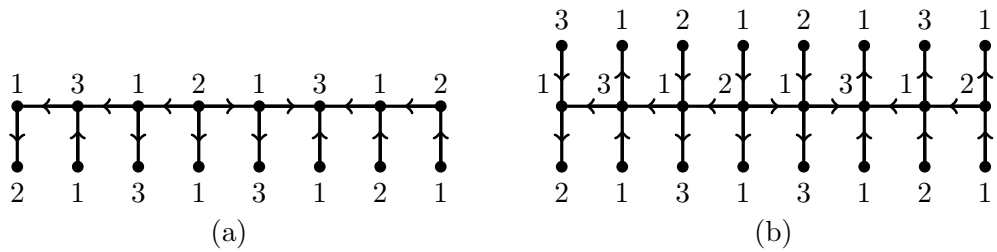

Figure 6. Packing colorings for the proof of Theorem 18.

Consider now the case $p \geq 2$. We first color the vertices $x_1, \ldots, x_n$ and one of their pendant neighbors using the procedure described above, and then color the remaining pendant vertices in such a way that property (P) is satisfied. Hence, all pendant neighbors of a vertex with color 2 or 3 will be colored 1, and all pendant neighbors of a vertex with color 1 will be colored 2 or 3, depending on the orientation of the corresponding arc (see Figure 6(b) for an example).

The last claim directly follows from Proposition 15. ■

Finally, for orientations of p-coronae of cycles, we have the following.

Theorem 19. *Let $\overrightarrow{G}$ be any orientation of a p-corona $C_n \odot pK_1$, with $p \geq 1$ and $C_n = x_0 \cdots x_{n-1}$. Then, for every $n \geq 3$, $2 \leq \chi_\rho(\overrightarrow{G}) \leq 4$. Moreover,*

(1) *$\chi_\rho(\overrightarrow{G}) = 2$ if and only if $C_n \odot pK_1$ is bipartite (that is, n is even) and one part of the bipartition contains only sources or sinks in $\overrightarrow{G}$.*

(2) *$\chi_\rho(\overrightarrow{G}) = 4$ if and only if either:*

  (2.1) *$\overrightarrow{C_n}$ is a directed cycle, $n \geq 5$ and $n \not\equiv 0 \pmod 4$, or*

  (2.2) *$\overrightarrow{G}$ contains the oriented graph depicted in Figure 7 as a subgraph, or*

  (2.3) *$n \equiv 0 \pmod 4$ and there exists a vertex x_i, $0 \leq i \leq n-1$, such that the paths $x_i x_{i+1} x_{i+2} x_{i+3}$ and $x_{i+4} \cdots x_{i-1}$ (indices are taken modulo n) are both directed paths, but in opposite direction.*

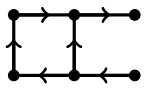

Figure 7. Configuration for the proof of Theorem 19.

Before proving this theorem, we introduce a useful coloring procedure, called *standard coloring procedure* (SCP for short), that produces a coloring π of an orientation of the path $P_n = x_1 \cdots x_n$.

1. Assume $(c, c') \in \{1, 2, 3\}^2$, with $|\{c, c'\} \cap \{1\}| = 1$, and $S \subseteq V(P_n)$ are given.
2. Set $\pi(x_1) := c$ and $\pi(x_2) := c'$.
3. For $j = 3, \ldots, n$, set $\pi(x_j) := 1$ if $\pi(x_{j-1}) \neq 1$, $\pi(x_j) := \pi(x_{j-2})$ if $\pi(x_{j-1}) = 1$ and $x_{j-1} \in S$, and $\pi(x_j) := 5 - \pi(x_{j-2})$ otherwise.

Figure 8 shows colorings of two orientations of $P_8 = x_1 \cdots x_8$ produced by SCP, with $(c, c') = (1, 2)$ and $S = \{x_3\}$, and with $(c, c') = (3, 1)$ and $S = \{x_4, x_8\}$, respectively. Note that SCP always produces a packing 3-coloring of the path $x_1 \ldots x_n$, but not necessarily a packing 3-coloring of $\overrightarrow{C_n}$, and that the only possible conflicts lie on the path $x_{n-2}x_{n-1}x_nx_1x_2x_3$ (such conflicts may appear when a directed path of length 2 or 3 contains x_1 as an internal vertex). For instance, the second example depicted in Figure 8 is a packing 3-coloring of $\overrightarrow{C_8}$, while the first one is not.

$$1 \longrightarrow 2 \longrightarrow 1 \longleftarrow 2 \longleftarrow 1 \longleftarrow 3 \longrightarrow 1 \longrightarrow 2 \longrightarrow (1)$$

$$3 \longrightarrow 1 \longrightarrow 2 \longrightarrow 1 \longleftarrow 2 \longleftarrow 1 \longleftarrow 3 \longleftarrow 1 \longrightarrow (3)$$

Figure 8. Sample colorings produced by SCP.

