Discussiones Mathematicae
Graph Theory 37 (2017) 315–336
doi:10.7151/dmgt.1958

FORBIDDEN STRUCTURES FOR PLANAR PERFECT CONSECUTIVELY COLOURABLE GRAPHS

Marta Borowiecka-Olszewska

and

Ewa Drgas-Burchardt

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Z. Szafrana 4a, 65–516 Zielona Góra, Poland

e-mail: M.Borowiecka-Olszewska@wmie.uz.zgora.pl
E.Drgas-Burchardt@wmie.uz.zgora.pl

Abstract

A *consecutive colouring* of a graph is a proper edge colouring with positive integers in which the colours of edges incident with each vertex form an interval of integers. The idea of this colouring was introduced in 1987 by Asratian and Kamalian under the name of *interval colouring*. Sevastjanov showed that the corresponding decision problem is NP-complete even restricted to the class of bipartite graphs. We focus our attention on the class of consecutively colourable graphs whose all induced subgraphs are consecutively colourable, too. We call elements of this class *perfect consecutively colourable* to emphasise the conceptual similarity to perfect graphs. Obviously, the class of perfect consecutively colourable graphs is induced hereditary, so it can be characterized by the family of induced forbidden graphs. In this work we give a necessary and sufficient conditions that must be satisfied by the generalized Sevastjanov rosette to be an induced forbidden graph for the class of perfect consecutively colourable graphs. Along the way, we show the exact values of the deficiency of all generalized Sevastjanov rosettes, which improves the earlier known estimating result. It should be mentioned that the deficiency of a graph measures its closeness to the class of consecutively colourable graphs. We motivate the investigation of graphs considered here by showing their connection to the class of planar perfect consecutively colourable graphs.

Keywords: edge colouring, consecutive (interval) colouring, deficiency, Sevastjanov graph, forbidden graph.

2010 Mathematics Subject Classification: 05C75, 05C15, 05C35.

1. Introduction

A proper edge colouring of a graph with integers such that the colours of edges incident with each vertex form an interval of integers is called *consecutive*, and the graph for which there exists a consecutive colouring is said to be *consecutively colourable*. The idea of this colouring called first an *interval colouring* of a graph was introduced in 1987 by Asratian and Kamalian [2]. The interval colouring of a graph under the name of consecutive colouring was also investigated by Giaro and Kubale in [9]. Sevastjanov [22] showed that the existence of a consecutive colouring of a given graph is an NP-complete problem even in a class of bipartite graphs. Many problems concerning arranging tasks and creating schedules which do not allow any pauses in work may be solved by the construction of a consecutive graph colouring. Doubtlessly, this fact has resulted in several papers dealing with this topic. Most of them concern bipartite graphs in connection with applications [1, 3, 5, 8, 13–19, 21, 22].

It is worth mentioning that consecutive colourability is one of the nonhereditary concepts in graph theory. It means that there are graphs having a consecutive colouring whose some induced subgraphs do not have such a property. For example all complete graphs of even order are consecutively colourable unlike any complete graph of odd order. It provoked searching for graphs that are not consecutively colourable and are minimal with respect to some graph invariants. The wide literature on this subject is addressed to the minimality with respect to the number of vertices, maximum degree, cyclomatic number or other parameters (e.g. [16, 20]).

Note that many nonhereditary concepts in graph theory are very popular subjects of scientific research. In addition, some of them inspired new concepts that are hereditary and seem to be even more popular. The most notable example is the class of graphs G for which the equality $\omega(G) = \chi(G)$ holds, where $\omega(G)$ and $\chi(G)$ are the clique and chromatic numbers of G, respectively. Evidently this class is not hereditary, which is confirmed, for instance, by facts $\omega(C_5 \cup K_3) = 3 = \chi(C_5 \cup K_3)$ and $\omega(C_5) = 2 \neq 3 = \chi(C_5)$. Moreover, $\omega(K_2) = 2 = \chi(K_2)$, which means that even the class of graphs G for which $\omega(G)$ and $\chi(G)$ are different numbers is not hereditary. On the other hand, the class of perfect graphs, that consists of all graphs G for which the equality $\omega(G) = \chi(G)$ holds for all its induced subgraphs, is induced hereditary. Moreover, it is one of the most frequently quoted classes of graphs in the literature.

We focus our attention on the class of consecutively colourable graphs whose all induced subgraphs are consecutively colourable, too. We denote this class by $\mathfrak{N}_{\leq}$ and call their elements *perfect consecutively colourable* graphs to emphasise the conceptual similarity to perfect graphs. Obviously, the class of perfect consecutively colourable graphs is induced hereditary, so it can be characterized by the family $\mathcal{C}(\mathfrak{N}_{\leq})$ of minimal induced forbidden graphs. Such a family consists of all graphs which are not consecutively colourable and are very close to having such a colouring.

The main subject considered in this paper was introduced by ourselves in [6]. We continue the study in this field with the aim of characterizing all minimal induced forbidden graphs in $\mathcal{C}(\mathfrak{N}_{\leq})$. In general, the characterization of all forbidden graphs for the class $\mathfrak{N}_{\leq}$ seems to be a very difficult problem. We shall show in Remark 2 that the investigation can be restricted to bipartite graphs, but even that, the problem remains difficult. The knowledge on consecutive colourability of bipartite graphs is not wide. Frequently, the corresponding results solve problems only in some subclasses such as complete bipartite graphs, forests, (α, β)-biregular graphs [1, 15, 21, 22]. It has to be mentioned that even these narrow statements have been paid for with great effort. One can find (e.g. in [10]) that all outerplanar bipartite graphs are consecutively colourable. It implies that $\mathcal{C}(\mathfrak{N}_{\leq})$ does not contain any outerplanar bipartite graph and motivates asking about planar graphs in $\mathcal{C}(\mathfrak{N}_{\leq})$ (some results on consecutive colourability of planar graphs one can find in [4]).

The additional motivation for this study comes from Lemma 4 presented herein. We know that planarity of graphs is closed under graph homeomorphism. Although consecutive colourability does not have such a property, Lemma 4 shows some connection with this feature. It makes intuition to consider both these concepts simultaneously.

The first class of planar bipartite graphs that have no consecutive colouring whose all proper induced subgraphs have consecutive colourings was considered in [6]. We would like to mention that generalized Sevastjanov rosettes from $\mathcal{C}(\mathfrak{N}_{\leq})$, introduced and characterized in this paper, create a second infinite class of graphs of this type described in the literature.

Our main contribution is presented in Theorem 4, which characterize all generalized Sevastjanov rosettes in $\mathcal{C}(\mathfrak{N}_{\leq})$. Along the way, we show in Theorem 3 the exact values of the deficiency of graphs analysed in the paper. It improves the corresponding theorem from [11]. We believe that the supporting results, presented in Theorem 1 and Lemma 4 , can play a role of tools to obtain many other facts connected with the subject of the paper.

2. Basic Definitions and Concepts

In general, we follow the notation and terminology of [7]. The graphs which we consider are finite, undirected, without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any set $S \subseteq V(G)$ the symbol $G[S]$ denotes the subgraph of G induced by S. By $deg_G(v)$ and $\Delta(G)$ we denote the degree of a vertex v in a graph G and the maximum degree over all vertices of G. The set of neighbours of a vertex $v \in G$ is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. An *attachment of a pendant edge* to a graph G is an operation resulting in a graph G' such that $V(G') = V(G) \cup \{v\}$, $E(G') = E(G) \cup \{vx\}$, where x is an arbitrary vertex of G and $v \notin V(G)$. If H is an induced subgraph of the graph G, then we write $H \leq G$, and respectively $H < G$ to emphasize that the graph H is a proper induced subgraph of G. The equality $G = H$ means that graphs G and H are isomorphic. Let G_1, G_2 be graphs such that $V(G_1) \cap V(G_2)$ can be nonempty. By $G_1 \cup G_2$ we mean a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. A *planar* graph is a graph that can be drawn on the plane in such a way that no its edges cross each other. A planar graph in which every vertex lies on the boundary of the outer face is called *outerplanar*. A complete graph, a cycle and a path of order n are denoted by K_n, C_n and P_n, respectively. A complete bipartite graph with partite sets of order n_1 and n_2 is denoted by K_{n_1,n_2}.

The symbols $\mathbb{Z}$ and $\mathbb{N}$ stand for the set of integers and positive integers, respectively, and moreover $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $a, b \in \mathbb{N}_0$, $k \in \mathbb{N}$. If $a \leq b$, then $[a, b]$ denotes the set $\{a, a+1, \ldots, b-1, b\}$. If $a > b$, then $[a, b] = \emptyset$ and $\sum_{i=a}^{b} c_i = 0$ for any c_i. We adopt the convention $[1, b] = [b]$. Moreover, we assume that the last element of an ordered set $S = \{a, a+k, a+2k, \ldots, b\}$ is b, and $S = \emptyset$ if $a > b$.

