# INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS ${ }^{1}$ 

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#### Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4 -coloring problem is polynomially solvable and the interval incidence 5 -coloring problem is $\mathcal{N} \mathcal{P}$-complete, and they asked if $\chi_{i i}(G) \leq 2 \Delta(G)$ holds for an arbitrary graph $G$. In this paper, we prove that an interval incidence 6 -coloring always exists for any subcubic graph $G$ with $\Delta(G)=3$.


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## 1. Introduction

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let $G=(V, E)$ be a simple graph, and let $X \subset V$ be a non-empty set. By $N_{G}(X)=\left\{v \in V: \exists_{u \in X}\{v, u\} \in E\right\}$ we mean the open

[^0]neighborhood of $X$, by $G[X]$ we mean the subgraph of $G$ induced by the set $X$, and by $G \backslash X$ we mean the graph $G[V \backslash X]$. We say that $X$ is a dominating set of $G$ if $V=N_{G}(X) \cup X$, and we say that $X$ is a total dominating set if $V=N_{G}(X)$. In what follows we use $N_{G}(v)$ instead of $N_{G}(\{v\})$. Let $\operatorname{deg}_{G}(v)=\left|N_{G}(v)\right|$ be the degree of a vertex $v \in V(G)$. By $n(G), \Delta(G)$ and $\delta(G)$ we denote the number of vertices of $G$, the maximum and the minimum degree of a vertex of $G$, respectively. By a subcubic graph $G$ we mean a graph with $\Delta(G) \leq 3$. By an isolated vertex (in a graph $G$ ) we mean a vertex $v \in V(G)$ with $\operatorname{deg}_{G}(v)=0$, and by an isolated edge (in a graph $G$ ) we mean an edge $e=\{u, v\}$ such that $\operatorname{deg}_{G}(u)=\operatorname{deg}_{G}(v)=1$. We say that $X \subset V(G)$ is an independent set if each vertex of $G[X]$ is isolated in $G[X]$. By a pendant vertex we mean a vertex of degree 1 .

For a given graph $G=(V, E)$, we define an incidence as a pair $(v, e)$, where vertex $v \in V$ is one of the endpoints of edge $e \in E$, i.e., $v \in e$. The set of all incidences of $G$ will be denoted by $I(G)$, thus $I(G)=\{(v, e): v \in V \wedge e \in$ $E \wedge v \in e\}$. We say that two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds: (1) $v=w$ and $e \neq f$; (2) $e=f$ and $v \neq w$; (3) $e=\{v, w\}$, $f=\{w, u\}$ and $v \neq u$.

By an incidence coloring of $G$ we mean a function $c: I(G) \rightarrow \mathbb{N}$ such that $c((v, e)) \neq c((w, f))$ for any two adjacent incidences $(v, e)$ and $(w, f)$. The incidence coloring number of $G$, denoted by $\chi_{i}(G)$, is the smallest number of colors in an incidence coloring of $G$. In what follows we use the simplified notation $c(v, e)$ instead of $c((v, e))$.

A finite nonempty set $A \subset \mathbb{N}$ is an interval if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring $c$ of graph $G$ and $v \in V(G)$ let $A_{c}(v)=\{c(v, e): v \in e \wedge e \in E(G)\}$. By an interval incidence coloring of a graph $G$ we mean an incidence coloring $c$ of $G$ such that for each vertex $v \in V(G)$ the set $A_{c}(v)$ is an interval. By an interval incidence $k$-coloring we mean an interval incidence coloring using all colors from the set $\{1, \ldots, k\}$. The interval incidence coloring number of $G$, denoted by $\chi_{i i}(G)$, is the smallest number of colors in an interval incidence coloring of $G$.

### 1.1. Background and previous results

Alon et al. [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that $\chi_{i}(G) \leq \Delta(G)+2$ holds for every graph $G$ (incidence coloring conjecture, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at
least $\Delta+\Omega(\log \Delta)$. In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most $\Delta+2$, e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2 -trees (i.e., $K_{4}$-minor free graphs) [4], hypercubes [18], complete $k$-partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4 -colorability problem is $\mathcal{N} \mathcal{P}$-complete for semicubic graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., $[2,6,8]$ ), which can be applied to the open-shop scheduling problem $[6,7]$. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence $k$-coloring of bipartite graphs is polynomial for each $k \leq 6$ and $\Delta \leq 3$, polynomial for $k=5$ and $\Delta=4$, and $\mathcal{N} \mathcal{P}$-complete for $k=6$ and $\Delta=4$. In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g., $\Delta(G)+1 \leq \chi_{i i}(G) \leq \chi(G) \cdot \Delta(G)$ for an arbitrary graph $G$, and they determined the exact values of $\chi_{i i}$ for some basic classes of graphs (e.g., complete $k$-partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether $\chi_{i i} \leq 4$ is easy, and $\chi_{i i} \leq 5$ is $\mathcal{N} \mathcal{P}$-complete. The problem of interval incidence 6 -coloring of subcubic graphs remained unsolved.

### 1.2. Main results

Our main result in the paper is Theorem 21 which states $\chi_{i i}(G) \leq 6$ for every subcubic graph $G$. To prove it, we state and prove Theorem 8: in any subcubic graph $G$ with $\delta(G) \geq 2$ there is a maximal induced bipartite subgraph of $G$ without isolated vertices, or equivalently, $G$ has a total dominating set $S$ such that $G[S]$ is a bipartite graph.

## 2. Maximal Induced Bipartite Subgraphs Without Isolated Vertices

In this section we prove (in Theorem 8) that any subcubic graph $G$ with $\delta(G) \geq 2$ contains a maximal induced bipartite subgraph without isolated vertices.

### 2.1. Introductory properties

By $H \subset G$ we mean that $H$ is a subgraph of $G$. By $H \sqsubset G$ we mean that $H$ is an induced subgraph of $G$, i.e., $H=G[V(H)]$.

Observation 1. If $G_{1} \sqsubset G_{2}$ and $G_{2} \sqsubset G_{3}$, then $G_{1} \sqsubset G_{3}$.

Observation 2. Let $G_{1} \sqsubset G$ and $G_{2} \sqsubset G$. If $G_{1} \subset G_{2}$, then $G_{1} \sqsubset G_{2}$.
Let $\mathcal{B}(G)=\left\{H \sqsubset G: N_{G}(V(H))=V(G) \wedge H\right.$ is bipartite $\}$, i.e., the set of all induced bipartite subgraphs of a given graph $G$ such that $V(H)$ is a total dominating set of $G$. If $H \in \mathcal{B}(G)$, then $V(H)$ is a total dominating set of $G$ and, obviously, $H$ has no isolated vertices.

In the following, let $G$ be any graph. Let $\hat{\mathcal{B}}(G)$ be the subfamily of $\mathcal{B}(G)$ consisting of all the elements (graphs) in $\mathcal{B}(G)$ that are maximal with respect to the subgraph relation $(\subset)$.

Observation 3. If $H \in \mathcal{B}(G)$, then there is $H^{\prime} \in \hat{\mathcal{B}}(G)$ such that $H \subset H^{\prime}$.
By Observations 2 and 3 we have
Observation 4. Let $H \in \mathcal{B}(G)$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if for each $v \in$ $V(G) \backslash V(H)$ the subgraph $G[V(H) \cup\{v\}]$ is not bipartite.

Observation 5. If $H \in \mathcal{B}(G) \backslash \hat{\mathcal{B}}(G)$, then there is a vertex $v \in V(G) \backslash V(H)$ such that $G[V(H) \cup\{v\}] \in \mathcal{B}(G)$.