Observe that if $c = 1$ (resp. $c' = 1$) SCP assigns color 1 to every vertex x_j such that j is odd (resp. even), and colors 2 and 3 alternate on other vertices whenever S is empty. If S is not empty, we have $|S|$, or $|S| - 1$ if $x_1 \in S$ and $c = 1$ (resp. $x_2 \in S$ and $c' = 1$), places where the color 2 or 3 is duplicated. Hence, we have the following.

Proposition 20. *Let $\overrightarrow{P_n}$ be any orientation of the path $P_n = x_1 \cdots x_n$ of odd length $n-1$ and S be a set of sources or sinks in $\overrightarrow{P_n}$ with odd indices not containing x_1. Consider the coloring π of $\overrightarrow{P_n}$ produced by SCP with $(c, c') = (1, \alpha)$ for some $\alpha \in \{2, 3\}$ and S. Then we have*

(i) $\pi(x_n) = \alpha$ *if* $|S|$ *is even* (*respectively odd*) *and* $n \equiv 2 \pmod 4$ (*respectively* $n \equiv 0 \pmod 4$),

(ii) $\pi(x_n) = 5 - \alpha$ *otherwise.*

Proof. This directly follows from the above discussion. ■

***Proof of Theorem* 19.** Since a packing coloring is a proper coloring, we clearly have $\chi_\rho(\overrightarrow{G}) \geq 2$ for every orientation $\overrightarrow{G}$ of $C_n \odot pK_1$, $n \geq 3$, $p \geq 1$.

Let $\overrightarrow{G}$ be any orientation of $C_n \odot pK_1$ and $\overrightarrow{C_n}$ be the orientation of the cycle C_n induced by $\overrightarrow{G}$. Denote by z_i^j, $1 \leq j \leq p$, the pendant neighbors of x_i, $0 \leq i \leq n-1$. We consider two cases.

If $\overrightarrow{C_n}$ contains a source vertex, say x_0 without loss of generality, then, by Theorem 18, there exists a packing 3-coloring of $\overrightarrow{G} \setminus \{x_0, z_0^1, \ldots, z_0^p\}$. Since x_0 is

a source, this packing coloring can be extended to a packing 4-coloring of $\overrightarrow{G}$ by coloring x_0 with color 4 and all vertices z_0^j, $1 \leq j \leq p$, with color 1.

If $\overrightarrow{C_n}$ does not contain any source vertex then $\overrightarrow{C_n}$ is a directed cycle. By Theorem 17, we know that there exists a packing 4-coloring π of $\overrightarrow{C_n}$. This packing coloring can be extended to a packing 4-coloring of $\overrightarrow{G}$ by coloring every pendant vertex z_i^j, $0 \leq i \leq n-1$, $1 \leq j \leq p$, by $\pi(x_{i-1})$ if $z_i^j x_i$ is an arc in $\overrightarrow{G}$ and by $\pi(x_{i+1})$ otherwise (indices are taken modulo n). Hence, $\chi_\rho(\overrightarrow{G}) \leq 4$ for every orientation $\overrightarrow{G}$ of $C_n \odot pK_1$, $n \geq 3$, $p \geq 1$.

Claim (1) directly follows from Proposition 15.

We now consider claim (2). If $\chi_\rho(\overrightarrow{C_n}) = 4$ (which happens, by Theorem 17, if and only if $\overrightarrow{C_n}$ is a directed cycle, $n \geq 5$ and $n \not\equiv 0 \pmod 4$) then, by Proposition 14, $\chi_\rho(\overrightarrow{G}) = 4$ (condition (2.1) of the theorem).

If $\chi_\rho(\overrightarrow{C_n}) = 2$ (which happens, by Theorem 17, if and only if n is even and the orientation $\overrightarrow{C_n}$ of C_n is alternating) then we clearly have $\chi_\rho(\overrightarrow{G}) \leq 3$ since both colors 2 and 3 are available for pendant neighbors of vertices colored 1.