In the paper we use a concept of a *proper edge colouring* of G, i.e., a mapping $c: E(G) \to \mathbb{Z}$ such that for any adjacent $e_1, e_2 \in E(G)$ the values $c(e_1)$ and $c(e_2)$ are different. Since the values of edge colourings that are considered herein are numbers, we use the words colour and number interchangeably.

Definition 1. Let A be a finite set of integers. The *deficiency* $\mathit{def}(A)$ of A is the number of integers between $\min A$ and $\max A$ not belonging to A.

Clearly, $\mathit{def}(A) = \max A - \min A - |A| + 1$. A set $A \in \mathbb{Z}$ for which $\mathit{def}(A) = 0$ is called an *interval*.

A proper edge colouring of a graph with integers such that the colours of edges incident with each vertex form an interval of integers is called *consecutive*, and the graph for which there exists a consecutive colouring is said to be *consecutively colourable*.

Definition 2. Let G be a graph, and let $c: E(G) \to \mathbb{N}$ be its proper edge colouring. The *deficiency of c at vertex $v \in V(G)$*, denoted by $\mathit{def}(G, c, v)$, is the deficiency of the set of colours of edges incident with v. Furthermore, $\sum_{v \in V(G)} \mathit{def}(G, c, v)$ is called the *deficiency of the colouring c* and denoted by $\mathit{def}(G, c)$. Next, $\min_c \mathit{def}(G, c)$, where min is taken over all proper edge colourings of G, is denoted by $\mathit{def}(G)$ and called the *deficiency of the graph G*.

Clearly, if $def(G) = 0$, then the graph G is consecutively colourable. It is also easy to see that $def(G)$ is equal to the minimum number of pendant edges whose attachment to G makes it consecutively colourable. From now on, the class of consecutively colourable graphs will be denoted by $\mathfrak{N}$.

Remark 1. The class $\mathfrak{N}$ has e.g. the following properties.

(a) If G_1, G_2 are disjoint graphs from $\mathfrak{N}$, then the graph G obtained by adjoining G_1 and G_2 in some vertex v, i.e., obtained from $G_1 \cup G_2$ by the identification of some vertex of G_1 with some vertex of G_2, belongs to $\mathfrak{N}$. In particular, $\mathfrak{N}$ is closed under sequential attachment of pendant edges.

(b) Every odd cycle does not belong to $\mathfrak{N}$.

(c) Every bipartite graph with the cyclomatic number at most 8 belongs to $\mathfrak{N}$ (the cyclomatic number is the minimum number of edges which must be removed from a graph to destroy all its cycles).

(d) Every bipartite graph with bipartition (V_1, V_2) for which $\min\{|V_1|, |V_2|\} \leq 3$ belongs to $\mathfrak{N}$.

(e) Every bipartite graph G with $|V(G)| \leq 14$ belongs to $\mathfrak{N}$.

(f) Every bipartite graph G with $\Delta(G) \leq 3$ belongs to $\mathfrak{N}$.

(g) Every bipartite graph G without a vertex of degree 3 that satisfies $\Delta(G) = 4$ belongs to $\mathfrak{N}$.

Proof. The appreciate theorems with references and proofs one can find e.g. in [16] (Chapter 8). ■

Among graphs which do not have any consecutive colouring it is worth finding these ones that lacks little to be consecutive colourable. For this purpose we defined in [6] an induced hereditary class of graphs

$$\mathfrak{N}_{\leq} = \{G\colon \text{each } H \leq G \text{ satisfies } H \in \mathfrak{N}\}.$$

Elements of the class $\mathfrak{N}_{\leq}$ are called *perfect consecutively colourable* graphs.

For completeness we recall that a class of graphs $\mathcal{P}$ is *induced hereditary* if it is closed with respect to taking induced subgraphs. It is known that the induced hereditary class of graphs $\mathcal{P}$ can be uniquely characterized by the family $\mathcal{C}(\mathcal{P})$ of its *minimal induced forbidden graphs* [12], i.e.,

$$\mathcal{C}(\mathcal{P}) = \{G \notin \mathcal{P}\colon \text{each } H < G \text{ satisfies } H \in \mathcal{P}\}.$$

By the above definition, the graphs from $\mathcal{C}(\mathcal{P})$ are not in $\mathcal{P}$, but are very close to $\mathcal{P}$. Since the class $\mathfrak{N}_{\leq}$ is induced hereditary, it can be uniquely characterized by the family

$$\mathcal{C}(\mathfrak{N}_{\leq}) = \{G \notin \mathfrak{N}_{\leq}\colon \text{each } H < G \text{ satisfies } H \in \mathfrak{N}_{\leq}\}.$$

Remark 2 [6]**.** The set of odd cycles $\{C_{2k+1}: k \in \mathbb{N}\}$ is contained in $\mathcal{C}(\mathfrak{N}_{\leq})$. Moreover, any other non-bipartite graph does not belong to $\mathcal{C}(\mathfrak{N}_{\leq})$.

By Remark 2 we see that in the class $\mathcal{C}(\mathfrak{N}_{\leq})$ there are only odd cycles and bipartite graphs. The problem is to characterize bipartite graphs that belong to $\mathcal{C}(\mathfrak{N}_{\leq})$.

Remark 3 [6]**.** If $G \in \mathcal{C}(\mathfrak{N}_{\leq})$ is a bipartite graph with bipartition (V_1, V_2), then G has the following properties.

(a) G is 2 -connected (i.e., it has more than two vertices and the graph resulting by the removal of any vertex is connected).

(b) The cyclomatic number of G is greater than 8.

(c) $|V_1| \geq 4$ and $|V_2| \geq 4$.

(d) $\Delta(G) \geq 4$ and $|V(G)| \geq 15$.

(e) If $\Delta(G) = 4$, then G has a vertex of degree 3.

Definition 3. Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. The graph denoted by $S_{\mathbf{l},\mathbf{s}}$ and presented in Figure 1 is called the *generalized Sevastjanov rosette*. It means that $V(S_{\mathbf{l},\mathbf{s}}) = \{x, y_1, y_2, y_3\} \cup \{v_{i,j}: i \in [3], j \in [l_i]\} \cup \{a_{i,j}: i \in [3], j \in [s_i - 1]\}$, and $E(S_{\mathbf{l},\mathbf{s}}) = \{xv_{i,j}: i \in [3], j \in [l_i]\} \cup \{v_{i,j}y_i: i \in [3], j \in [l_i]\} \cup \{a_{i,j}a_{i,j+1}: i \in [3], j \in [s_1 - 2]\} \cup \{y_1a_{3,1}, y_1a_{2,s_2-1}, y_2a_{1,1}, y_2a_{3,s_3-1}, y_3a_{2,1}, y_3a_{1,s_1-1}\}$.

Note that s_i is the length of the path in front of y_i that joins y_j and y_k, where $i, j, k \in [3]$ are distinct indices.

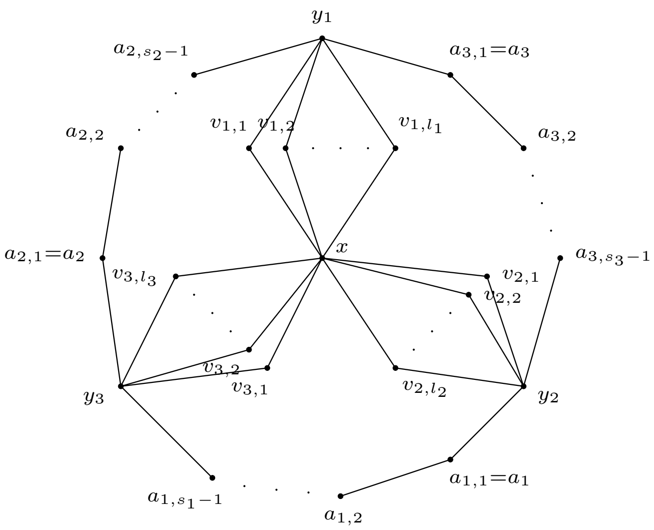

Figure 1. The graph $S_{\mathbf{l},\mathbf{s}}$.

Let us recall that Giaro *et al.* in [11] considered only graphs $S_{\mathbf{l},\mathbf{s}}$ with $\mathbf{l} = (l,l,l)$ and $\mathbf{s} = (2,2,2)$, which were denoted by S_l. For the simplicity of the notation, let $S_{(l,l,l),(s,s,s)} = S_{l,s}$ and $S_{l,2} = S_l$.

From now on, when we consider the generalized Sevastjanov rosette we use the notation and labels of the vertices used in Definition 3 and presented in Figure 1. In the whole paper, especially in proofs of the next lemmas and theorems , we consider an induced subgraph H_i of the graph $G = S_{\mathbf{l},\mathbf{s}}$, where $H_i = G\,[N_G[y_i] \cup x]$, $i \in [3]$. For the simplicity of the notation in the case $\mathbf{s} = (2,2,2)$, the only interior vertex of the path in front of y_i is denoted by a_i, $i \in [3]$.