Since any dominating set $S \subset V(G)$ is a total dominating set if and only if $G[S]$ has no isolated vertices, we have

Observation 6. Let $G$ be an arbitrary graph and let $H \subset G$. Then, $H \in \hat{\mathcal{B}}(G)$ if and only if $H$ is a maximal induced bipartite subgraph (of $G$ ) without isolated vertices.

Let $\mathcal{G}_{3}^{2}$ be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3 . Let $\mathcal{M}_{3}^{2}$ be the subfamily of $\mathcal{G}_{3}^{2}$ consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by $\mathcal{M}$ the set of elements in $\mathcal{M}_{3}^{2}$ that are minimal with respect to the subgraph relation $(\subset)$. By Observation 6 we have

Observation 7. Let $G \in \mathcal{G}_{3}^{2}$. Then, $G \in \mathcal{M}_{3}^{2} \Leftrightarrow \mathcal{B}(G)=\emptyset \Leftrightarrow \hat{\mathcal{B}}(G)=\emptyset$.

### 2.2. Main Theorem

Theorem 8. Let $G$ be a subcubic graph with $\delta(G) \geq 2$. Then, $G$ has a maximal induced bipartite subgraph without isolated vertices.

By Observation 7, Theorem 8 is equivalent to $\mathcal{M}=\emptyset$. First, we prove some structural properties of graphs from $\mathcal{M}$.

Lemma 9. Let $G \in \mathcal{M}$. Then, $G$ is a connected graph and $\Delta(G)=3$.
Proof. Let $G \in \mathcal{M}$. Let us assume to the contrary that $G=G_{1} \cup G_{2}$, where $G_{1}$ and $G_{2}$ are disjoint graphs (without common vertices). Since $G_{i} \subsetneq G \in \mathcal{M}$ and $G_{i} \in \mathcal{G}_{3}^{2}$, we have $G_{i} \notin \mathcal{M}_{3}^{2}$, for $i \in\{1,2\}$. Hence, there exist $H_{1} \in \hat{\mathcal{B}}\left(G_{1}\right)$ and $H_{2} \in \hat{\mathcal{B}}\left(G_{2}\right)$. Thus, $H_{1} \cup H_{2} \in \hat{\mathcal{B}}(G)$, a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex, $G$ is not a cycle, which implies $\Delta(G)=3$.

Lemma 10. Let $G \in \mathcal{M}$ and let $v$ be a vertex of degree 2 in $G$. Then, every neighbor of $v$ in $G$ has degree 3 .

Proof. Let $G \in \mathcal{M}$. Suppose to the contrary that there are two adjacent vertices of degree 2. Since $G$ is not a cycle (by Lemma 9), there is a subgraph $P$ of $G$ with vertex set $\left\{v_{0}, \ldots, v_{k+1}\right\}$ and edges $\left\{v_{i}, v_{i+1}\right\}$, for $i \in\{0, \ldots, k\}$, such that $\operatorname{deg}_{G}\left(v_{0}\right)=\operatorname{deg}_{G}\left(v_{k+1}\right)=3$, and $\operatorname{deg}_{G}\left(v_{i}\right)=2$ for $i \in\{1, \ldots, k\}$, where $k \geq 2$.

Suppose $v_{0} \neq v_{k+1}$. Since $G^{\prime}=G \backslash\left\{v_{1}, \ldots, v_{k}\right\} \sqsubset G \in \mathcal{M}$ and $G^{\prime} \in \mathcal{G}_{3}^{2}$, we have $G^{\prime} \notin \mathcal{M}_{3}^{2}$. Hence, there exists $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$, and $H^{\prime} \sqsubset G$ by Observation 1. If $v_{0} \in V\left(H^{\prime}\right)$, then let $H=G\left[V\left(H^{\prime}\right) \cup\left\{v_{1}, \ldots, v_{k-1}\right\}\right]$, otherwise, let $H=$ $G\left[V\left(H^{\prime}\right) \cup\left\{v_{1}, \ldots, v_{k}\right\}\right]$. In both cases, $H \sqsubset G, H$ is a bipartite graph, and $V(H)$ is a total dominating set, i.e., $H \in \mathcal{B}(G)$. By Observation 7 we get a contradiction.

Suppose $v_{0}=v_{k+1}$. Since $\operatorname{deg}_{G}\left(v_{0}\right)=3$, there is $c \in N_{G}\left(v_{0}\right) \backslash\left\{v_{1}, v_{k}\right\}$. If $\operatorname{deg}_{G}(c)=3$, then let $G^{\prime}=G \backslash\left\{v_{0}, \ldots, v_{k}\right\}$. If $\operatorname{deg}_{G}(c)=2$, then let $G^{\prime}=$ $G \backslash\left\{v_{0}, \ldots, v_{k}, c\right\}$. In both cases, $G^{\prime} \sqsubset G$ and $G \neq G^{\prime} \in \mathcal{G}_{3}^{2}$. Hence, there is $H^{\prime} \in$ $\hat{\mathcal{B}}\left(G^{\prime}\right)$. Let $H=G\left[V\left(H^{\prime}\right) \cup\left\{v_{0}, \ldots, v_{k-1}\right\}\right]$. Thus, $H \in \mathcal{B}(G)$, a contradiction.

Lemma 11. If $G \in \mathcal{G}_{3}^{2}$ contains $G_{0}$ as a subgraph (see Figure 1), where vertices $v_{2}, v_{3} \in V\left(G_{0}\right)$ are of degree 2 in $G$, then $G \notin \mathcal{M}$.


Figure 1. The subgraph $G_{0}$ of a graph $G$.

Proof. Suppose to the contrary that $G \in \mathcal{M}$. Suppose $G_{0} \subset G$. The other possible edges in $G$ are marked by the dotted lines (in Figure 1).

By $\operatorname{deg}_{G}\left(v_{2}\right)=\operatorname{deg}_{G}\left(v_{3}\right)=2$, from Lemma 10 we have $\operatorname{deg}_{G}\left(v_{1}\right)=\operatorname{deg}_{G}\left(v_{4}\right)$ $=3$. Since $G^{\prime}=G \backslash\left\{v_{3}\right\} \in \mathcal{G}_{3}^{2} \backslash \mathcal{M}_{3}^{2}$, there is $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$. Hence, $v_{1} \in V\left(H^{\prime}\right)$ or $v_{4} \in V\left(H^{\prime}\right)$. Thus, $H^{\prime} \in \mathcal{B}(G)$, a contradiction.

Lemma 12. Let $G \in \mathcal{M}$ and let $v$ be a vertex of degree 3 in $G$. Then, at most one neighbor of $v$ has degree 2 .
Proof. Let $G \in \mathcal{M}$ and let $N_{G}(v)=\{x, y, z\}$. Suppose to the contrary that at least two vertices from $N_{G}(v)$ have degree 2 . Let $\operatorname{deg}_{G}(x)=\operatorname{deg}_{G}(y)=2$. Let $\left\{v_{x}\right\}=N_{G}(x) \backslash\{v\}$ and $\left\{v_{y}\right\}=N_{G}(y) \backslash\{v\}$. By Lemma 10, $\operatorname{deg}_{G}\left(v_{x}\right)=$ $\operatorname{deg}_{G}\left(v_{y}\right)=3$.

