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# ON THE CROSSING NUMBERS OF CARTESIAN PRODUCTS OF WHEELS AND TREES

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**Abstract**

Bokal developed an innovative method for finding the crossing numbers of Cartesian product of two arbitrarily large graphs. In this article, the crossing number of the join product of stars and cycles are given. Afterwards, using Bokal's zip product operation, the crossing numbers of the Cartesian products of the wheel $W_n$ and all trees $T$ with maximum degree at most five are established.

**Keywords:** graph, drawing, crossing number, join product, Cartesian product.

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## 1. Introduction

Let $G$ be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of $G$ is a representation of $G$ in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding pair of points. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The *crossing number* $\mathrm{cr}(G)$ is the smallest number of edge crossings in any drawing of $G$. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

The repetitive patterns in Cartesian products of graphs reflects in their drawings and makes Cartesian products one of the first graph classes for which the crossing numbers were studied (for a definition of Cartesian product see [15]). Let $C_n$ be the cycle of length $n$, $P_n$ be the path on $n$ vertices, and $S_n$ be the star isomorphic to $K_{1,n}$. The crossing numbers of Cartesian products of graphs have been studied since 1973, when Harary *et al.* established the crossing number of $C_3 \square C_3$ and conjectured that $\mathrm{cr}(C_m \square C_n) = (m-2)n$ for $3 \leq m \leq n$, [11]. This hypothesis has been proved for $m \leq 7$ by other authors, but it has not yet been proved for $3 \leq m \leq n$ in general. The best result was given by Glebsky and Salazar in [10].

Several authors were researching the crossing numbers of the Cartesian products $G \square C_n$, $G \square P_n$, $G \square S_n$ for some specific graphs $G$. Beineke and Ringeisen in [3] as well as Jendroľ and Ščerbová in [14] determined the crossing numbers of the Cartesian products of four-vertex graphs with the cycle $C_n$. Some other results concerning the crossing numbers of Cartesian products of special small graphs with paths, cycles and stars one can find in [15, 16, 19, 20], and [25]. Moreover, Jendroľ and Ščerbová in [14] conjectured that the crossing number of $S_n \square P_m$ is $(m-2) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. This conjecture was proved by Bokal using zip product operation, see [4]. It was the first exact result which gives the crossing number of Cartesian product of two graphs, where both graphs are arbitrarily large. In [5], Bokal extended the properties of zip product operation and established the crossing numbers of Cartesian products of trees with stars as well as of paths with wheels. Another results concerning Cartesian products of two arbitrarily large graphs were obtained in [26] and [27]. In the paper, we extend these results by giving the crossing numbers of the Cartesian products of trees with wheels. Our methods are based on some properties of zip product operation and of known values of crossing numbers of join of stars and cycles.

The *join product* $G_i + G_j$ of graphs $G_i$ and $G_j$ is created from vertex-disjoint copies of $G_i$ and $G_j$ by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of the graph $G_i + G_j$ is the union of $E(G_i)$, $E(G_j)$ and $E(K_{m,n})$. The first results on crossing numbers of join of paths and cycles as well as of two cycles appeared in [17]. There are also some known results concerning crossing numbers of join products of discrete graphs, paths and cycles with special graphs, see for example [18, 21, 22] and [23].

We denote the number of crossings of a graph $G$ in a drawing $D$ by $\mathrm{cr}_D(G)$. Let $G_i$ and $G_j$ be edge-disjoint subgraphs of a graph $G$. Then we denote by

$\mathrm{cr}_D(G_i, G_j)$ the number of crossings between the edges of $G_i$ and the edges of $G_j$ in the drawing $D$. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map". In a good drawing $D$ of the graph $G$, we say that a cycle $C$ *separates* the vertices of a subgraph $G_i$ not containing vertices of $C$ if the vertices of $G_i$ are contained in different components of $\mathbb{R}^2 \setminus C$.

In the paper, some proofs are based on Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, in [24] he proved that

$$\mathrm{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor, \quad \text{if } \min\{p,q\} \leq 6.$$

In Section 2, we give some special properties concerning drawings of join products of graphs with cycles, especially of join products of discrete graphs with cycles and of their subgraphs. Using these results, in Section 3 we find the exact values of crossing numbers for join products of stars with cycles. Let $W_n$ be the wheel on $n+1$ vertices and let $D_m$ denote the discrete graph on $m$ vertices. In Section 4, the isomorphism between $S_m + C_n$ and $W_n + D_m$, together with the Bokal's results on zip product operation, allows us to establish the crossing number of Cartesian product of all trees with maximum degree at most five with all wheels.