Let $H = S_{(l_1,l_2,l_3),(s_1,s_2,s_3)}$ be the generalized Sevastjanov rosette and $\widetilde{H} = H[V(H_1) \cup V(H_2)]$. By $S_{(l_1,l_2),(s_1,s_2)}$ we mean a graph obtained from $\widetilde{H}$ by joining the vertices $a_{3,1}$ and a_{3,s_3-1} by a path of the length $s_1 - 2$, and the vertices $a_{1,1}$ and a_{2,s_2-1} by a path of the length $s_2 - 2$. The graph $S_{(l_1,l_2),(s_1,s_2)}$ is also called the generalized Sevastjanov rosette and is denoted by $S_{\mathbf{l},\mathbf{s}}$ with $\mathbf{l} = (l_1, l_2)$ and $\mathbf{s} = (s_1, s_2)$.

We also write $S_{(l_1,0,0),(s_1,s_2,s_3)}$ or $S_{(0,0,0),(s_1,s_2,s_3)}$ when we consider a proper induced subgraph of the graph $S_{\mathbf{l},\mathbf{s}}$, but such a graph is not the generalized Sevastjanov rosette.

3. Preliminary Results

For a fixed graph G, let $\gamma\colon E(G) \to \mathbb{Z}$ be its proper edge colouring. By $S(v,\gamma)$ we denote the set of colours of the edges incident with a vertex $v \in V(G)$ in the colouring γ.

In the next three lemmas and theorem we consider a graph G such that $V(G) = \{v_i\colon i \in [l]\} \cup \{x, y, z_1, z_2\}$, $E(G) = \{xv_i, v_iy\colon i \in [l]\} \cup \{yz_1, yz_2\}$ (see Figure 2).

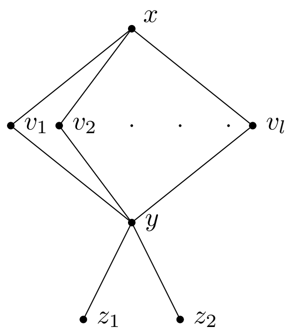

Figure 2. The graph G.

Lemma 1. *Let $l \in \mathbb{N}$, $i, j \in \mathbb{Z}$, $0 \leq i < j \leq l+1$. Let G be the graph presented in Figure 2. If γ is a precolouring of G such that $\gamma(xv_k) = k$ for $k \in [l]$, $\gamma(yz_1) = i$ and $\gamma(yz_2) = j$, then there exists a consecutive colouring β of G that is an extension of γ if and only if $j - i$ is an odd integer from the set $[l+1]$.*

Proof. Since $0 \leq i < j \leq l+1$, we have that $j - i \in [l+1]$. First we shall show that such an extension β does not exist for $j - i = 2s$, where $s \in \mathbb{N}$. Assume, to the contrary, that β can be constructed. We know that for each $k \in [l]$ the value $\beta(v_k y)$ differs by one from $\beta(xv_k)$. Moreover, $\beta(xv_k) = \gamma(xv_k)$ and $\beta(v_k y) \neq i$. Thus step by step the following sequence of equalities is forced:

$$\begin{aligned} &\beta(v_{i+1}y) = i+2, \text{ and if } s \geq 2, \text{ then} \\ &\beta(v_{i+3}y) = i+4, \ldots, \beta(v_{i+2s-1}y) = i+2s = j. \end{aligned}$$

Since $i+1 \geq 1$ and $j-1 \leq l$, all the edges $v_{i+1}y, \ldots, v_{i+2s-1}y = v_{j-1}y$ exist in G and hence all the above equalities are well described. Hence, in particular, $\beta(v_{j-1}y) = j$ and $\beta(yz_2) = j$, giving a contradiction to the assumption that β is consecutive.

Now we consider the case $j - i = 2s - 1$ for some $s \in \mathbb{N}$, in which the required consecutive extension β of γ is defined as follows:

$$\beta(v_k y) = \begin{cases} k-1, & \text{if } k \in [i] \text{ or } k \in \{i+2, i+4, \ldots, j-1\}, \\ k+1, & \text{if } k \in [j, l] \text{ or } k \in \{i+1, i+3, \ldots, j-2\}. \end{cases}$$

To observe that β is consecutive it is enough to analyse this property in the vertex y. Because $j - i$ is an odd number we see that $[i+1, j-1]$ is an empty set (for $s = 1$) or of the even cardinality (for $s \geq 2$). Thus, if $s \geq 2$, then the edges from the set $\{v_k y\colon k \in [i+1, j-1]\}$ have in the colouring β different colours from the set $[i+1, j-1]$. If $s = 1$, then $j = i+1$. Hence y is incident in the colouring β with edges coloured with all different elements from $[0, l+1]$. ■

Lemma 2. *Let $l \in \mathbb{N}$ and let $i, j \in \mathbb{Z}$ be such that either $i = -1$ and $0 \leq j \leq l$ or $j = l+2$ and $1 \leq i \leq l+1$. Let G be the graph presented in Figure 2. If γ is a precolouring of G such that $\gamma(xv_k) = k$ for $k \in [l]$, $\gamma(yz_1) = i$ and $\gamma(yz_2) = j$, then there exists a consecutive colouring β of G that is an extension of γ if and only if $j - i$ and l have different parity.*

Proof. First we prove the assertion in the case $i = -1$. Now the condition that $j+1$ and l have different parity is equivalent to the condition that $l - j$ is even. Keeping in mind that $l - j$ is even, we define β in the following way:

$$\beta(v_k y) = \begin{cases} k-1, & \text{if } k \in [j] \text{ or } k \in \{j+2, j+4, \ldots, l\}, \\ k+1, & \text{if } k \in \{j+1, j+3, \ldots, l-1\}. \end{cases}$$

Note that y is incident in the colouring β with edges coloured with all different elements from $[-1, l]$.

Next, suppose that $l - j$ is odd and the required extension β exists. As in the proof of the previous lemma, step by step, we obtain a sequence of forced equalities:

$$\beta(v_{j+1}y) = j + 2, \text{ and if } j \leq l - 3, \text{ then } \beta(v_{j+3}y) = j + 4, \ldots, \beta(v_l y) = l + 1.$$

Recall that β is consecutive, $\beta(yz_1) = -1$ and $\beta(v_l y) = l + 1$. Thus y should be incident in the colouring β with edges coloured with all different elements from $[-1, l+1]$. As a consequence we obtain $deg_G(y) \geq l+1-(-1)+1 = l+3$, which contradicts the construction of G.

In the case $j = l+2$ we apply to the first case the symmetry of both the colouring γ and the graph G. ■

Lemma 3. *Let $l \in \mathbb{N}$ and let $i, j \in \mathbb{Z}$ be such that either $i = -2$ and $-1 \leq j \leq l-1$ or $j = l + 3$ and $2 \leq i \leq l + 2$. Let G be the graph presented in Figure 2. If γ is a precolouring of G such that $\gamma(xv_k) = k$ for $k \in [l]$, $\gamma(yz_1) = i$ and $\gamma(yz_2) = j$, then there exists a consecutive colouring β of G that is an extension of γ if and only if $j - i = 1$.*

Proof. First we prove the assertion in the case $i = -2$. Assuming that $j - i = 1$, we have $j = -1$ and define β in the following way:

$$\beta(v_k y) = k - 1, \text{ for } k \in [l].$$

Note that y is incident in the colouring β with edges coloured with all different elements from $[-2, l - 1]$.

Next, suppose that $j \neq -1$ and the required extension β exists. Since β is consecutive and $j \neq -1$, there exists k such that $\beta(v_k y) = -1$. But $\beta(xv_k) \in [l]$ implies that $\beta(v_k y) \geq 0$ for every $k \in [l]$, and we obtain a contradiction.

In the case $j = l+3$ we apply to the first case the symmetry of both the colouring γ and the graph G. ■

All the previous lemmas are used in the proof of the next theorem which could be a very useful tool in a search for consecutive colourings of some graphs.

Theorem 1. *Let $l \in \mathbb{N}$, $i, j, a \in \mathbb{Z}$, $i < j$. Let G be the graph presented in Figure 2. If γ is a precolouring of G such that $\gamma(xv_k) = a + k$ for $k \in [l]$, $\gamma(yz_1) = i$ and $\gamma(yz_2) = j$, then there exists a consecutive colouring β of G that is an extension of γ if and only if one of the following conditions is satisfied.*

(a) *$i, j \in [a, a + l + 1]$ and $j - i$ is an odd integer;*

(b) *$i = a - 1$ and $j \in [a, a + l]$, and $j - i$, l have different parity;*

(c) *$i \in [a + 1, a + l + 1]$ and $j = a + l + 2$, and $j - i$, l have different parity;*

(d) $i = a - 2$ *and* $j = a - 1$;

(e) $i = a + l + 2$ *and* $j = a + l + 3$.

Proof. To obtain a consecutive colouring of the graph G it is enough to show the conclusion only for $a = 0$ with a consecutive colouring β', and next define $\beta(e) = \beta'(e) + a$ for every $e \in E(G)$.

Of course all the conditions (a) $\div$ (e) imply the existence of the extension β by Lemmas 1, 2 and 3.