Suppose $\operatorname{deg}_{G}(z)=2$. Let $\left\{v_{z}\right\}=N_{G}(z) \backslash\{v\}$. By Lemma 10, $\operatorname{deg}_{G}\left(v_{z}\right)=3$. If any two of the vertices $v_{x}, v_{y}, v_{z}$ are equal, then by Lemma 11 (i.e., because $\left.G_{0} \sqsubset G\right)$ we get a contradiction. Hence, vertices $v_{x}, v_{y}, v_{z}$ are different. Since $G^{\prime}=G \backslash\{x, y, z, v\} \in \mathcal{G}_{3}^{2} \backslash M_{3}^{2}$, there is $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$. Thus, $G\left[V\left(H^{\prime}\right) \cup\{v, x\}\right] \in$ $\mathcal{B}(G)$, a contradiction.

Suppose $\operatorname{deg}_{G}(z)=3$. If $v_{x}=v_{y}$, then by Lemma 11 we get a contradiction. Hence, $v_{x} \neq v_{y}$. Suppose $z=v_{x}$ (the case $z=v_{y}$ can be treated analogously). Since $G_{x}=G \backslash\{x\} \in \mathcal{G}_{3}^{2} \backslash \mathcal{M}_{3}^{2}$, there is $H_{x} \in \hat{\mathcal{B}}\left(G_{x}\right)$. Since $H_{x}$ is maximal in $\mathcal{B}(G)$, we have $v \in V\left(H_{x}\right)$ or $z \in V\left(H_{x}\right)$. Thus, $H_{x} \in \mathcal{B}(G)$, a contradiction. Then, vertices $v_{x}, v_{y}, z$ are different. Since $G^{\prime}=G \backslash\{x, y, v\} \in \mathcal{G}_{3}^{2} \backslash \mathcal{M}_{3}^{2}$, there is $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$. If $z \in V\left(H^{\prime}\right)$, then let $A=V\left(H^{\prime}\right) \cup\{v\}$. If $z \notin V\left(H^{\prime}\right)$, then let $A=V\left(H^{\prime}\right) \cup\{v, x\}$. In both cases, $G[A] \in \mathcal{B}(G)$, a contradiction.

Let $G$ be any subcubic graph. We say that $H \subset G$ is a $Q$-cycle (of $G$ ) if: ( $q_{1}$ ) for each $v \in V(H), \operatorname{deg}_{G}(v)=3$, and
$\left(q_{2}\right) H \sqsubset G$ and $H$ is isomorphic to a cycle, i.e., $H$ is an induced cycle, and
$\left(q_{3}\right)$ for each vertex $v \in V(G) \backslash V(H),\left|N_{G}(v) \cap V(H)\right| \leq 1$.
Lemma 13. Let $G \in \mathcal{M}$. Let $v \in V(G)$ have all neighbors of degree 3. Then, for each $x \in N_{G}(v)$ there is a $Q$-cycle $C_{x}$ such that $x \in V\left(C_{x}\right), v \notin V\left(C_{x}\right)$ and $N_{G}(v) \cap V\left(C_{x}\right)=\{x\}$.
Proof. Let $G \in \mathcal{M}$ and let $v \in V(G)$ be a vertex with all neighbors of degree 3 . Since $G^{\prime}=G \backslash\{v\} \in \mathcal{G}_{3}^{2} \backslash \mathcal{M}_{3}^{2}$, there is $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$. Hence, $N_{G}(v) \cap V\left(H^{\prime}\right)=\emptyset$.

Let $x \in N_{G}(v)$ and let $N_{G}(x)=\{a, b, v\}$. Since $H^{\prime}$ is bipartite and maximal in $\mathcal{B}(G)$, we have that $a$ and $b$ belong to the same connected component of $H^{\prime}$, and the length of each path in $H^{\prime}$ from $a$ to $b$ is odd. Let $P \subset H^{\prime}$ be a path joining $x_{1}=a$ and $x_{s-1}=b$ ( $s$ is odd), with vertex set $\left\{x_{1}, \ldots, x_{s-1}\right\}$ and edges $\left\{x_{i}, x_{i+1}\right\}$, for $i \in\{1, \ldots, s-2\}$. Let $x_{0}=x$ and let $C_{x}$ be the graph with $V\left(C_{x}\right)=V(P) \cup\left\{x_{0}\right\}$, and $E\left(C_{x}\right)=E(P) \cup\left\{\left\{x_{s-1}, x_{0}\right\},\left\{x_{0}, x_{1}\right\}\right\}$. Since $P \subset H^{\prime}$, we have $N_{G}(v) \cap V\left(C_{x}\right)=\{x\}$, and $v \notin V\left(C_{x}\right)$.

Claim 14. For each $i \in\{1, \ldots, s-1\}$, the following properties are satisfied:
$\left(p_{1}\right) \operatorname{deg}_{G}\left(x_{i}\right)=3$,
$\left(p_{2}\right) N_{G}\left(x_{i}\right)=\left\{a_{i}, x_{(i-1) \bmod s}, x_{(i+1) \bmod s}\right\}$, where $a_{i} \in V\left(H^{\prime}\right) \backslash V\left(C_{x}\right)$,
$\left(p_{3}\right) N_{G}\left(a_{i}\right) \cap V\left(H^{\prime}\right)=\left\{x_{i}\right\}$.
Proof. We proceed by induction on $i$. Suppose $i=1$. Let $X=V\left(H^{\prime}\right) \backslash\left\{x_{i}\right\} \cup$ $\{x, v\}$. Hence, $G[X]$ is bipartite. If $\operatorname{deg}_{G}\left(x_{i}\right)=2$ or $N_{G}\left(a_{i}\right) \cap V\left(H^{\prime}\right) \neq\left\{x_{i}\right\}$, then $G[X] \in \mathcal{B}(G)$, a contradiction. If $a_{i} \notin V\left(H^{\prime}\right) \backslash V\left(C_{x}\right)$, then $a_{i} \notin V\left(H^{\prime}\right)$ or $a_{i} \in V\left(C_{x}\right)$. If $a_{i} \notin V\left(H^{\prime}\right)$, then $N_{G}\left(a_{i}\right) \cap V\left(H^{\prime}\right) \neq\left\{x_{i}\right\}$ (otherwise $H^{\prime}$ is not maximal in $\mathcal{B}(G)$ ), a contradiction. If $a_{i} \in V\left(C_{x}\right)$, then $G[X] \in \mathcal{B}(G)$, a contradiction.

Suppose the properties $\left(p_{1}\right),\left(p_{2}\right),\left(p_{3}\right)$ hold for $1, \ldots, i-1(2 \leq i \leq s-1)$. Hence, each path joining $x_{1}$ and $x_{s-1}$ in $H^{\prime}$ contains $x_{1}, \ldots, x_{i}$. Let $X=V\left(H^{\prime}\right) \backslash$ $\left\{x_{i}\right\} \cup\{x, v\}$. Hence, $G[X]$ is bipartite. The rest of the proof of properties $\left(p_{1}\right),\left(p_{2}\right),\left(p_{3}\right)$ for $i$ is literally the same as in the case $i=1$.

We show that $C_{x}$ is a $Q$-cycle. Since $\operatorname{deg}_{G}(x)=3$, by $\left(p_{1}\right)$ we have $\left(q_{1}\right)$. Since $v \notin V\left(C_{x}\right)$ and $a_{i} \notin V\left(C_{x}\right)$ (by $\left(p_{2}\right)$ ), for $i \in\{1, \ldots, s-1\}$, we have that $C_{x}$ is an induced cycle of $G$. Since $a_{i} \in V\left(H^{\prime}\right)$ (by $\left(p_{2}\right)$ ), we have $a_{i} \neq v$. Thus, by ( $p_{3}$ ) we get $\left|N_{G}\left(a_{i}\right) \cap V\left(C_{x}\right)\right| \leq 1$, for $i \in\{1, \ldots, s-1\}$.