## 2. Preliminary Results

Let us consider a graph $G$ with $V(G) = \{v_1, v_2, \dots, v_m\}$ and the cycle $C_n$ with the vertices $c_1, c_2, \dots, c_n$. The join product $G + C_n$ consists of one copy of the graph $G$, one copy of the cycle $C_n$, and of the edges joining every vertex of $G$ with every vertex of $C_n$. The edges joining the vertices of $G$ with the vertices of $C_n$ form the complete bipartite graph $K_{m,n}$. For the vertices $v_1, v_2, \dots, v_m$ of the graph $G$, let $T^i$ denote the subgraph induced by $n$ edges joining the vertex $v_i$ with the vertices $c_1, c_2, \dots, c_n$ of the cycle $C_n$. So,

$$G + C_n = G \cup K_{m,n} \cup C_n = G \cup \left( \bigcup_{i=1}^{m} T^i \right) \cup C_n.$$

**Lemma 1.** *Let $G$ be a graph of order $m$, $m \geq 1$. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of $C_n$ do not cross each other.*

***Proof.*** Assume an optimal drawing of the graph $G + C_n$ in which two edges of $C_n$ cross. Any optimal drawing is good and in such a drawing, the edges of a 3-cycle cannot cross. Thus, assume $n \geq 4$ in the rest of the proof. Let $x$ be the point of the plane in which two edges, say $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$, of $C_n$ cross. The plane is a normal space. Hence, in the plane there is an open set $A_x$ such that $A_x$ contains $x$ together with the corresponding segments of the crossed edges. All remaining edges of the drawing are disjoint with $A_x$, see Figure 1(a). Figure 1(b) shows that the edges $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$ can be redrawn into new edges $\{c_i, c_j\}$ and $\{c_{i+1}, c_{j+1}\}$ which do not cross. The vertices $c_1, \ldots, c_{i-1}, c_i, c_j, c_{j-1}, \ldots, c_{i+2}, c_{i+1}, c_{j+1}, c_{j+2}, \ldots, c_n, c_1$ form the $n$-cycle again. Since every vertex of the cycle $C_n$ is adjacent to every vertex of the graph $G$, the new drawing of the graph $G + C_n$ with less number of crossings is obtained. This contradiction completes the proof. ■

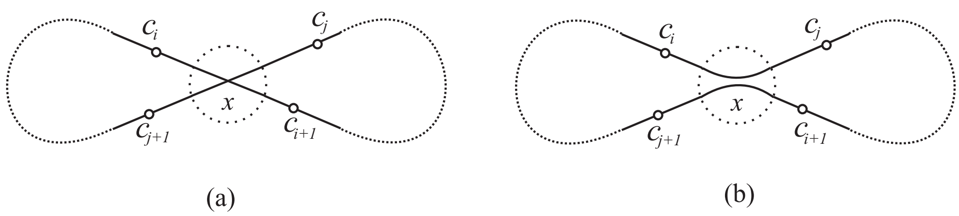


Figure 1. Elimination of a crossing in $C_n$.

We note that, in Section 3, only optimal drawings of stars with cycles are considered and therefore, in the whole section it will be assumed that the cycle $C_n$ does not cross itself.

Assume now the discrete graph $D_m$ on $m$ vertices. Clearly, $D_m + C_n$ is a subgraph of the graph $G + C_n$, where $|V(G)| = m$. In the proofs of the paper, several special properties of the graph $D_m + C_n$ are used. Let us present some of them. Several results of the paper are based on the next Lemma 2. Although Lemma 2 was proved in [17], we prove it again. This proof is simpler than the previous one.