Now, assume that the required extension β exists. First suppose that $i \leq -3$. Since β is consecutive, we obtain that there exists k such that $\beta(v_k y) = -2$ or $\beta(v_k y) = -1$. But $\beta(xv_k) \in [l]$ implies that $\beta(v_k y) \geq 0$ for every $k \in [l]$, a contradiction. If $j \geq l + 4$, then applying the symmetry of the colouring γ and of the graph G we also obtain the contradiction. Hence $-2 \leq i < j \leq l + 3$.

Next, facts that $deg_G(y) = l + 2$, $i < j$ and β is consecutive imply that $j - i \leq l + 1$. Hence, if $i = -2$ we have $-1 \leq j \leq l - 1$, and if $j = l + 3$ we have $2 \leq i \leq l+2$. In both cases by Lemma 3 we obtain $j = i+1$. Thus, either $i = -2$ and $j = -1$ or $i = l + 2$ and $j = l + 3$, and we obtain conditions (d) and (e).

Similarly, if $i = -1$ we have $0 \leq j \leq l$, and if $j = l + 2$ we have $1 \leq i \leq l + 1$. In both cases by Lemma 2 we obtain that $j - i$ and l have different parity. Thus, we obtain conditions (b) and (c).

It is enough to consider the case $0 \leq i < j \leq l + 1$. Then directly from Lemma 1 we have that $j - i$ is an odd integer. ■

Lemma 4. *Let $s \in \mathbb{N}, s \geq 2$ and $p \in \mathbb{N}_0$. Let G be an arbitrary graph and let $x \in V(G)$ be a vertex of degree two having neighbours w, z. Next, let G^* be a graph resulting from G by the substitution of the path w, x, z with the path $w, x_1, \ldots, x_{s-1}, z$. If there exists a proper edge colouring β of a graph G such that $def(G, \beta, x) = p$, then there exists a proper edge colouring α of G^* such that*

(a) $def(G^*, \alpha, x_{s-1})$
$$= \begin{cases} 0, & \textit{if } p, s \textit{ have the same parity and } s \geq p + 2, \\ 1, & \textit{if } p, s \textit{ have different parity and } s \geq p + 2, \ \textit{ and} \\ p - s + 2, & \textit{if } s < p + 2, \end{cases}$$

(b) $def(G^*, \alpha, x_i) = 0$ *for each* $i \in [s - 2]$, *and*

(c) $def(G^*, \alpha, v) = def(G, \beta, v)$ *for each* $v \in V(G) \setminus \{x\}$.

Proof. Assume, without loss of generality, that $\beta(wx) < \beta(xz)$. We define an edge colouring α of the graph G^* in the following way:

$\alpha(e) = \beta(e)$ for every $e \in E(G) \setminus \{wx, xz\}$,

$\alpha(wx_1) = \beta(wx)$,

$\alpha(x_i x_{i+1}) = \alpha(wx_1) + i$, for $i \in [\min\{s - 2, p\}]$,

$\alpha(x_{s-1} z) = \beta(xz)$.

Moreover, if $s \geq p+2$, then the edges $x_i x_{i+1}$ for $i \in [p+1, s-2]$ obtain alternately colours of the edges $x_{p-2}x_{p-1}$ and $x_{p-1}x_p$, i.e.,

$$\alpha(x_i x_{i+1}) = \begin{cases} \alpha(x_p x_{p+1}) - 1, & \text{if } i \in [p+1, s-2], \text{ where } i, p \text{ have different parity,} \\ \alpha(x_p x_{p+1}), & \text{if } i \in [p+2, s-2], \text{ where } i, p \text{ have the same parity.} \end{cases}$$ ■

Lemma 5. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. If $\min\{l_i : i \in [3]\} \leq 6$, then $S_{\mathbf{l},\mathbf{s}} \in \mathfrak{N}$.*

Proof. Let $H = S_{\mathbf{l},(2,2,2)}$, $G = S_{\mathbf{l},\mathbf{s}}$. Without loss of generality, we can assume that $l_2 \leq 6$. First we show that $H \in \mathfrak{N}$ for every $l_2 \in [6]$.

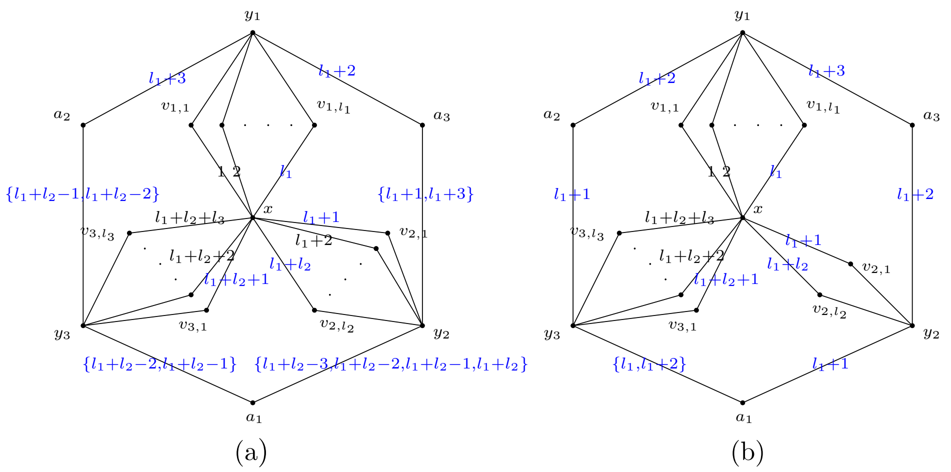

Figure 3. (a) The consecutive colouring γ of $H = S_{\mathbf{l},(2,2,2)}$ for $l_2 \in [3,6]$, (b) the consecutive colouring γ of $H = S_{\mathbf{l},(2,2,2)}$ for $l_2 \in [2]$.

For $l_2 \in [3,6]$ we define a precolouring γ of the graph H such that $\gamma(xv_{1,r}) = r$ for $r \in [l_1]$, $\gamma(xv_{2,r}) = l_1 + r$ for $r \in [l_2]$, $\gamma(xv_{3,r}) = l_1 + l_2 + r$ for $r \in [l_3]$. Moreover $\gamma(a_2y_1) = l_1 + 3$, $\gamma(y_1a_3) = l_1 + 2$ (see Figure 3(a)). Thus, Theorem 1(e) implies that there exists a consecutive colouring β_1 of an induced subgraph H_1 of the graph H that is an extension of $\gamma_{|H_1}$. Let $i = \gamma(xv_{3,1}) - 3 = l_1 + l_2 - 2$, $j = \gamma(xv_{3,1}) - 2 = l_1 + l_2 - 1$. Then $|i - \gamma(a_2y_1)| = |l_1 + l_2 - 2 - (l_1 + 3)| = |l_2 - 5|$, $|j - \gamma(a_2y_1)| = |l_2 - 4|$. Notice that for every $l_2 \in [3,6]$, either $|l_2 - 5| = 1$ or $|l_2 - 4| = 1$. Hence, let us define

$$\gamma(y_3a_2) = \begin{cases} l_1 + l_2 - 2, & \text{if } |l_2 - 5| = 1, \\ l_1 + l_2 - 1, & \text{if } |l_2 - 4| = 1, \end{cases}$$

$$\gamma(a_1y_3) = \begin{cases} l_1 + l_2 - 1, & \text{if } |l_2 - 5| = 1, \\ l_1 + l_2 - 2, & \text{if } |l_2 - 4| = 1. \end{cases}$$

Thus, Theorem 1(d) implies that there exists a consecutive colouring β_3 of an induced subgraph H_3 that is an extension of $\gamma_{|H_3}$. From the construction it follows that an edge colouring obtained by taking together colourings β_1 and β_3 is a consecutive colouring of $H_1 \cup H_3$. Since $\gamma(xv_{2,r}) \in [l_1+1, l_1+l_2]$ for $r \in [l_2]$, $\gamma(y_1a_3) = l_1+2$, $\gamma(a_1y_3) \in \{l_1+l_2-2, l_1+l_2-1\}$, we have $l_1 \le \gamma(a_1y_3) \pm 1 \le l_1+l_2$ for every $l_2 \in [3,6]$. Hence, it is enough to define $\gamma(a_3y_2) = i$ and $\gamma(y_2a_1) = j$ for any i, j such that

$$i \in \{l_1+1, l_1+3\},\ j \in \begin{cases} \{l_1+l_2-2, l_1+l_2\}, & \text{if } |l_2-5| = 1, \\ \{l_1+l_2-3, l_1+l_2-1\}, & \text{if } |l_2-4| = 1. \end{cases}$$

Thus, Theorem 1(a) implies that there exists a consecutive colouring β_2 of an induced subgraph H_2 that is an extension of $\gamma_{|H_2}$. Hence, an edge colouring β of the graph H being an extension of β_1, β_2 and β_3 such that $\beta_{|H_i} = \beta_i$ for $i \in [3]$ is also consecutive in the case $l_2 \in [3,6]$.