Suppose therefore that $\chi_\rho(\overrightarrow{C_n}) = 3$. If $\overrightarrow{C_n}$ is a directed cycle, which implies $n \equiv 0 \pmod 4$, then the packing 3-coloring given by the circular pattern [1213] can be extended to a packing 3-coloring of $\overrightarrow{G}$, as in the proof of Theorem 18.

Assume now that $\overrightarrow{C_n}$ is not a directed cycle and let π be a packing 3-coloring of $\overrightarrow{C_n}$. This coloring can be extended to a packing 3-coloring of $\overrightarrow{G}$ except if there exists three consecutive vertices $x_{i-1}x_ix_{i+1}$ (indices are taken modulo n) such that (i) x_i is a source (resp. a sink) in $\overrightarrow{C_n}$ but not in $\overrightarrow{G}$, and (ii) $\pi(x_i) = 1$ and $\{\pi(x_{i-1}), \pi(x_{i+1})\} = \{2, 3\}$. Indeed, if such a case occurs, none of the colors from the set $\{1, 2, 3\}$ can be assigned to a pendant out-neighbor (resp. in-neighbor) of x_i. Otherwise, the packing 3-coloring of $\overrightarrow{C_n}$ can be extended to a packing 3-coloring of $\overrightarrow{G}$ by (i) assigning color 1 to all pendant neighbors of vertices colored 2 or 3, (ii) assigning the color $\pi(x_{i-1})$ to every pendant out-neighbor (resp. in-neighbor) of a source (resp. a sink) vertex x_i of $\overrightarrow{G}$ and the color $5-\pi(x_{i-1})$ to its in-neighbors (resp. out-neighbors), and (iii) assigning the color $\pi(x_{i-1})$ to every pendant in-neighbor (resp. out-neighbor) of a vertex x_i which is neither a source nor a sink in $\overrightarrow{G}$, and the color $\pi(x_{i+1})$ to its out-neighbors (resp. in-neighbors), whenever $x_{i-1}x_ix_{i+1}$ (resp. $x_{i+1}x_ix_{i-1}$) is a directed path.

We thus need to determine in which cases the orientation $\overrightarrow{C_n}$ of C_n can be colored in such a way that such a situation does not occur. Such colorings will be called *good* packing colorings.

For any subset X of $V(C_n)$, we denote by $S(X)$ the subset of X containing all the vertices that are either a source or a sink in $\overrightarrow{C_n}$, and by $S^*(X)$ the subset of $S(X)$ containing all the vertices that are neither a source nor a sink in $\overrightarrow{G}$.

Hence, $S^*(V(C_n))$ is precisely the set of vertices we must care about. Obviously, if $S^*(V(C_n))$ is empty, every packing 3-coloring of $\overrightarrow{C_n}$ is good. We thus assume in the rest of the proof that $S^*(V(C_n))$ is not empty. Note also that $|S(V(C_n))|$ is even for every orientation $\overrightarrow{C_n}$ of C_n.

In the following, we will construct good packing 3-colorings, when this is possible, using SCP with an adequate set S either on the whole cycle $\overrightarrow{C_n}$ or on part of it.

We consider four cases, according to the value of n mod 4.

Case 1. $n \equiv 0 \pmod 4$. Consider first the case $n = 4$. The only possible packing 3-coloring of any orientation $\overrightarrow{C_4}$ of C_n with $\chi_\rho(\overrightarrow{C_4}) = 3$ is 1213. It is then easy to check that the only orientation $\overrightarrow{C_4}$ of C_4 for which we cannot produce a good packing 3-coloring is the one given in Figure 7. In the following, we can thus assume $n \geq 8$.

Since n is even, C_n is bipartite. Let (A, B) denote the bipartition of $V(C_n)$. If $|S^*(A)|$ is even or $|S^*(B)|$ is even, a good coloring can be obtained by means of SCP. Suppose without loss of generality that $A = \{x_0, x_2, \ldots, x_{n-2}\}$ and $|S^*(A)|$ is even. Consider the coloring π produced by SCP, starting at x_0, with $(c, c') = (1, 2)$ and $S = S^*(A)$. Since $n \equiv 0 \pmod 4$ and $|S^*(A)|$ is even, by Proposition 20, π is a good packing 3-coloring of $\overrightarrow{C_n}$.