**Lemma 2.** *Let $D$ be a good drawing of the join product $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which no edge of $C_n$ is crossed and $C_n$ does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \ldots, m$, two different subgraphs $T^i$ and $T^j$ cross each other at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times in $D$.*

***Proof.*** By hypothesis, the cycle $C_n$ divides the plane into two regions in such a way that all $m$ subgraphs $T^1, T^2, \ldots, T^m$ are placed in one of these regions, say in the interior region of $C_n$. Assume that two different subgraphs $T^i$ and $T^j$ cross each other less than $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times. As both are placed inside $C_n$, it is possible to add a new vertex $w$ into the unbounded region of $C_n$ together with the edges $\{w, c_1\}$, $\{w, c_2\}, \ldots, \{w, c_n\}$ such that none of them crosses an edge of $T^i \cup T^j$. Now the edges of the new graph incident with the vertices $v_i$, $v_j$, and $w$ form the subgraph isomorphic to $K_{3,n}$ and cross each other less than $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times. This contradiction with $\mathrm{cr}(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ completes the proof. ■

For the case when the cycle $C_n$ separates some of the other vertices of the graph, or when some subgraphs $T^i$, $i \in \{1, 2, \ldots, m\}$, cross $C_n$, we have the next corollary.

**Corollary 3.** *Let $D$ be a good drawing of the join product $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which the edges of $C_n$ do not cross each other and $C_n$ does not separate $r$ vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$, $2 \leq r \leq m$. If none of the subgraphs $T^{i_1}, T^{i_2}, \ldots, T^{i_r}$ induced on the edges incident with the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ crosses $C_n$, then two different subgraphs $T^{i_j}$ and $T^{i_k}$, $j, k = 1, 2, \ldots, r$, cross each other at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times in $D$.*

***Proof.*** By the assumptions, the drawing of $D_m + C_n$ contains the subdrawing of $D_r + C_n$ satisfying the conditions of Lemma 2. Thus, the result follows. ∎

**Corollary 4.** *Let $D$ be a good drawing of the join product $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which the edges of $C_n$ do not cross each other and $C_n$ does not separate $r$ vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$, $2 \leq r \leq m$. Let $T^{i_1}, T^{i_2}, \ldots, T^{i_s}$, $s < r$, be the subgraphs induced on the edges incident with the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ that do not cross $C_n$. If $k$ edges of some subgraph $T^{i_j}$ induced on the edges incident with the vertex $v_{i_j}$, $j \in \{s+1, s+2, \ldots, r\}$, cross the cycle $C_n$, then the subgraph $T^{i_j}$ crosses each of the subgraphs $T^{i_1}, T^{i_2}, \ldots, T^{i_s}$ at least $\left\lfloor \frac{n-k}{2} \right\rfloor \left\lfloor \frac{(n-k)-1}{2} \right\rfloor$ times in $D$.*

***Proof.*** When $k > n-3$, the assertion is true. For $k \leq n-3$, consider $n-k$ vertices of the cycle $C_n$ incident with the edges of $T^{i_j}$ which do not cross $C_n$. Let us delete all edges of $T^1, T^2, \ldots, T^m$ which are not incident with these $n-k$ vertices. The resulting subgraph is homeomorphic to the graph $D_m + C_{n-k}$ and, in its subdrawing induced by $D$, all subgraphs $T^{i_1}, T^{i_2}, \ldots, T^{i_s}$ as well as $T^{i_j}$ of $D_m + C_{n-k}$ satisfies the conditions of Corollary 3. Hence, the subgraph $T^{i_j}$ crosses each of the subgraphs $T^{i_1}, T^{i_2}, \ldots, T^{i_s}$ at least $\left\lfloor \frac{n-k}{2} \right\rfloor \left\lfloor \frac{(n-k)-1}{2} \right\rfloor$ times. ∎

## 3. JOIN OF STARS WITH CYCLES

Our aim in this section is to give the crossing numbers of the join products $S_m + C_n$ for $m \leq 5$ and $n \geq 3$. Clearly, the graph $K_1 + C_n$ is planar. It is shown in [17] that $\mathrm{cr}(P_k + C_n) = \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ for $k \geq 2$. As $S_1 + C_n$ is isomorphic to $P_2 + C_n$ and $S_2 + C_n$ is isomorphic to $P_3 + C_n$, the crossing number of $S_1 + C_n$ is one, and the crossing number of $S_2 + C_n$ is $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$. For $m \geq 3$, our proofs are based on Kleitman's result $\mathrm{cr}(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$ for $\min\{p, q\} \leq 6$. This is the reason that, unfortunately, we are not able to give the crossing number of the graph $S_m + C_n$ for $m > 5$ and arbitrarily large $n$, because the graph $S_6 + C_n$ contains $K_{7,n}$ as a subgraph.