We say that $H$ is a $Q_{2}$-cycle (of $G$ ) if $H$ is a $Q$-cycle of $G$, and it holds $\left(q_{4}\right)$ for each $v \in N_{G}(V(H)) \backslash V(H), \operatorname{deg}_{G}(v)=2$.

Lemma 15. Let $G \in \mathcal{M}$ and let $C$ be a $Q$-cycle of $G$. Then, $C$ is a $Q_{2}$-cycle.
Proof. Let $G \in \mathcal{M}$. Let $C$ be a $Q$-cycle of $G$ with the vertex set $\left\{x_{0}, \ldots, x_{s-1}\right\}$, and edges $\left\{x_{0}, x_{1}\right\}, \ldots,\left\{x_{s-2}, x_{s-1}\right\},\left\{x_{s-1}, x_{0}\right\}$. Let $S=\{0, \ldots, s-1\}$. Let $\left\{a_{i}\right\}=N_{G}\left(x_{i}\right) \backslash V(C)$, for $i \in S$. If $\operatorname{deg}_{G}\left(a_{i}\right)=2$, then let $\left\{b_{i}\right\}=N_{G}\left(a_{i}\right) \backslash\left\{x_{i}\right\}$. Hence, $b_{i} \notin V(C)$. By Lemma 10 we have $\operatorname{deg}_{G}\left(b_{i}\right)=3$. Let $G^{\prime}=G \backslash(V(C) \cup$ $\left.\left\{a_{i}: \operatorname{deg}_{G}\left(a_{i}\right)=2 \wedge i \in S\right\}\right)$. Since $G^{\prime} \in \mathcal{G}_{3}^{2} \backslash \mathcal{M}_{3}^{2}$, there is $H^{\prime} \in \hat{\mathcal{B}}\left(G^{\prime}\right)$.

Suppose to the contrary that $C$ is not a $Q_{2}$-cycle, i.e., there exists $r \in S$ such that $\operatorname{deg}_{G}\left(a_{r}\right)=3$. Let $f: V\left(G^{\prime}\right) \rightarrow\{0,1\}$ be the characteristic function of $V\left(H^{\prime}\right)$, i.e., $f(u)=1$ if and only if $u \in V\left(H^{\prime}\right)$. Let us consider two cases.
(i) For each $i \in S: \operatorname{deg}_{G}\left(a_{i}\right)=2 \Rightarrow f\left(b_{i}\right)=0$ and $\operatorname{deg}_{G}\left(a_{i}\right)=3 \Rightarrow f\left(a_{i}\right)=0$.
(ii) For some $t \in S$ : $\operatorname{deg}_{G}\left(a_{t}\right)=2 \wedge f\left(b_{t}\right)=1 \operatorname{or~}_{\operatorname{deg}_{G}}\left(a_{t}\right)=3 \wedge f\left(a_{t}\right)=1$.

We construct a function $\tilde{f}: V(G) \rightarrow\{0,1\}$ such that $\tilde{f}(u)=f(u)$ for each $u \in V\left(G^{\prime}\right)$. Let $u \in V(G) \backslash V\left(G^{\prime}\right)$. We define $\tilde{f}(u)$ depending on cases (i), (ii).
(i) Let $\tilde{f}\left(x_{r}\right)=0$ and let $\tilde{f}\left(x_{j}\right)=1$, for each $j \in S \backslash\{r\}$. For each $j \in S$, if $\operatorname{deg}_{G}\left(a_{j}\right)=2$, then $\tilde{f}\left(a_{j}\right)=1$,
(ii) Take any $t \in S$, if exists, such that $\operatorname{deg}_{G}\left(a_{t}\right)=2 \wedge f\left(b_{t}\right)=1$ and let $\tilde{f}\left(a_{t}\right)=1$. Then, for each $j \in S, j \neq t$, if $\operatorname{deg}_{G}\left(a_{j}\right)=2$, then $\tilde{f}\left(a_{j}\right)=1-f\left(b_{j}\right)$. Next, for each $j \in S$, if $\operatorname{deg}_{G}\left(a_{j}\right)=2 \wedge f\left(b_{j}\right)=0$, then $\tilde{f}\left(x_{j}\right)=1$. Finally, for each $j \in S$, if $\operatorname{deg}_{G}\left(a_{j}\right)=3$ or $\operatorname{deg}_{G}\left(a_{j}\right)=2 \wedge f\left(b_{j}\right)=1$, then $\tilde{f}\left(x_{j}\right)=$ $1-\tilde{f}\left(a_{(j+1)} \bmod s\right)$.
Let $H=G[\{u \in V(G): \tilde{f}(u)=1\}]$. In the case $(\mathrm{i}), x_{r} \notin V(H) \cap V(C)$. Hence, $H$ is a bipartite graph. For each $u \in V(G) \backslash V\left(G^{\prime}\right), u \neq x_{r}$, we have that $u \in V(H)$. Thus, $V(H)$ is a total dominating set of $G$ and $H \in \mathcal{B}(G)$, a contradiction.

In case (ii), if there is no $t \in S$ such that $\operatorname{deg}_{G}\left(a_{t}\right)=2 \wedge f\left(b_{t}\right)=1$, then, by assumption, there is $t \in S$ such that $\operatorname{deg}_{G}\left(a_{t}\right)=3 \wedge f\left(a_{t}\right)=1$, so finally $\tilde{f}\left(a_{t}\right)=1$ for some $t \in S$. Hence, there is $p \in S$ such that $\tilde{f}\left(x_{p}\right)=0$. Thus, $V(C) \backslash V(H) \neq \emptyset$.

Let us remind that for each $i \in S \backslash\{t\}$, if $\operatorname{deg}_{G}\left(a_{i}\right)=2$ and $\tilde{f}\left(a_{i}\right)=1$, then $f\left(b_{i}\right)=0$. Let $X=\left\{i \in S: \operatorname{deg}_{G}\left(a_{i}\right)=3 \wedge \tilde{f}\left(a_{i}\right)=1\right\} \cup\{t\}$. Suppose that for some two $i, j \in X$, there is a path in $H$ between $a_{i}$ and $a_{j}$ with successive vertices $x_{i}, x_{(i+1) \bmod s}, \ldots, x_{j}$. Hence, $\tilde{f}\left(x_{i}\right)=\tilde{f}\left(x_{(i+1) \bmod s}\right)=\cdots=\tilde{f}\left(x_{j}\right)=1$, which implies that $\tilde{f}\left(a_{(i+1) \bmod s}\right)=0, \tilde{f}\left(a_{(i+2) \bmod s}\right)=0, \ldots, \tilde{f}\left(a_{j}\right)=0$, a contradiction. Thus, $H$ is a bipartite graph.

For every $j \in S$ we have $N_{G}\left(a_{j}\right) \cap V(H) \neq \emptyset$, and $\tilde{f}\left(a_{j}\right)=1$ or $\tilde{f}\left(a_{j}\right)=$ $0 \wedge \tilde{f}\left(x_{(j-1) \bmod s}\right)=1$. Hence, we get $N_{G}\left(x_{j}\right) \cap V(H) \neq \emptyset$. Thus, $V(H)$ is a total dominating set and $H \in \mathcal{B}(G)$, a contradiction.

By Lemmas $10,12,13$ and Lemma 15 , and by the definition of $Q_{2}$-cycle we have the following corollary.

Corollary 16. Let $G \in \mathcal{M}$ and $v \in V(G)$. The following properties are satisfied:
(i) $\operatorname{deg}_{G}(v)=2$ if and only if vertex $v$ has all neighbors of degree 3,
(ii) $\operatorname{deg}_{G}(v)=3$ if and only if exactly one neighbor of $v$ has degree 2,
(iii) if $\operatorname{deg}_{G}(v)=3$, then there is exactly one $Q_{2}$-cycle containing $v$,
(iv) if $\operatorname{deg}_{G}(v)=2$, then vertex $v$ has two neighbors from disjoint $Q_{2}$-cycles.