If both $|S^*(A)|$ and $|S^*(B)|$ are odd, but $S(A) \setminus S^*(A) \neq \emptyset$ or $S(B) \setminus S^*(B) \neq \emptyset$, we can proceed in a similar way by using, without loss of generality, the set $S'(A) = S^*(A) \cup \{x_{2j}\}$, for some vertex $x_{2j} \in S(A) \setminus S^*(A)$, instead of the set $S^*(A)$ in SCP since $|S'(A)|$ is even.

Finally, suppose that both $|S^*(A)|$ and $|S^*(B)|$ are odd, $S(A) = S^*(A)$ and $S(B) = S^*(B)$, that is, every source or sink in $\overrightarrow{C_n}$ is neither a source nor a sink in $\overrightarrow{G}$. We consider two cases.

• $|S^*(A)| = |S^*(B)| = 1$. Without loss of generality, we may assume that x_0 is a source and x_i, for some odd i, $1 \leq i \leq n-1$, is a sink. Hence, $x_0 \cdots x_i$ and $x_{n-1} \cdots x_i$ are both directed paths of odd length in $\overrightarrow{C_n}$. Suppose first that $i = 1$, that is, x_0 is a source and x_1 is a sink. A good packing 3-coloring of $\overrightarrow{C_n}$ is then given by the following pattern (the colors of x_0 and $x_i = x_1$ are dotted):

$$[\dot{1}\dot{2}\ 3121\ \ldots\ 3121\ 32].$$

Similarly, if $i \equiv 1 \pmod 4$, a good packing 3-coloring of $\overrightarrow{C_n}$ is then given by:

$$[\dot{1}\ 2131\ \ldots\ 2131\ \dot{2}\ 3121\ \ldots\ 3121\ 32].$$

Now, if $i \equiv 3 \pmod 4$, $i \geq 7$, a good packing 3-coloring of $\overrightarrow{C_n}$ is given by:

$$[\dot{1}23\ 1213\ \ldots\ 1213\ \dot{2}\ 1312\ \ldots\ 1312].$$

The remaining case is $i = 3$, which corresponds to condition (2.3) of the theorem. We will prove that in that case $\overrightarrow{C_n}$ does not admit any good packing 3-coloring, which implies $\chi_\rho(\overrightarrow{C_n}) = 4$. Note first that the directed path $\overrightarrow{P} = x_0 x_{n-1} \cdots x_3$ has length $n - 3 \equiv 1 \pmod 4$. Let us consider the possible packing 3-colorings of $\overrightarrow{P}$. Clearly, the pattern 123 can only be used on the left end of $\overrightarrow{P}$, while the pattern 321 can only be used on the right end of $\overrightarrow{P}$. Moreover, the only circular good pattern is [1213]. Therefore, up to mirror symmetry (reversing the orientation of every arc of $\overrightarrow{C_n}$ gives the same oriented graph), there are six possible packing 3-colorings of $\overrightarrow{P}$, given by the following patterns:

$$
\begin{aligned}
&1213 \ \ldots\ 1213\ 12,\\
&1213 \ \ldots\ 1213\ 21,\\
123\ &1213 \ \ldots\ 1213\ 121,\\
&2131 \ \ldots\ 2131\ 21,\\
&3121 \ \ldots\ 3121\ 31,\\
&3121 \ \ldots\ 3121\ 32.
\end{aligned}
$$

It is then not difficult to check that none of these colorings can be extended to a good packing 3-coloring of $\overrightarrow{C_n}$, as shown by the following diagrams (the colors of x_0 and x_3 are dotted).

$$
\begin{array}{ccccccccc}
2 & \longleftarrow & \dot{1} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{2} \longleftarrow 1\\[6pt]
2 & \longleftarrow & \dot{1} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{1} \longleftarrow 2\\
2 & \longleftarrow & \dot{1} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{1} \longleftarrow 2\\
1 & \longleftarrow & \dot{2} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{1} \longleftarrow 2\\
1 & \longleftarrow & \dot{3} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{1} \longleftarrow 3\\
1 & \longleftarrow & \dot{3} & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \dot{2} \longleftarrow 3
\end{array}
$$

• $|S^*(A)| \geq 3$ or $|S^*(B)| \geq 3$. Suppose $|S^*(A)| \geq 3$, without loss of generality. Since $n \equiv 0 \pmod 4$ and both $|S^*(A)|$ and $|S^*(B)|$ are odd, by Proposition 20, applying SCP starting at x_0 leads in all cases to a "bad" coloring that assigns to $x_{n-1}x_0x_1$ either the pattern 213 or 312 if $x_0 \in S^*(A)$, or the pattern 212 or 313 otherwise (in that case, $x_{n-1}x_0x_1$ is a directed path, in either direction). We thus need to "correct" this bad coloring, which can be done by replacing a sequence $1\alpha \cdots \beta 1$ of the coloring produced by SCP by $1\alpha \cdots \beta' 1$ with $\beta' = 5 - \beta$.