For $l_2 \in [2]$ we define a precolouring γ of the graph H such that $\gamma(xv_{1,r}) = r$ for $r \in [l_1]$, $\gamma(xv_{2,r}) = l_1 + r$ for $r \in [l_2]$, $\gamma(xv_{3,r}) = l_1 + l_2 + r$ for $r \in [l_3]$. Moreover $\gamma(y_3a_2) = l_1+1$, $\gamma(a_2y_1) = l_1+2$, $\gamma(y_1a_3) = l_1+3$, $\gamma(a_3y_2) = l_1+2$, $\gamma(y_2a_1) = l_1+1$, and

$$\gamma(a_1y_3) = \begin{cases} l_1, & \text{if } l_2 = 2, \\ l_1+2, & \text{if } l_2 = 1, \end{cases}$$

(see Figure 3(b), we assume that $v_{2,1} = v_{2,l_2}$ if $l_2 = 1$). Thus, Theorem 1(e) implies that there exists a consecutive colouring β_1 of an induced subgraph H_1 that is an extension of $\gamma_{|H_1}$, and Theorem 1(a) implies that there exists a consecutive colouring β_2 of an induced subgraph H_2 that is an extension of $\gamma_{|H_2}$. Next, we construct a consecutive colouring β_3 of an induced subgraph H_3 that is an extension of $\gamma_{|H_3}$. To do it for $l_2 = 2$ we use Theorem 1(d) because $\gamma(y_3a_2) - \gamma(a_1y_3) = 1$ and $\gamma(xv_{3,1}) - \gamma(a_1y_3) = 3$; for $l_2 = 1$ we use Theorem 1(a) because $\gamma(a_1y_3) - \gamma(y_3a_2) = 1$ is an odd number and $\gamma(xv_{3,1}) - \gamma(y_3a_2) = 1$. Hence, an edge colouring β of the graph H being an extension of β_1, β_2 and β_3 such that $\beta_{|H_i} = \beta_i$ for $i \in [3]$ is also consecutive in the case $l_2 \in [2]$.

Thus, $H \in \mathfrak{N}$ for every $l_2 \in [6]$. Now, we use three times in turn Lemma 4 for the graph H with $p = 0$ and the substitution of paths y_1, a_3, y_2; y_2, a_1, y_3; y_3, a_2, y_1 with paths $y_1, a_{3,1}, \ldots, a_{3,s_3-1}, y_2$; $y_2, a_{1,1}, \ldots, a_{1,s_1-1}, y_3$; $y_3, a_{2,1}, \ldots, a_{2,s_2-1}, y_1$, respectively. We obtain an edge colouring α of G such that $def(G, \alpha) = def(H, \beta)$ $= 0$. Hence, $G \in \mathfrak{N}$ for every $l_2 \in [6]$. ■

Remark 4. If $l_i, s_i \in \mathbb{N}$ for $i \in [2]$, s_1, s_2 are even, $\mathbf{l} = (l_1, l_2)$ and $\mathbf{s} = (s_1, s_2)$, then $S_{\mathbf{l},\mathbf{s}} \in \mathfrak{N}$.

Proof. Let $H^* = S_{(l_1,l_2),(2,2)} = S_{\mathbf{l},\mathbf{s}'}$, $H = S_{(l_1,l_2,l_3),(2,2,2)}$, where l_3 is an arbitrary positive integer, and $G = S_{\mathbf{l},\mathbf{s}}$. Since $\mathbf{s}' = (2,2)$, from the definition of H^* it follows that it is obtained from the graph $\widetilde{H} = H[V(H_1) \cup V(H_2)]$ by the identification of the vertex $a_1 \in V(\widetilde{H})$ with the vertex $a_2 \in V(\widetilde{H})$. Hence, it is enough to use a consecutive colouring $\beta^* = \beta_{|H_1 \cup H_2}$ of the graph H obtained from a precolouring γ, which is presented in Figure 3(b). Notice that β^* is also a consecutive colouring of the graph H^* obtained from $\widetilde{H}$ by joining the vertices a_1 and a_2. Hence $H^* \in \mathfrak{N}$.

Now, we use two times in turn Lemma 4 for the graph H^* with $p = 0$ and the substitution of paths y_1, a_3, y_2 and y_2, a_1, y_3 with paths $y_1, a_{1,1}, \ldots, a_{1,s_1-1}, y_2$ and $y_2, a_{2,1}, \ldots, a_{2,s_2-1}, y_3$, respectively. We obtain an edge colouring α of G such that $def(G, \alpha) = def(H^*, \beta^*) = 0$. Hence, $G \in \mathfrak{N}$. ■

The next theorem of Giaro *et al.* [11] holds for general graphs. We use it to obtain a lower bound on the deficiency of certain bipartite graphs.

Theorem 2 [11]. *If there are sequences of integers $p_1, p_2, \ldots, p_l$ and $q_1, q_2, \ldots, q_l$, and there is a vertex $v \in V(G)$ with $deg_G(v) > 1$ such that for any two different vertices $u, w \in V(G)$ adjacent to v there is a path $u = v_1, v_2, \ldots, v_{p_i} = w$, where p_i is one of the numbers of the first sequence, such that*

$$deg_G(v) + p_i - 1 - \sum_{r=1}^{p_i} deg_G(v_r) \geq q_i,$$

then $def(G) \geq \min\{q_i : i \in [l]\}$.

Lemma 6. *If $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and all s_i are even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$, then*

$$def(S_{\mathbf{l},\mathbf{s}}) \geq \min\{l_i - s_i - 4 : i \in [3]\}.$$

Proof. We apply Theorem 2 to $G = S_{\mathbf{l},\mathbf{s}}$ with $v = x$. Notice that $deg_G(x) = l_1 + l_2 + l_3 > 1$ and any two different vertices adjacent to x can be connected by a path in which the degrees of succeeding vertices are either $2, l_i + 2, 2$ for some $i \in [3]$ or $2, l_j + 2, \underbrace{2, \ldots, 2}_{s_i - 1}, l_k + 2, 2$, where $i, j, k \in [3]$ are different indices. Hence, Theorem 2 holds for 6-element sequences ($l = 6$) with $(p_i)_{i=1}^6 = (3, 3, 3, s_1 + 3, s_2 + 3, s_3 + 3)$ and $(q_i)_{i=1}^6 = (l_2 + l_3 - 4, l_1 + l_3 - 4, l_1 + l_2 - 4, l_1 - s_1 - 4, l_2 - s_2 - 4, l_3 - s_3 - 4)$, since for every $i \in [3]$ and different indices i and $j, k \in [3]$ we have

$$\begin{aligned} deg_G(x) + p_i - 1 - \sum_{r=1}^{p_i} deg_G(v_r) &= l_1 + l_2 + l_3 + 3 - 1 - (l_i + 6) \\ &= l_1 + l_2 + l_3 - l_i - 4 = l_j + l_k - 4, \end{aligned}$$

and

$$
\begin{aligned}
deg_G(x) + p_{i+3} - 1 \; - & \sum_{r=1}^{p_{i+3}} deg_G(v_r) \\
&= l_1 + l_2 + l_3 + s_i + 3 - 1 - [2\,(s_i+1) + l_j + l_k + 4] \\
&= l_1 + l_2 + l_3 - l_j - l_k - s_i - 4 = l_i - s_i - 4.
\end{aligned}
$$

Since $l_i - s_i - 4 < l_i + l_j - 4$ for every different $i, j \in [3]$, we conclude that $def(G) \geq \min\{q_i : i \in [6]\} = \min\{l_i - s_i - 4 : i \in [3]\}$. ■

Lemma 7. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. Let $m = \min\{l_i - s_i - 4 : i \in [3]\}$, and let i_0 be such that $m = l_{i_0} - s_{i_0} - 4$. If $l_{i_0} \geq 7$, then there exists a proper edge colouring α of the graph $S_{\mathbf{l},\mathbf{s}}$ such that*

$$def(S_{\mathbf{l},\mathbf{s}}, \alpha) = \max\{0, l_{i_0} - s_{i_0} - 4\}.$$

Proof. Let $H = S_{\mathbf{l},(2,2,2)}$, $G = S_{\mathbf{l},\mathbf{s}}$. Without loss of generality, we can assume that $i_0 = 2$.