By Corollary 16 we have the next corollary.
Corollary 17. Let $G \in \mathcal{M}$. The graph $G$ satisfies the following properties:
(i) there is an integer $q \geq 1$ such that $V(G)=D \cup \bigcup_{i=1}^{q} V\left(C_{i}\right)$, where for each $i \in\{1, \ldots, q\}$ the graph $C_{i}$ is a $Q_{2}$-cycle and $D$ is the set of all vertices of degree 2 ,
(ii) $E(G)=\left\{\{u, v\}: \exists_{i \in\{1, \ldots, q\}}\left(\{u, v\} \in E\left(C_{i}\right) \vee\left(u \in V\left(C_{i}\right) \wedge v \in D\right)\right)\right\}$.

Proof of Theorem 8. Suppose to the contrary that $G \in \mathcal{M}$.

By Corollary 17 , there is $q \geq 1$ such that $V(G)=D \cup \bigcup_{i=1}^{q} V\left(C_{i}\right)$, where for each $i \in\{1, \ldots, q\}$ the graph $C_{i}$ is a $Q_{2}$-cycle and $D$ is the set of all vertices of degree 2, and

$$
E(G)=\left\{\{u, v\}: \exists_{i \in\{1, \ldots, q\}}\left(\{u, v\} \in E\left(C_{i}\right) \vee\left(u \in V\left(C_{i}\right) \wedge v \in D\right)\right)\right\}
$$

Let $Q=\left(D \cup \bigcup_{i=1}^{q}\left\{c_{i}\right\}, E_{Q}\right)$, where for each $i \in\{1, \ldots, q\}$ vertex $c_{i}$ corresponds to the cycle $C_{i}$ and

$$
E_{Q}=\left\{\left\{v, c_{i}\right\}: i \in\{1, \ldots, q\} \wedge v \in D \wedge \exists_{x \in V\left(C_{i}\right)}\{v, x\} \in E(G)\right\}
$$

By Corollary 16 and Corollary 17 we have that $Q$ is a simple bipartite graph with partitions $D$ and $C=\bigcup_{i=1}^{q}\left\{c_{i}\right\}$. Obviously, for all vertices $v \in D$ and $c \in C$ we have that $\operatorname{deg}_{Q}(v)=2<\operatorname{deg}_{Q}(c)$. Thus, by Hall's Marriage Theorem [10] there is a matching $S$ in $Q$ covering all vertices from partition $C$.

Let

$$
S^{\prime}=\left\{\{v, x\} \in E(G): v \in D \wedge \exists_{i \in\{1, \ldots, q\}}\left\{v, c_{i}\right\} \in S \wedge x \in V\left(C_{i}\right)\right\}
$$

and let

$$
V^{\prime}=\left\{x \in \bigcup_{i=1}^{q} V\left(C_{i}\right): \exists_{e \in S^{\prime}} x \in e\right\}
$$

Let $H=G\left[V(G) \backslash\left(D \cup V^{\prime}\right)\right]$. For each $i \in\{1, \ldots, q\}$ there is $x$ such that $\{x\}$ $=V\left(C_{i}\right) \cap V^{\prime}$ and $N_{G}(x) \cap V(H) \neq \emptyset$. If $y \in V\left(C_{i}\right)$ and $x \neq y$, then $N_{G}(y) \cap$ $V(H) \neq \emptyset$. Hence, $H$ is an induced bipartite graph without isolated vertices. Since for each $v \in D$ at most one neighbor of $v$ belongs to $V^{\prime}$, we have $N_{G}(v) \cap V(H) \neq \emptyset$. Thus, $N_{G}(V(H))=V(G)$ and $H \in \mathcal{B}(G)$, a contradiction.

## 3. Interval Incidence 6-Coloring of Subcubic Graphs

In this section we prove our main result, i.e., Theorem 21 , which states $\chi_{i i}(G) \leq$ $2 \Delta(G)$ for each subcubic graph $G$. By Theorem 8 we have the following lemma.
Lemma 18. Let $G$ be a connected graph and $G \in \mathcal{G}_{3}^{2}$. Let $H \in \hat{\mathcal{B}}(G)$ and let $A, B \subset V(H)$ be any partition of $V(H)$, such that $A$ and $B$ are disjoint independent sets and $A \cup B=V(H)$. Then, $A$ and $B$ are disjoint independent dominating sets, and the graph $G[V(G) \backslash V(H)]$ has only isolated vertices and isolated edges.

Proof. Let $v \in V(G) \backslash V(H)$. If $N_{G}(v) \cap V(H) \subset A$ or $N_{G}(v) \cap V(H) \subset B$, then $G[V(H) \cup\{v\}]$ is a bipartite graph, a contradiction. Thus, $N_{G}(v) \cap A \neq \emptyset$ and $N_{G}(v) \cap B \neq \emptyset$. Let $v \in A(v \in B)$. Since $H$ is an induced graph without isolated vertices, we have $v \in N_{G}(B)\left(v \in N_{G}(A)\right)$. Hence, $A$ and $B$ are disjoint independent dominating sets.

Since $G$ is subcubic and $\left|N_{G}(v) \cap V(H)\right| \geq 2$ for any $v \in V(G) \backslash V(H)$, graph $G[V(G) \backslash V(H)]$ has only isolated vertices and isolated edges.

Lemma 19. Let $G$ be a subcubic non-bipartite graph with $\Delta(G)=3$. Then, there is a vertex coloring $c: V(G) \rightarrow\{1,2,3,4\}$ such that for each $v \in V(G)$ the following properties hold:
(i) if $\operatorname{deg}_{G}(v)=1$, then $c(v) \in\{1,4\}$,
(ii) if $\operatorname{deg}_{G}(v) \geq 2$ and $c(v) \neq p$, then $a_{p}(v) \geq 1$, for $p \in\{1,4\}$,
(iii) $a_{i}(v) \leq|c(v)-i|$, for $i \in\{1,2,3,4\}$,
where $a_{i}(v)=\left|\left\{w \in N_{G}(v): c(w)=i\right\}\right|$, for $i \in\{1,2,3,4\}$.
Proof. If $\delta(G)=1$, then we successively remove pendant vertices from graph $G$, until there is no pendant vertex. Let us denote the resulting graph by $G^{\prime}$. Obviously, $\delta\left(G^{\prime}\right) \geq 2$. Let us observe that we cut off all trees attached to $G$.

By Theorem 8 we have $\hat{\mathcal{B}}\left(G^{\prime}\right) \neq \emptyset$. Let $H$ be any element of $\hat{\mathcal{B}}\left(G^{\prime}\right)$ with the largest possible number of vertices.

Let $A, B \subset V(H)$ be any two partite sets of $V(H)$, i.e., $A$ and $B$ are disjoint independent sets and $A \cup B=V(H)$. By Lemma 18, $A$ and $B$ are disjoint independent dominating sets of $G^{\prime}$, and the graph $G\left[V\left(G^{\prime}\right) \backslash V(H)\right]$ has only isolated vertices and isolated edges. Let $I_{i} \subset V\left(G^{\prime}\right) \backslash V(H)$ be the set of all vertices of degree $i$ in $G^{\prime}$, for $i \in\{2,3\}$. Let us define the partition $I_{3}=I_{3}^{A} \cup I_{3}^{B} \cup I_{3}^{2}$ :

- $I_{3}^{A}=\left\{v \in I_{3}:\left|N_{G^{\prime}}(v) \cap A\right|=2 \wedge\left|N_{G^{\prime}}(v) \cap B\right|=1\right\}$,
- $I_{3}^{B}=\left\{v \in I_{3}:\left|N_{G^{\prime}}(v) \cap A\right|=1 \wedge\left|N_{G^{\prime}}(v) \cap B\right|=2\right\}$,
- $I_{3}^{2}=\left\{v \in I_{3}:\left|N_{G^{\prime}}(v) \cap A\right|=1 \wedge\left|N_{G^{\prime}}(v) \cap B\right|=1\right\}$.