We consider three subcases.

1. There exist $a \in S^*(A)$ and $b \in S^*(B)$ with $d_{\overrightarrow{C_n}}(a,b) = 1$. We may assume without loss of generality that $a = x_i$ is a source and $b = x_{i+1}$ is a sink. Hence, we have the following configuration (— stands for an arc in either direction):

$$— \quad \longleftarrow \quad a \quad \longrightarrow \quad b \quad \longleftarrow \quad — \; —$$

Consider the following coloring of this configuration (the colors of a and b are dotted):

$$1 — 3 \longleftarrow \dot{2} \longrightarrow \dot{1} \longleftarrow 2 — 3 — 1$$

If the vertex to the right of b is not a source, the remaining part of the cycle is not empty (since $n \equiv 0 \pmod 4$) and this coloring can be extended to a good packing 3-coloring of $\overrightarrow{C_n}$ by means of SCP. To see this, observe that SCP would have produced the following bad coloring on the same configuration (the bad color which implies our claim, since our coloring has modified this color, appears in bold):

$$1 — 3 \longleftarrow \dot{1} \longrightarrow \dot{3} \longleftarrow 1 \longleftarrow \mathbf{2} — 1$$

We finally claim that we can always find some i such that $x_i \in S^*(A)$ (resp. $x_i \in S^*(B)$), $x_{i+1} \in S^*(B)$ (resp. $x_{i+1} \in S^*(A)$) and $x_{i+2} \notin S^*(A)$ (resp. $x_{i+2} \notin S^*(B)$). This simply follows from the fact that if no such i exists, then the orientation $\overrightarrow{C_n}$ of C_n is alternating, which implies $\chi_\rho(\overrightarrow{C_n}) = 2$, contrary to our assumption.

2. There exist $a \in S^*(A)$ and $b \in S^*(B)$ with $d_{\overrightarrow{C_n}}(a,b) \equiv 1 \pmod 4$, $d_{\overrightarrow{C_n}}(a,b) \geq 5$, and subcase 1 does not occur.

Again, we assume without loss of generality that $a = x_i$ is a source and $b = x_j$ is a sink. Since subcase 1 does not occur, we necessarily have the following configuration:

$$\longleftarrow \longleftarrow a \longrightarrow \longrightarrow \cdots \longrightarrow \longrightarrow \longrightarrow b \longleftarrow \longleftarrow$$

We then color this configuration as follows (the pattern 2131 is repeated as many times as necessary):

$$1 — 3 \longleftarrow 2 \longleftarrow \dot{1} \longrightarrow (2131)^* \longrightarrow \dot{2} \longleftarrow 3 \longleftarrow 1$$

As in the previous subcase, the remaining part of the cycle is not empty. Hence, this coloring can be extended to a good packing 3-coloring of $\overrightarrow{C_n}$ by means of SCP, since SCP would have produced the following bad coloring on the same configuration:

$$1 — 3 \longleftarrow 1 \longleftarrow \dot{2} \longrightarrow (1312)^* \longrightarrow \dot{1} \longleftarrow \mathbf{2} \longleftarrow 1$$

3. There exist $a \in S^*(A)$ and $b \in S^*(B)$ with $d_{\overrightarrow{C_n}}(a,b) \equiv 3 \pmod 4$, $d_{\overrightarrow{C_n}}(a,b) \geq 7$, and subcases 1 and 2 do not occur.