First suppose that l_2 is even. In this case we define a precolouring γ of the graph H such that $\gamma(xv_{1,r}) = r$ for $r \in [l_1]$, $\gamma(xv_{2,r}) = l_1 + r$ for $r \in [l_2]$, $\gamma(xv_{3,r}) = l_1 + l_2 + r$ for $r \in [l_3]$. Moreover $\gamma(a_2y_1) = l_1 + 3$, $\gamma(y_1a_3) = l_1 + 2$, $\gamma(a_3y_2) = l_1 + 3$, $\gamma(y_2a_1) = l_1 + l_2 - 2$, $\gamma(a_1y_3) = l_1 + l_2 - 1$, $\gamma(y_3a_2) = l_1 + l_2 - 2$ (see Figure 4(a)). Thus, Theorem 1(e) implies that there exists a consecutive colouring β_1 of an induced subgraph H_1 of the graph H that is an extension of $\gamma_{|H_1}$, and Theorem 1(d) implies that there exists a consecutive colouring β_3 of H_3 that is an extension of $\gamma_{|H_3}$. Since $\gamma(y_2a_1) - \gamma(a_3y_2) = l_1 + l_2 - 2 - (l_1 + 3) = l_2 - 5$ is odd, Theorem 1(a) implies that there exists a consecutive colouring β_2 of H_2 that is an extension of $\gamma_{|H_2}$. Let β be an edge colouring of the graph H being an extension of β_1, β_2 and β_3 such that $\beta_{|H_i} = \beta_i$ for $i \in [3]$. This implies that

$$
\begin{aligned}
&def(H, \beta, v) = 0 \text{ for every } v \neq a_2, \\
&def(H, \beta, a_2) = l_1 + l_2 - 2 - (l_1 + 3) - 1 = l_2 - 6 > 0.
\end{aligned}
$$

Now, we use twice in turn Lemma 4 for the graph H with $p = 0$ and the substitution of paths y_1, a_3, y_2 and y_2, a_1, y_3 with paths $y_1, a_{3,1}, \ldots, a_{3,s_3-1}, y_2$ and $y_2, a_{1,1}, \ldots, a_{1,s_1-1}, y_3$, respectively. We obtain an edge colouring α^* of the graph $H^* = S_{\mathbf{l},(s_1,2,s_3)}$ such that $def(H^*, \alpha^*) = def(H, \beta)$. Using Lemma 4 once again for $p = l_2 - 6$ and the substitution of a path y_3, a_2, y_1 with a path $y_3, a_{2,1}, \ldots, a_{2,s_2-1}, y_1$ we obtain an edge colouring α of the graph G such that

$def(G, \alpha, v) = def(H^*, \alpha^*, v) = 0$ for every $v \in V(H^*) \setminus \{a_2\}$,

$def(G, \alpha, a_{2,i}) = 0$ for every $i \in [s_2 - 2]$,

$$\begin{aligned} def(G,\alpha,a_{2,s_2-1}) &= \begin{cases} 0, & \text{if } s_2 \geq l_2-6+2,\\ l_2-6-s_2+2, & \text{if } s_2 < l_2-6+2, \end{cases}\\ &= \begin{cases} 0, & \text{if } l_2-s_2-4 \leq 0,\\ l_2-s_2-4, & \text{if } l_2-s_2-4 > 0, \end{cases}\\ &= \max\{0, l_2-s_2-4\}. \end{aligned}$$

Hence, $def(G,\alpha) = \max\{0, l_2-s_2-4\}$.

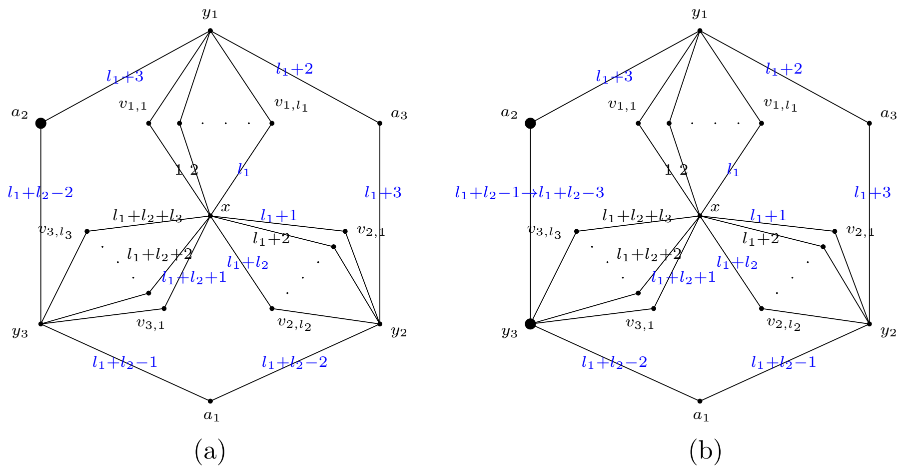

Figure 4. (a) The edge colouring γ of $H = S_{\mathbf{l},(2,2,2)}$ for even l_2, (b) the edge colouring γ of $H = S_{\mathbf{l},(2,2,2)}$ for odd l_2.

Now, suppose that l_2 is odd. Then $l_2 - s_2 - 4$ is also odd. Let $\gamma(y_2a_1) = l_1 + l_2 - 1$, $\gamma(a_1y_3) = l_1 + l_2 - 2$, $\gamma(y_3a_2) = l_1 + l_2 - 1$, and let the remaining edges of the graph H receive the same colours as in the previous case when l_2 is even (see Figure 4(b)). Similarly to the previous case, Theorems 1(e) and 1(d) imply that there exist consecutive colourings β_1 and β_3 of induced subgraphs H_1 and H_3 of the graph H that are extensions of $\gamma_{|H_1}$ and $\gamma_{|H_3}$, respectively. Since $\gamma(y_2a_1) - \gamma(a_3y_2) = l_1 + l_2 - 1 - (l_1 + 3) = l_2 - 4$ is odd, Theorem 1(a) implies that there exists a consecutive colouring β_2 of H_2 that is an extension of $\gamma_{|H_2}$. Let β' be an edge colouring of the graph H being an extension of β_1, β_2 and β_3 such that $\beta'_{|H_i} = \beta_i$ for $i \in [3]$, and let an edge colouring β be such that

$\beta(e) = \beta'(e)$ for every $e \in E(H) \setminus \{y_3a_2\}$,

$\beta(y_3a_2) = l_1 + l_2 - 3$.

It follows that

$def(H,\beta,v) = def(H,\beta',v) = 0$ for every $v \in V(H) \setminus \{y_3, a_2\}$,

$def(H,\beta',y_3) = 0$,

$def(H,\beta,y_3) = def(H,\beta',y_3) + 1 = 1$ (the colour $l_1 + l_2 - 1$ is omitted),

$def(H,\beta',a_2) = l_1 + l_2 - 1 - (l_1 + 3) - 1 = l_2 - 5 > 0$,

$def(H, \beta, a_2) = l_1 + l_2 - 3 - (l_1 + 3) - 1 = l_2 - 7 \geq 0.$
Now, similarly to the previous case, we use twice in turn Lemma 4 for the graph H with the same parameters and the edge colouring β if $l_2 - s_2 - 4 \geq 1$, and with the same parameters and the edge colouring β' if $l_2 - s_2 - 4 \leq -1$. We obtain an edge colouring α^* of the graph $H^* = S_{\mathbf{l},(s_1,2,s_3)}$ such that

$$def(H^*, \alpha^*) = \begin{cases} def(H, \beta), & \text{if } l_2 - s_2 - 4 \geq 1, \\ def(H, \beta'), & \text{if } l_2 - s_2 - 4 \leq -1. \end{cases}$$

Using Lemma 4 once again for $p = \begin{cases} l_2 - 7, & \text{if } l_2 - s_2 - 4 \geq 1, \\ l_2 - 5, & \text{if } l_2 - s_2 - 4 \leq -1, \end{cases}$ and substituting a path y_3, a_2, y_1 with a path $y_3, a_{2,1}, \ldots, a_{2,s_2-1}, y_1$ we obtain an edge colouring α of the graph G such that

$def(G, \alpha, v) = def(H^*, \alpha^*, v) = 0$ for every $v \in V(H^*) \setminus \{y_3, a_2\}$,

$def(G, \alpha, a_{2,i}) = 0$ for every $i \in [s_2 - 2]$,

$def(G, \alpha, y_3) = def(H^*, \alpha^*, y_3) = \begin{cases} def(H, \beta, y_3) = 1, & \text{if } l_2 - s_2 - 4 \geq 1, \\ def(H, \beta', y_3) = 0, & \text{if } l_2 - s_2 - 4 \leq -1, \end{cases}$

- if $l_2 - s_2 - 4 \geq 1$, then

$$\begin{aligned} def(G, \alpha, a_{2,s_2-1}) &= \begin{cases} 0, & \text{if } s_2 \geq l_2 - 7 + 2, \\ l_2 - 7 - s_2 + 2, & \text{if } s_2 < l_2 - 7 + 2, \end{cases} \\ &= \begin{cases} 0, & \text{if } l_2 - s_2 - 4 \leq 1, \\ l_2 - s_2 - 5, & \text{if } l_2 - s_2 - 4 > 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } l_2 - s_2 - 4 = 1, \\ l_2 - s_2 - 5, & \text{if } l_2 - s_2 - 4 > 1, \end{cases} = l_2 - s_2 - 5, \end{aligned}$$

- if $l_2 - s_2 - 4 \leq -1$, then $s_2 \geq l_2 - 5 + 2$ and $def(G, \alpha, a_{2,s_2-1}) = 0$.