In this section, the crossing numbers of the graphs $S_m + C_n$ for $m = 3, 4, 5$, and for all $n \geq 3$ are established. The methods used in the general proofs are successful for $n \geq 5$. For the small graphs $S_3 + C_3$, $S_3 + C_4$, $S_4 + C_3$, $S_4 + C_4$, $S_5 + C_3$, and $S_5 + C_4$, the crossing numbers must be done separately. We tried to compute these values of crossing numbers using algorithm located on the webside `http://crossings.uos.de/`. This algorithm can find the crossing numbers of small undirected graphs. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price, see [6, 7] and [8]. The system also generates verifiable formal proofs, as described in [9]. Unfortunately, the capacity of this system is restricted and it does not establish crossing numbers for large graphs.

In the reasonable time, the algorithm found the exact values of crossing numbers for the graphs $S_3+C_3$, $S_3+C_4$, $S_4+C_3$, $S_4+C_4$, and $S_5+C_3$. The next Lemma 5 lists these results. We remark that we can prove that all these results are correct. All these proofs are similar to the proof of Lemma 6. Moreover, all of them are much easier. So, we omit these proofs and the reader can find them on his own. Since for the graph $S_5+C_4$ the algorithm in `http://crossings.uos.de/` found only a lower bound of 20 and an upper bound of 23, we prove in Lemma 6 that the crossing number of the graph $S_5 + C_4$ is 23.

**Lemma 5.** $\mathrm{cr}(S_3 + C_3) = 5$*,* $\mathrm{cr}(S_3 + C_4) = \mathrm{cr}(S_4 + C_3) = 8$*,* $\mathrm{cr}(S_4 + C_4) = 14$*, and* $\mathrm{cr}(S_5 + C_3) = 13$.

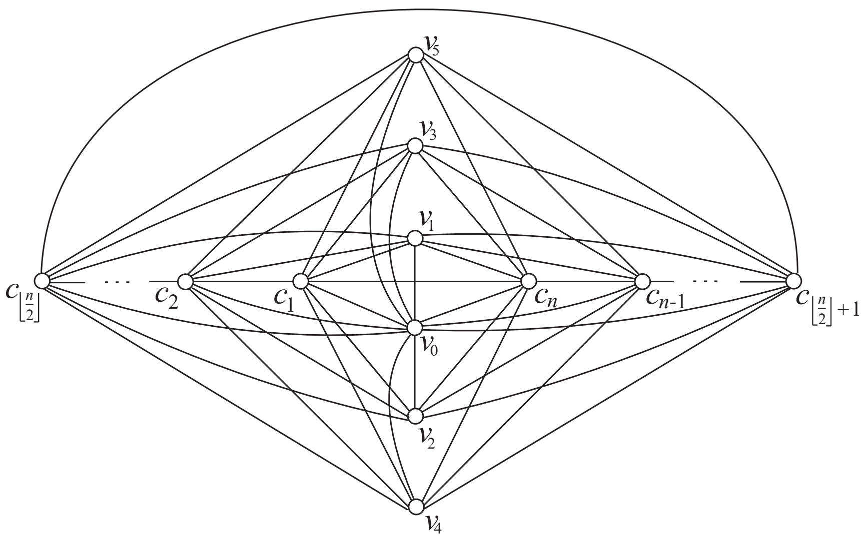


Figure 2. The drawing of the graph graph $S_5 + C_n$.

The drawing in Figure 2 shows the graph $S_5 + C_n$. Let us denote by $v_0$, $v_1$, $v_2, v_3, v_4$, and $v_5$ the vertices of the star $S_5$, where $v_0$ is the central vertex, and let

$c_1, c_2, \ldots, c_n$ be the vertices of the cycle $C_n$. For the graph $S_5 + C_n$, as well as for its subgraphs $S_3 + C_n$ and $S_4 + C_n$, let $T^{v_i}$ denote the subgraph induced on the $n$ edges incident with the vertex $v_i$, $i = 0, 1, 2, \ldots, 5$. The edges of $\bigcup_{i=0}^{5} T^{v_i} \cong K_{6,n}$ cross each other $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times. For every vertex $c_j$, $j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, the edges incident with $c_j$ cross the edges of the star $S_5$ four times, and the cycle $C_n$ crosses $S_5$ three times. So, in Figure 2, exactly $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4 \left\lfloor \frac{n}{2} \right\rfloor + 3$ crossings appear among the edges of the graph $S_5 + C_n$ in this drawing.