This subcase can be solved similarly as the previous one. We have the following configuration:

$$\longleftarrow \; a \; \longrightarrow \; \longrightarrow \; \longrightarrow \; \cdots \; \longrightarrow \; \longrightarrow \; \longrightarrow \; \longrightarrow \; b \; \longleftarrow$$

for which we use the following coloring:

$$1 \longleftarrow \dot{2} \longrightarrow 3 \longrightarrow (1213)^* \longrightarrow 2 \longrightarrow \dot{1} \longleftarrow 2$$

Again, the remaining part of the cycle is not empty and this coloring can be extended to a good packing 3-coloring of $\overrightarrow{C_n}$ by means of SCP, since SCP would have produced the following bad coloring on the configuration:

$$1 \longleftarrow \dot{2} \longrightarrow 1 \longrightarrow (3121)^* \longrightarrow 3 \longrightarrow \dot{1} \longleftarrow \mathbf{3}$$

4. None of the previous cases occurs. If none of the previous cases occurs, then the vertices of $S^*(A)$ and $S^*(B)$ necessarily alternate on $\overrightarrow{C_n}$ and the weak directed distance between any two consecutive such vertices equals 3. Hence, $\overrightarrow{C_n}$ is a sequence of directed paths of length 3 in opposite directions. Since $|S^*(A)| = |S(A)|$ is odd, the length of C_n equals $6k$ for some odd k, which contradicts the assumption $n \equiv 0 \pmod 4$. Therefore, this last subcase cannot occur.

Case 2. $n \equiv 2 \pmod 4$. In this case C_n is again bipartite and, using the same procedure as in Case 1, a good packing 3-coloring of $\overrightarrow{C_n}$ can be produced whenever (i) $|S^*(A)|$ or $|S^*(B)|$ is odd, or (ii) both $|S^*(A)|$ and $|S^*(B)|$ are even, but $S(A) \setminus S^*(A) \neq \emptyset$ or $S(B) \setminus S^*(B) \neq \emptyset$, where (A,B) denotes the bipartition of $V(C_n)$.

Suppose now that both $|S^*(A)|$ and $|S^*(B)|$ are even (they cannot be both equal to 0), $S(A) = S^*(A)$ and $S(B) = S^*(B)$. In that case, SCP produces a bad coloring of $\overrightarrow{C_n}$ and this coloring can be "corrected" in exactly the same way as in Case 1 since, for doing that, we only need n to be even.

Case 3. n is odd. Consider the set $S = S(V(C_n))$, that is the set of vertices that are either a source or a sink in $\overrightarrow{C_n}$. Without loss of generality, suppose that x_0 is a source and consider the coloring π produced by SCP on the path $x_0x_1 \cdots x_{n-1}$, starting at x_0, with $(c,c') = (2,1)$ and S. If $\pi(x_{n-1}) = 3$, π is a packing 3-coloring of $\overrightarrow{G}$, of the form $21\ldots13$, and we are done.

If $\pi(x_{n-1}) = 2$ (π is not a packing coloring of $\overrightarrow{G}$), consider the coloring π' produced by SCP on the path $x_1x_2 \cdots x_{n-1}x_0$, starting at x_1, with $(c,c') = (3,1)$ and S. Let now X denote the set of sources or sinks which are assigned color 1

by π, and X' the set of sources or sinks which are assigned color 1 by π'. We clearly have $X \cap X' = \emptyset$ and $X \cup X' = S \setminus \{x_0\}$ (since x_0 is a source, $\pi(x_0) \neq 1$ and $\pi'(x_0) \neq 1$). Therefore, since $|S|$ is even, we get that $|X[$ and $|X'|$ do not have the same parity. Hence, since $\pi(x_0) = 2$ and $\pi(x_{n-1}) = 2$, starting with $\pi'(x_1) = 3$ necessarily gives $\pi'(x_0) = 2$. This proves that π' is a good packing 3-coloring of $\overrightarrow{G}$, of the form $231\ldots1$.

This concludes the proof. ■

5. Discussion

In this paper, we have determined the packing chromatic number of coronae and p-coronae of paths and cycles. We also extended to digraphs the notion of packing coloring and determined the packing chromatic number of orientations of such graphs.

In particular, we have proved that every orientation of a p-corona of a path admits a packing 3-coloring. Using a similar proof, it is not difficult to extend this result to the more general case of oriented trees (we can inductively construct a packing coloring satisfying the property (P) such that vertices with color 1 correspond to one part of the bipartition of the tree). Hence, we also have

Theorem 21. *Let T be a tree. For any orientation $\overrightarrow{T}$ of T, $\chi_\rho(\overrightarrow{T}) \leq 3$.*

Since every caterpillar is a tree, we get that every oriented caterpillar has packing chromatic number at most 3. However, we leave as an open question the characterization of undirected caterpillars with packing chromatic number at most 4, 5 and 6 (by Theorem 3 we know that every caterpillar has packing chromatic number at most 7 and characterizing caterpillars with packing chromatic number at most 2 or 3 is easy).

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