Note that $I_{2}, I_{3}^{A}, I_{3}^{B}$ are independent sets in $G^{\prime}$, each vertex $v \in I_{3}^{2}$ belongs to an isolated edge in $G^{\prime}\left[I_{3}^{2}\right]$, and each vertex from $I_{2}$ has neighbors from $A$ and $B$.

Let us define a coloring $c: V(G) \rightarrow\{1,2,3,4\}$ in the following steps.
$\left(C_{1}\right)$ If $v \in A$, then $c(v)=1$, and if $v \in B$, then $c(v)=4$.
$\left(C_{2}\right)$ If $v \in I_{3}^{B}$, then $c(v)=2$, and if $v \in I_{3}^{A}$, then $c(v)=3$.
$\left(C_{3}\right)$ For each successive $v \in I_{2}$ we assign a color following the algorithm: if $c(v)$ is not determined, then let $\{u\}=N_{G^{\prime}}(v) \cap A$. If there is $x \in N_{G^{\prime}}(u)$ such that $c(x)=2$, then let $c(v)=3$. Otherwise, for each vertex $x \in N_{G^{\prime}}(u)$ either $c(x) \in\{3,4\}$ or $c(x)$ is not determined, and then let $c(v)=2$.
$\left(C_{4}\right)$ For each successive $\{v, w\} \in E\left(G^{\prime}\left[I_{3}^{2}\right]\right)$ we assign colors to both $v$ and $w$ following the algorithm: if $c(v)$ and $c(w)$ are not determined, then let $\{u\}=$ $N_{G^{\prime}}(v) \cap A$. If there is $x \in N_{G^{\prime}}(u)$ such that $c(x)=2$, then let $c(v)=3$ and $c(w)=2$. Otherwise, for each vertex $x \in N_{G^{\prime}}(u)$ either $c(x) \in\{3,4\}$ or $c(x)$ is not determined, and then let $c(v)=2$ and $c(w)=3$.
$\left(C_{5}\right)$ For each $v \in V\left(G^{\prime}\right)$ such that $\operatorname{deg}_{G^{\prime}}(v)<\operatorname{deg}_{G}(v)$, there is a tree $T_{v}$ such that $V\left(T_{v}\right) \subset V(G) \backslash V\left(G^{\prime}\right)$ and let $\{w\}=V\left(T_{v}\right) \cap N_{G}(v)$. Let $d: V\left(T_{v}\right) \rightarrow$
$\{a, b\}$ be a 2 -coloring of $T_{v}$ such that $d(w)=a$. Suppose $c(v) \leq 2$. For each $u \in V\left(T_{v}\right)$, if $d(u)=a$, then let $c(u)=4$, and if $d(u)=b$, then let $c(u)=1$. Suppose $c(v) \geq 3$. For each $u \in V\left(T_{v}\right)$, if $d(u)=a$, then let $c(u)=1$, and if $d(u)=b$, then let $c(u)=4$.
In step $\left(C_{1}\right)$ we colored $V(H)=A \cup B$ with colors 1 and 4 , in steps $\left(C_{2}\right)-\left(C_{4}\right)$ we colored vertices from $I_{2} \cup I_{3}$ with colors 2 or 3 , and in step $\left(C_{5}\right)$ we colored vertices from $V(G) \backslash V\left(G^{\prime}\right)$ with colors 1 or 4 . Since vertices colored with an arbitrary color form an independent set, $c$ is a vertex 4 -coloring of $G$.

Let $v \in V(G)$ and let $\operatorname{deg}_{G}(v)=1$. Then, $v \in V(G) \backslash V\left(G^{\prime}\right)$ and, by $\left(C_{5}\right)$, $c(v) \in\{1,4\}$. Thus, we get the property (i). Let $\operatorname{deg}_{G}(v) \geq 2$. If $v \in V(G) \backslash V\left(G^{\prime}\right)$, then, by $\left(C_{5}\right)$, the property (ii) holds. Let $v \in V\left(G^{\prime}\right)$. Since $A$ and $B$ are disjoint independent dominating sets of $G^{\prime}$, the property (ii) holds.

Since $c$ is a proper coloring of $G$, there is $a_{c(v)}(v)=0$ for each $v \in V(G)$.
Let $v \in V(G) \backslash V\left(G^{\prime}\right)$. By step $\left(C_{5}\right), c(v) \in\{1,4\}$. If $c(v)=1$, then $a_{2}(v)=0$, $a_{3}(v) \leq 1$ and $a_{4}(v) \leq 3$. If $c(v)=4$, then $a_{3}(v)=0, a_{2}(v) \leq 1$ and $a_{1}(v) \leq 3$.

Let $v \in V\left(G^{\prime}\right) \backslash V(H)$. If $v \in I_{3}^{A}$, then $c(v)=3, a_{1}(v)=2, a_{2}(v)=0, a_{4}(v)=$ 1. If $v \in I_{3}^{B}$, then $c(v)=2, a_{1}(v)=1, a_{3}(v)=0, a_{4}(v)=2$. If $v \in I_{2}$, then $c(v) \in$ $\{2,3\}$. If $\operatorname{deg}_{G^{\prime}}(v)=\operatorname{deg}_{G}(v)$, then $a_{1}(v)=a_{4}(v)=1$, and $a_{2}(v)=a_{3}(v)=0$. If $\operatorname{deg}_{G^{\prime}}(v)<\operatorname{deg}_{G}(v)$, then if $c(v)=2$, then $a_{1}(v)=1, a_{2}(v)=a_{3}(v)=0, a_{4}(v)=$ 2 , and if $c(v)=3$, then $a_{1}(v)=2, a_{2}(v)=a_{3}(v)=0, a_{4}(v)=1$. If $v \in I_{3}^{2}$, then $c(v) \in\{2,3\}$. If $c(v)=2$, then $a_{1}(v)=a_{3}(v)=a_{4}(v)=1$. If $c(v)=3$, then $a_{1}(v)=a_{2}(v)=a_{4}(v)=1$.

Let $v \in A \cup B$. Since $A$ and $B$ are disjoint dominating sets of $G^{\prime}$ and $H \in \hat{\mathcal{B}}\left(G^{\prime}\right)$, it suffices to prove that if $c(v)=1$, then $a_{2}(v) \leq 1$, and if $c(v)=4$, then $a_{3}(v) \leq 1$.

Suppose to the contrary that $c(v)=1$ and $a_{2}(v)=2$ for some $v \in A$. The case $c(v)=4$ and $a_{3}(v)=2$, for some $v \in B$, is analogous. Let $x, y \in N_{G^{\prime}}(v)$ such that $c(x)=c(y)=2$. Since $B$ is a dominating set of $G^{\prime}$, there is $w \in N_{G^{\prime}}(v) \cap B$ with $c(w)=4$. By the definition of coloring $c$, we have $v, x, y, w \in V\left(G^{\prime}\right)$ and $v, w \in V(H)$.