Hence, $def(G, \alpha, a_{2,s_2-1}) = \begin{cases} l_2 - s_2 - 5, & \text{if } l_2 - s_2 - 4 \geq 1, \\ 0, & \text{if } l_2 - s_2 - 4 \leq -1, \end{cases}$ which implies that

$$def(G, \alpha) = \begin{cases} l_2 - s_2 - 4, & \text{if } l_2 - s_2 - 4 \geq 1 > 0, \\ 0, & \text{if } l_2 - s_2 - 4 \leq -1 \leq 0, \end{cases} = \max\{0, l_2 - s_2 - 4\}. \qquad \blacksquare$$

Let $S_{\mathbf{l},\mathbf{s}} = S_{(l_1,l_2,l_3),(s_1,s_2,s_3)}$, where $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and all s_i are even. Let $i, j, k \in [3]$ be different indices. By $P_{i,j,k}$ we denote a path $y_i, a_{j,1}, a_{j,2}, \ldots, a_{j,s_j-1}$, y_k in the graph $S_{\mathbf{l},\mathbf{s}}$. Next, let C^* be a cycle $C_{s_1+s_2+s_3}$ included in the graph $S_{\mathbf{l},\mathbf{s}}$, i.e., $C^* = P_{1,3,2} \cup P_{2,1,3} \cup P_{3,2,1}$.

Lemma 8. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. Let $A \subseteq V(C^*)$. If F is any proper induced subgraph of the graph $S_{\mathbf{l},\mathbf{s}}$ such that $F = S_{\mathbf{l},\mathbf{s}}[V(S_{\mathbf{l},\mathbf{s}}) - A]$, then*

$$F \in \mathfrak{N}.$$

Proof. Let $H = S_{\mathbf{l},(2,2,2)}$, $G = S_{\mathbf{l},\mathbf{s}}$. Without loss of generality, we can assume that F is connected. The assumption that F is a proper induced subgraph of the graph G implies $A \neq \emptyset$. Thus, without loss of generality, we can assume that there exists a vertex $v \in A \cap V(P_{3,2,1})$.

First we define a precolouring γ of the graph H such that $\gamma(xv_{1,r}) = r$ for $r \in [l_1]$, $\gamma(xv_{2,r}) = l_1 + r$ for $r \in [l_2]$, $\gamma(xv_{3,r}) = l_1 + l_2 + r$ for $r \in [l_3]$. Moreover $\gamma(a_2y_1) = l_1 + 1$, $\gamma(y_1a_3) = l_1$, $\gamma(a_3y_2) = l_1 - 1$, $\gamma(y_2a_1) = l_1 + l_2$, $\gamma(a_1y_3) = l_1 + l_2 - 1$, $\gamma(y_3a_2) = l_1 + l_2 - 2$ (see Figure 5). Thus, Theorems 1(a) and 1(d) imply that there exist consecutive colourings β_1 and β_3 of induced subgraphs H_1 and H_3 of the graph H that are extensions of $\gamma_{|H_1}$ and $\gamma_{|H_3}$, respectively. Since $\gamma(y_2a_1) - \gamma(a_3y_2) = l_1 + l_2 - (l_1 - 1) = l_2 + 1$, we have that $\gamma(y_2a_1) - \gamma(a_3y_2)$ and l_2 have different parity. Thus, Theorem 1(b) implies that there exists a consecutive colouring β_2 of H_2 that is an extension of $\gamma_{|H_2}$. Let β be an edge colouring of the graph H being an extension of β_1, β_2 and β_3 such that $\beta_{|H_i} = \beta_i$ for $i \in [3]$. This implies that

$$\begin{aligned} &\mathit{def}(H, \beta, v) = 0 \text{ for every } v \neq a_2, \\ &\mathit{def}(H, \beta) = \mathit{def}(H, \beta, a_2) = l_1 + l_2 - 2 - (l_1 + 1) - 1 = l_2 - 4. \end{aligned}$$

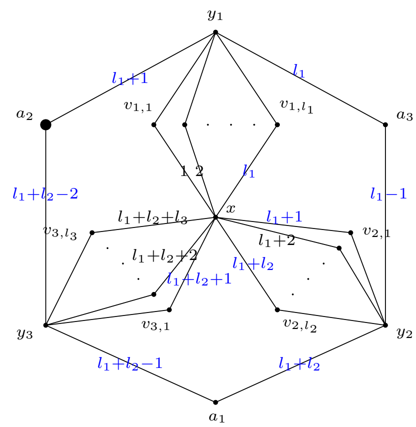

Figure 5. The edge colourings γ of $H = S_{\mathbf{l},(2,2,2)}$.

Now, similarly to the proof of Lemma 7, we use twice in turn Lemma 4 for the graph H with $p = 0$ and the substitution of paths y_1, a_3, y_2 and y_2, a_1, y_3 with paths $y_1, a_{3,1}, \ldots, a_{3,s_3-1}, y_2$ and $y_2, a_{1,1}, \ldots, a_{1,s_1-1}, y_3$, respectively. We obtain an edge colouring α^* of the graph $H^* = S_{\mathbf{l},(s_1,2,s_3)}$ such that $\mathit{def}(H^*, \alpha^*) = \mathit{def}(H, \beta)$. Moreover, in this colouring α^* edges of new paths receive alternately colours of the edges y_1a_3 and a_3y_2, y_2a_1 and a_1y_3, respectively. Since $\alpha^*(y_3a_2) = \min\{c: c \in S(y_3, \alpha^*)\}$ and $\alpha^*(a_2y_1) = \max\{c: c \in S(y_1, \alpha^*)\}$, we moreover obtain that induced subgraph $G^* = H^*[V(H^*) - a_2]$ of the graph G is consecutively colourable.

Notice that $\alpha^*(a_{1,s_1-1}y_3) = \min\{c: c \in S(y_3, {\alpha^*}_{|G^*})\}$, $\alpha^*(y_1a_{3,1}) = \max\{c: c \in S(y_1, {\alpha^*}_{|G^*})\}$, and $\alpha^*(a_{3,s_3-1}y_2) = \min\{c: c \in S(y_2, {\alpha^*}_{|G^*})\}$, $\alpha^*(y_2a_{1,1}) = \max\{c: c \in S(y_2, {\alpha^*}_{|G^*})\}$. Thus $F^* = G^*[V(G^*) - B] \in \mathfrak{N}$, where $B = A \cap V(G^*)$. Since the graph F is isomorphic to a graph obtained by adjoining some paths to the graph $F^* \in \mathfrak{N}$ in the vertex $y_1 \in V(G)$ or $y_3 \in V(G)$, Remark 1(a) implies $F \in \mathfrak{N}$. ■

Let $S_{\mathbf{l},\mathbf{s}} = S_{(l_1,l_2),(s_1,s_2)}$, where $l_i, s_i \in \mathbb{N}$ for $i \in [2]$, and all s_i are even. Let $P_{1,2}$ and $P_{2,1}$ be paths $y_1, a_{1,1}, a_{1,2}, \ldots, a_{1,s_1-1}, y_2$ and $y_2, a_{2,1}, a_{2,2}, \ldots, a_{2,s_2-1}, y_1$ in the graph $S_{\mathbf{l},\mathbf{s}}$, respectively. Of course, $C^* = P_{1,2} \cup P_{2,1}$.

Remark 5. Let $l_i, s_i \in \mathbb{N}$ for $i \in [2]$, s_1, s_2 are even, $\mathbf{l} = (l_1, l_2)$ and $\mathbf{s} = (s_1, s_2)$. Let $A \subseteq V(C^*)$. If F is any proper induced subgraph of the graph $S_{\mathbf{l},\mathbf{s}}$ such that $F = S_{\mathbf{l},\mathbf{s}}[V(S_{\mathbf{l},\mathbf{s}}) - A]$, then

$$F \in \mathfrak{N}.$$

Proof. Similarly to the proof of Lemma 8, without loss of generality, we can assume that F is connected, and there exists a vertex $v \in A \cap V(P_{2,1})$.

Let $H^* = S_{(l_1,l_2),(2,2)} = S_{\mathbf{l},\mathbf{s}'}$, $H = S_{(l_1,l_2,l_3),(2,2,2)}$, where l_3 is an arbitrary positive integer, and $G = S_{\mathbf{l},\mathbf{s}}$. From the definition of H^* and the same arguments as in the proof of Remark 4 we obtain an edge colouring $\beta^* = \beta_{|H_1 \cup H_2}$ of the graph H being the extention of the precolouring γ, which is presented in Figure 5. Notice that β^* is an edge colouring of the graph H^* obtained from $\widetilde{H} = H[V(H_1) \cup V(H_2)]$ by joining the vertices $a_1 \in V(\widetilde{H})$ and $a_2 \in V(\widetilde{H})$, and such that $def(H^*, \beta^*) = def(H^*, \beta^*, a_2) = l_1 + l_2 - (l_1 + 1) - 1 = l_2 - 2$.

Now, we use Lemma 4 for the graph H^* with $p = 0$ and the substitution of a path y_1, a_3, y_2 with a path $y_1, a_{1,1}, \ldots, a_{1,s_1-1}, y_2$. We obtain an edge colouring α^* of the graph $H^{**} = S_{\mathbf{l},(s_1,2)}$ such that $def(H^{**}, \alpha^*) = def(H^*, \beta^*)$. Moreover, in the colouring α^* edges of a new path receive alternately colours of the edges y_1a_3 and a_3y_2. Since $a_1 = a_2$ in the graph H^{**}, $\alpha^*(y_2a_1) = \max\{c: c \in S(y_2, \alpha^*)\}$ and $\alpha^*(a_2y_1) = \max\{c: c \in S(y_1, \alpha^*)\}$, we obtain that induced subgraph $G^* = H^{**}[V(H^{**}) - a_2]$ of the graph G is consecutively colourable.