Since $c(x)=c(y)=2$, we have $a_{1}(x)=a_{1}(y)=1, a_{3}(x) \leq 1, a_{3}(y) \leq 1$, $1 \leq a_{4}(x) \leq 2$ and $1 \leq a_{4}(y) \leq 2$. Let us consider the following cases:

- $x \notin N_{G^{\prime}}(w)$ and $y \notin N_{G^{\prime}}(w)$. If edge $\{v, w\}$ is isolated in $H$, then let $W=V(H) \cup\{x, y\}$. Otherwise, let $W=V(H) \cup\{x, y\} \backslash\{v\}$.
- $x \in N_{G^{\prime}}(w)$ or $y \in N_{G^{\prime}}(w)$. Let $W=V(H) \cup\{x, y\} \backslash\{v\}$.

In both cases, the graph $G^{\prime}[W] \in \mathcal{B}\left(G^{\prime}\right)$ and $\left|V\left(G^{\prime}[W]\right)\right|>|V(H)|$, a contradiction. Thus, the coloring $c$ satisfies the property (iii).

Proposition 20. [14] For any graph $G, \Delta(G)+1 \leq \chi_{i i}(G) \leq \chi(G) \cdot \Delta(G)$.
We prove that an interval incidence 6 -coloring always exists for any subcubic graph $G$ with $\Delta(G)=3$.

Theorem 21. Let $G$ be a subcubic graph. Then, $\chi_{i i}(G) \leq 2 \Delta(G)$.
Proof. If $G$ is a subcubic bipartite graph, then by Proposition 20 we have $\chi_{i i}(G) \leq 2 \Delta(G)$. If $\Delta(G)=2$, then one can easily construct an interval incidence 4 -coloring. Thus, $\chi_{i i}(G) \leq 2 \Delta(G)$. Let $G$ be a subcubic non-bipartite graph with $\Delta(G)=3$. By Lemma 19 , there is a vertex coloring $c: V(G) \rightarrow\{1,2,3,4\}$ satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring $f: I(G) \rightarrow\{1,2,3,4,5,6\}$ in three steps.
In the first step, using the coloring $c$, we define the interval $A_{f}(v)$ for each vertex $v \in V(G)$, as follows. If $\operatorname{deg}_{G}(v)=2$ and $c(v) \in\{2,3\}$, then let $A_{f}(v)=$ $\{3,4\}$. If $c(v)=4$ and $\operatorname{deg}_{G}(v)=1$, then $A_{f}(v)=\{6\}$. If $c(v)=4$ and $\operatorname{deg}_{G}(v)=2$, then $A_{f}(v)=\{5,6\}$. In the other cases, let $A_{f}(v)=\{c(v), \ldots, c(v)+$ $\left.\operatorname{deg}_{G}(v)-1\right\}$. Thus, by Lemma 19 (i)-(iii) we get
$\left(a_{1}\right)$ if $\operatorname{deg}_{G}(v)=1$, then $c(v) \in\{1,4\}$ and $A_{f}(v)=\{c(v)\}$,
$\left(a_{2}\right)$ if $\operatorname{deg}_{G}(v)=2$, then if $c(v) \in\{1,3\}$, then $A_{f}(v)=\{c(v), c(v)+1\}$ and if $c(v) \in\{2,4\}$, then $A_{f}(v)=\{c(v)+1, c(v)+2\}$,
$\left(a_{3}\right)$ if $\operatorname{deg}_{G}(v)=3$, then $A_{f}(v)=\{c(v), c(v)+1, c(v)+2\}$.
In the second step, for each $v \in V(G)$, we construct a sequence $L_{f}(v)$ (i.e., a linear ordered set) from elements of $N_{G}(v)$, as follows (see Figure 2).
$\left(l_{1}\right)$ Suppose $\operatorname{deg}_{G}(v)=1$. If $N_{G}(v)=\{x\}$, then let $L_{f}(v)=(x)$.
$\left(l_{2}\right)$ Suppose $\operatorname{deg}_{G}(v)=2$. Let $N_{G}(v)=\{x, y\}$, where $c(x) \leq c(y)$. Then,

- if $c(v) \in\{1,4\}$, then let $L_{f}(v)=(x, y)$,
- if $c(v) \in\{2,3\}$, then let $L_{f}(v)=(y, x)$.
( $l_{3}$ ) Suppose $\operatorname{deg}_{G}(v)=3$. Let $N_{G}(v)=\{x, y, z\}$, where $c(x) \leq c(y) \leq c(z)$. Then,
- if $c(v) \in\{1,4\}$, then let $L_{f}(v)=(x, y, z)$,
- if $c(v)=2$, then let $L_{f}(v)=(y, z, x)$,
- if $c(v)=3$, then let $L_{f}(v)=(z, x, y)$.

By $v_{i}$ we mean the $i$-th element of the sequence $L_{f}(v)$, i.e., $L_{f}(v)=\left(v_{1}, \ldots\right)$.
In the final step, for each vertex $v$, we define the incidence coloring $f$ as follows: $f\left(v,\left\{v, v_{i}\right\}\right)=\min A_{f}(v)+i-1$, for $i \in\left\{1, \ldots, \operatorname{deg}_{G}(v)\right\}$.

In Figure 2 the white vertex is the vertex $v$, and the list above is $L_{f}(v)$. By Lemma 19 (i)-(iii), the set of all possible values of $c$ of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e., $v$ ) are given at the edges adjacent to $v$.

Obviously, all the incidences at vertex $v$ are colored with different colors from $A_{f}(v)$. Observe that the set of colors $A_{f}(v)$ is an interval of integers.

We prove that the coloring $f$ is an incidence coloring. It is enough to prove that for each vertex $v \in V(G)$ and each vertex $w \in N_{G}(v)$ we have $f(v,\{v, w\}) \notin$ $A_{f}(w)$, or, equivalently, $f(v,\{v, w\})<\min A_{f}(w)$ or $f(v,\{v, w\})>\max A_{f}(w)$.


Figure 2. Interval coloring of incidences at the white vertex $v$, according to its degree and the values of $c$ at the neighbors $x, y, z$ of $v$. The set of possible values of $c$ of a vertex is given in the curly brackets below the vertex. The list $L_{f}(v)$ is given above the white vertex $v$.

Suppose that $c(v)=1$. Then, $A_{f}(v) \subset\{1,2,3\}$ and $\min A_{f}(v)=1$. By the construction of $L_{f}(v)$ we have: if $\operatorname{deg}_{G}(v) \geq 1$, then $c\left(v_{1}\right) \in\{2,3,4\}$, and if $\operatorname{deg}_{G}(v)=2$, then $c\left(v_{2}\right)=4$, and if $\operatorname{deg}_{G}(v)=3$, then $c\left(v_{2}\right) \in\{3,4\}$ and $c\left(v_{3}\right)=4$ (see Figure 2). Hence, for each $i \in\left\{1, \ldots, \operatorname{deg}_{G}(v)\right\}$ we have $f\left(v,\left\{v, v_{i}\right\}\right)=$ $\min A_{f}(v)+i-1<i+1 \leq \min A_{f}\left(v_{i}\right)$.

Suppose that $c(v)=2$. Then, $A_{f}(v) \subset\{2,3,4\}$. Let $\operatorname{deg}_{G}(v)=3$. Hence, $\min A_{f}(v)=2$, and $c\left(v_{1}\right) \in\{3,4\}$ and $c\left(v_{2}\right)=4 \wedge c\left(v_{3}\right)=1$. Thus, $f\left(v,\left\{v, v_{i}\right\}\right)=$ $\min A_{f}(v)+i-1=i+1<i+2 \leq \min A_{f}\left(v_{i}\right)$, for $i \in\{1,2\}$, and $f\left(v,\left\{v, v_{3}\right\}\right)=$ $\min A_{f}(v)+2=4>3 \geq \max A_{f}\left(v_{3}\right)$. Let $\operatorname{deg}_{G}(v)=2$. Hence, $\min A_{f}(v)=3$, and $c\left(v_{1}\right)=4$ and $c\left(v_{2}\right)=1$. Thus, $f\left(v,\left\{v, v_{1}\right\}\right)=\min A_{f}(v)=3<4 \leq$ $\min A_{f}\left(v_{1}\right)$ and $f\left(v,\left\{v, v_{2}\right\}\right)=4>3 \geq \max A_{f}\left(v_{2}\right)$.

Suppose that $c(v)=3$. Then, $A_{f}(v) \subset\{3,4,5\}$ and $\min A_{f}(v)=3$. Let $\operatorname{deg}_{G}(v)=3$. Hence, $c\left(v_{1}\right)=4$ and $c\left(v_{2}\right)=1$ and $c\left(v_{3}\right) \in\{1,2\}$. Thus, $f\left(v,\left\{v, v_{1}\right\}\right)=\min A_{f}(v)=3<4 \leq \min A_{f}\left(v_{1}\right)$, and $f\left(v,\left\{v, v_{i}\right\}\right)=\min A_{f}(v)+$ $i-1>i+1 \geq \max A_{f}\left(v_{i}\right)$, for $i \in\{2,3\}$. Let $\operatorname{deg}_{G}(v)=2$. Hence, $c\left(v_{1}\right)=4$ and $c\left(v_{2}\right)=1$. Thus, $f\left(v,\left\{v, v_{1}\right\}\right)=3<4 \leq \min A_{f}\left(v_{1}\right)$ and $f\left(v,\left\{v, v_{2}\right\}\right)=4>3 \geq$ $\max A_{f}\left(v_{2}\right)$.

Suppose that $c(v)=4$. Then, $A_{f}(v) \subset\{4,5,6\}$. Let $\operatorname{deg}_{G}(v)=3$. Hence, $c\left(v_{1}\right)=1$ and $c\left(v_{2}\right) \in\{1,2\}$ and $c\left(v_{3}\right) \in\{1,2,3\}$ and $c\left(v_{2}\right) \leq c\left(v_{3}\right)$. Thus, $f\left(v,\left\{v, v_{i}\right\}\right)=\min A_{f}(v)+i-1 \geq i+3>i+2 \geq \max A_{f}\left(v_{i}\right)$, for each $i \in\{1,2,3\}$. Let $\operatorname{deg}_{G}(v)=2$. Hence, $c\left(v_{1}\right)=1$ and $c\left(v_{2}\right) \in\{1,2,3\}$, and $A_{f}(v)=\{5,6\}$. Thus, $f\left(v,\left\{v, v_{1}\right\}\right)=5>\max A_{f}\left(v_{1}\right)$ and $f\left(v,\left\{v, v_{2}\right\}\right)=6>\max A_{f}\left(v_{2}\right)$. Let $\operatorname{deg}_{G}(v)=1$. Hence, $c\left(v_{1}\right) \in\{1,2,3\}$. Thus, $f\left(v,\left\{v, v_{1}\right\}\right)=6>5 \geq \max A_{f}\left(v_{1}\right)$.

In all the cases we proved that $f\left(v,\left\{v, v_{i}\right\}\right) \notin A_{f}\left(v_{i}\right)$ for each $v_{i} \in N_{G}(v)$. Thus, $f$ is an interval incidence 6 -coloring of $G$.

## 4. Summary

In this paper we proved that for any subcubic graph $G$, $\chi_{i i}(G) \leq 2 \Delta(G)$. In [14] we proved that the upper bound of $2 \Delta(G)$ on $\chi_{i i}(G)$ holds for each complete $k$-partite graph $G$ and this bound is valid for other classes of graphs. Thus, we state the following

Conjecture 22 [Interval Incidence Coloring Conjecture (IICC)]. For any graph $G, \chi_{i i}(G) \leq 2 \Delta(G)$.

## References

[1] N. Alon, C. McDiarmid and B. Reed, Star arboricity, Combinatorica 12 (1992) 375-380.
doi:10.1007/BF01305230
[2] A. Asratian and R. Kamalian, Investigation on interval edge-colorings of graphs, J. Combin. Theory Ser. B 62 (1994) 34-43.
doi:10.1006/jctb.1994.1053
[3] R.A. Brualdi and J.Q. Massey, Incidence and strong edge colorings of graphs, Discrete Math. 122 (1993) 51-58. doi:10.1016/0012-365X(93)90286-3
[4] M. Hosseini Dolama, E. Sopena and X. Zhu, Incidence coloring of $k$-degenerated graphs, Discrete Math. 283 (2004) 121-128. doi:10.1016/j.disc.2004.01.015
[5] M. Hosseini Dolama and E. Sopena, On the maximum average degree and the incidence chromatic number of a graph, Discrete Math. Theor. Comput. Sci. 7 (2005) 203-216.
[6] K. Giaro, Interval edge-coloring, in: Graph Colorings, Contemporary Mathematics AMS, M. Kubale Ed. (2004) 105-121. doi:10.1090/conm/352/08
[7] K. Giaro, M. Kubale and M. Małafiejski, Compact scheduling in open shop with zero-one time operations, INFOR Inf. Syst. Oper. Res. 37 (1999) 37-47. doi:10.1080/03155986.1999.11732367
[8] K. Giaro, M. Kubale and M. Małafiejski, Consecutive colorings of the edges of general graphs, Discrete Math. 236 (2001) 131-143. doi:0.1016/S0012-365X(00)00437-4
[9] B. Guiduli, On incidence coloring and star arboricity of graphs, Discrete Math. 163 (1997) 275-278. doi:10.1016/0012-365X(95)00342-T
[10] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935) 26-30.
[11] R. Janczewski, A. Małafiejska and M. Małafiejski, Interval incidence coloring of graphs, Zesz. Nauk. Pol. Gd. 13 (2007) 481-488, in Polish.
[12] R. Janczewski, A. Małafiejska and M. Małafiejski, Interval wavelength assignment in all-optical star networks, in: PPAM 2009 (Springer Verlag, 2010) Lecture Notes in Comput. Sci. 6067 (2010) 11-20. doi:10.1007/978-3-642-14390-8_2
[13] R. Janczewski, A. Małafiejska and M. Małafiejski, Interval incidence coloring of bipartite graphs, Discrete Math. 166 (2014) 131-140. doi:10.1016/j.dam.2013.10.007
[14] R. Janczewski, A. Małafiejska and M. Małafiejski, Interval incidence graph coloring, Discrete Math. 182 (2015) 73-83. doi:10.1016/j.dam.2014.03.006
[15] R. Janczewski, A. Małafiejska and M. Małafiejski, On incidence coloring of complete multipartite and semicubic bipartite graphs, Discuss. Math. Graph Theory (2017), in press.
[16] X. Li and J. Tu, NP-completeness of 4-incidence colorability of semi-cubic graphs, Discrete Math. 308 (2008) 1334-1340. doi:10.1016/j.disc.2007.03.076
[17] M. Maydanskiy, The incidence coloring conjecture for graphs of maximum degree three, Discrete Math. 292 (2005) 131-141. doi:10.1016/j.disc.2005.02.003
[18] A.C. Shiau, T.-H. Shiau and Y.-L. Wang, Incidence coloring of Cartesian product graphs, Inform. Process. Lett. 115 (2015) 765-768. doi:10.1016/j.ipl.2015.05.002
[19] W.C. Shiu and P.K. Sun, Invalid proofs on incidence coloring, Discrete Math. 308 (2008) 6575-6580.
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