Notice that $\alpha^*(y_1a_{1,1}) = \max\{c: c \in S(y_1, {\alpha^*}_{|G^*})\}$ and $\alpha^*(a_{1,s_1-1}y_2) = \min\{c: c \in S(y_2, {\alpha^*}_{|G^*})\}$. Thus $F^* = G^*[V(G^*) - B] \in \mathfrak{N}$, where $B = A \cap V(G^*)$. Since the graph F is isomorphic to a graph obtained by adjoining some paths to the graph $F^* \in \mathfrak{N}$ in the vertex $y_1 \in V(G)$ or $y_2 \in V(G)$, Remark 1(a) implies $F \in \mathfrak{N}$. ■

4. Main Results

Giaro *et al.* [11] estimated the deficiency of special generalized Sevastjanov rosettes. More precisely, they proved that $l - 6 \le def(S_l) \le l - 5$, where $S_l =$

$S_{(l,l,l),(2,2,2)}$ and $l \geq 5$. In this section, we determine the deficiency of all the graphs $S_{\mathbf{l},\mathbf{s}}$, where $\mathbf{l} = (l_1, l_2, l_3)$, $\mathbf{s} = (s_1, s_2, s_3)$, $l_i, s_i \in \mathbb{N}$, s_i are even, which, in particular, gives the exact value of $def(S_l)$ for $l \in \mathbb{N}$.

Theorem 3. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. If $m = \min\{l_i - s_i - 4 \colon i \in [3]\}$, then*

$$def(S_{\mathbf{l},\mathbf{s}}) = \max\{0, m\}.$$

Proof. Let $G = S_{\mathbf{l},\mathbf{s}}$ and let i_0 be such that $m = l_{i_0} - s_{i_0} - 4$. Lemma 6 implies that $def(G) \geq l_{i_0} - s_{i_0} - 4$.

If $l_{i_0} \leq 6$, then $\min\{l_i \colon i \in [3]\} \leq 6$, and by Lemma 5 we have that $G \in \mathfrak{N}$. Then $l_{i_0} - s_{i_0} - 4 \leq 0$ for every $s_{i_0} \geq 2$, and $\max\{0, l_{i_0} - s_{i_0} - 4\} = 0$. If $l_{i_0} \geq 7$, then Lemma 7 implies that there exists an edge colouring α of the graph G such that $def(G) \leq def(G, \alpha) = \max\{0, l_{i_0} - s_{i_0} - 4\}$. Hence, $def(G) \leq \max\{0, l_{i_0} - s_{i_0} - 4\}$ for every $l_{i_0} \in \mathbb{N}$, which finishes the proof. ■

Corollary 6. *For every $l \in \mathbb{N}$,*

$$def(S_l) = \max\{0, l - 6\}.$$

The next theorem gives necessary and sufficient conditions for a graph $S_{\mathbf{l},\mathbf{s}}$ that guarantee that such a graph is in the class $\mathcal{C}(\mathfrak{N}_{\leq})$.

Theorem 4. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even, $\mathbf{l} = (l_1, l_2, l_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. The graph*

$$S_{\mathbf{l},\mathbf{s}} \in \mathcal{C}(\mathfrak{N}_{\leq})$$

if and only if $l_i = s_i + 5$ for every $i \in [3]$.

Proof. ($\Rightarrow$) Let us assume, by a contradiction, that $G = S_{\mathbf{l},\mathbf{s}} \in \mathcal{C}(\mathfrak{N}_{\leq})$ and there exists $i_0 \in [3]$ such that $l_{i_0} \neq s_{i_0} + 5$. If $l_{i_0} \leq s_{i_0} + 4$, then $\min\{l_i - s_i - 4 \colon i \in [3]\} \leq 0$, and Theorem 3 implies that $def(G) = 0$. Hence, $G \in \mathfrak{N}$, which leads us to a contradiction with the definition of $\mathcal{C}(\mathfrak{N}_{\leq})$.

Thus, $l_i \geq s_i + 5$ for every $i \in [3]$ and $l_{i_0} \geq s_{i_0} + 6$. But then the graph $H = S_{(l'_1, l'_2, l'_3),(s_1, s_2, s_3)}$, where $l'_i \in \mathbb{N}$, $l'_i = s_i + 5$ for $i \in [3]$, is a proper induced subgraph of the graph G. Thus, Theorem 3 implies that $def(H) = 1$ and $H \notin \mathfrak{N}$, which also leads us to a contradiction with the definition of $\mathcal{C}(\mathfrak{N}_{\leq})$.

($\Leftarrow$) Now, let us assume that $G = S_{\mathbf{l},\mathbf{s}}$ and $l_i = s_i + 5$ for each $i \in [3]$. Theorem 3 implies that $def(G) = 1$ and $G \notin \mathfrak{N}$.

Let H be any proper induced subgraph of the graph G. Without loss of generality, we can assume that H is connected. We will show that $H \in \mathfrak{N}$.

(a) If H does not contain the vertex x (perhaps other vertices, too), then H is isomorphic to a graph obtained by the attachment of pendant edges either to the outer cycle C^* from $\mathfrak{N}$ or to a path also from $\mathfrak{N}$. Hence, Remark 1(a) implies that $H \in \mathfrak{N}$.

(b) If H does not contain only some vertices $v_{i,j}$, $i \in [3]$, $j \in [l_i]$, then we can also prove that $H \in \mathfrak{N}$. Indeed, Theorem 3 implies that any subgraph $H = S_{(l'_1,l'_2,l'_3),(s_1,s_2,s_3)}$, where $l'_i \in \mathbb{N}$, $l'_i \le l_i$ for $i \in [3]$, and $l'_{i_0} < l_{i_0}$ for some $i_0 \in [3]$, is consecutively colourable. The fact that $H = S_{(l'_1,l'_2,0),(s_1,s_2,s_3)} = S_{(l'_1,l'_2),(s_3,s_1+s_2)} \in \mathfrak{N}$, where $l'_i \in \mathbb{N}$, $l'_i \le l_i$ for $i \in [2]$, follows from Remark 4. Notice also that $H = S_{(l'_1,0,0),(s_1,s_2,s_3)} \in \mathfrak{N}$, where $0 \le l'_1 \le l_1$, since H is isomorphic to either the cycle $C^* \in \mathfrak{N}$ or a graph obtained by adjoining the cycle $C^* \in \mathfrak{N}$ and a complete bipartite graph $K_{2,l'_1} \in \mathfrak{N}$ in the vertex y_i (see Remarks 1(d) and (a)).

(c) If H does not contain some vertices $a_{i,j}$ or y_i, $i \in [3]$, $j \in [l_i]$ (perhaps vertices $v_{i,j}$, $i \in [3]$, $j \in [l_i]$, too), then again we show $H \in \mathfrak{N}$. Indeed, let F be an induced subgraph of the graph G such that $F = G[V(G) - \{v_{i,j} \in V(G)\colon v_{i,j} \notin V(H), i \in [3], j \in [l_i]\}]$. Thus $H < F \le G$ and $H = F[V(F) - A]$, where $A \subseteq V(C^*)$, $A \ne \emptyset$. If $F = S_{(l'_1,l'_2,l'_3),(s_1,s_2,s_3)}$ or $F = S_{(l'_1,l'_2,0),(s_1,s_2,s_3)} = S_{(l'_1,l'_2),(s_3,s_1+s_2)}$, where $l'_i \in \mathbb{N}$, $l'_i \le l_i$ for all i, then directly from Lemma 8 or Remark 5 we have that $H \in \mathfrak{N}$. If $F = S_{(l'_1,0,0),(s_1,s_2,s_3)}$, where $0 \le l'_1 \le l_1$, then H is isomorphic to either a path from $\mathfrak{N}$ or a graph obtained by adjoining a path from $\mathfrak{N}$ and a complete bipartite graph $K_{2,l'_1} \in \mathfrak{N}$ in the vertex y_i. ■

Notice that directly by Theorem 3 and arguments in (b) of the second part of the proof of Theorem 4 we obtain the following.

Corollary 7. *Let $l_i, s_i \in \mathbb{N}$ for $i \in [3]$, and let all s_i be even. The graph*

$$S_{\mathbf{l},\mathbf{s}} \in \mathfrak{N}$$

if and only if one of the following conditions is satisfied:

(a) $\mathbf{l} = (l_1, l_2, l_3)$ *and* $\mathbf{s} = (s_1, s_2, s_3)$, *and there exists* $i \in [3]$ *such that* $l_i \le s_i + 4$,

(b) $\mathbf{l} = (l_1, l_2)$ *and* $\mathbf{s} = (s_1, s_2)$.

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Received 19 April 2016
Revised 1 October 2016
Accepted 1 October 2016