**Lemma 6.** $\mathrm{cr}(S_5 + C_4) = 23$.

***Proof.*** It follows from the drawing in Figure 2 that $\mathrm{cr}(S_5 + C_4) \leq 23$. To prove the reverse inequality assume that there is an optimal drawing $D$ of the graph $S_5 + C_4$ with at most 22 crossings. By Lemma 1, the edges of $C_4$ do not cross each other in $D$. Moreover, the edges of $C_4$ are crossed at most twice in $D$, otherwise by deleting the edges of $C_4$ from $D$ a drawing of the complete tripartite graph $K_{1,5,4}$ with less than 20 crossings is obtained. This contradicts the fact that $\mathrm{cr}(K_{1,5,4}) = 20$, see [13]. By Lemma 5, $\mathrm{cr}(S_4 + C_4) = 14$. This implies that in $D$ there are at most eight crossings on the edges of every subgraph $T^{v_i} \cup \{v_0, v_i\}$, $i = 1, 2, 3, 4, 5$. As $(S_5 + C_4) \setminus E(C_4 \cup T^{v_0})$ is isomorphic to the complete bipartite graph $K_{5,5}$ with crossing number 16, at most six crossings appear on the edges of $C_4 \cup T^{v_0}$. We note that $C_4$ is crossed at least once, because otherwise all subgraphs $T^{v_i}$, $i = 0, 1, 2, 3, 4, 5$, are placed in the same region of $C_4$ and, by Lemma 2, in $D$ there are at least $2\binom{6}{2} > 22$ crossings.

The cycle $C_4$ divides the plane into two regions and, in $D$, the vertex $v_0$ is placed in one of them, say in the interior region. Our next analysis depends on the number of crossings between the edges of $C_4$ and the edges of the subgraph $T^{v_0}$ in the drawing $D$. If $\mathrm{cr}_D(T^{v_0}, C_4) = 0$, then at least three subgraphs $T^{v_i}$, $i \in \{1, 2, 3, 4, 5\}$, are placed in the same region as $v_0$ such that $\mathrm{cr}(T^{v_i}, C_4) = 0$. By Corollary 3, each such subgraph $T^{v_i}$ crosses $T^{v_0}$ at least twice and, together with at least one crossing on $C_4$, on the edges of $C_4 \cup T^{v_0}$ there are more than six crossings. Thus, $\mathrm{cr}_D(T^{v_0}, C_4) \neq 0$.

If $\mathrm{cr}_D(T^{v_0}, C_4) = 2$, no other crossing appears on the edges of $C_4$. In this case, the star $S_5$ is placed in the interior region of $C_4$ such that none of $T^{v_i}$, $i = 1, 2, 3, 4, 5$, crosses $C_4$. Thus, by Corollary 3, in $D$ there are at least $2\binom{5}{2} + 2 = 22$ crossings among the edges of $(S_5 + C_4) \setminus E(S_5)$. This forces that $\mathrm{cr}_D(C_4 \cup T^{v_0}, T^{v_i} \cup \{v_0, v_i\}) = 0$ for all $i = 1, 2, 3, 4, 5$ and in the subdrawing induced by $C_4 \cup T^{v_0}$ there must exist a region containing all five vertices $c_1, c_2, c_3, c_4$, and $v_0$ on its boundary. The only possibility to obtain such a drawing is that one edge of $C_4$ is crossed by two edges of $T^{v_0}$. Without loss of generality, let the edge $\{c_1, c_2\}$ of $C_4$ is crossed by the edges $\{v_0, c_3\}$ and $\{v_0, c_4\}$. Thus, the 5-cycle $c_1 v_0 c_2 c_3 c_4 c_1$ forms the boundary of the region containing the vertices $v_1, v_2, v_3, v_4$, and $v_5$. This boundary together with the edges joining the vertices $v_1, v_2, v_3, v_4$, and $v_5$ with the vertices $c_1, c_2, c_3, c_4$, and $v_0$ form the graph $D_5 + C_5$. Hence, by Lemma 2, in $D$ there are at least $4\binom{5}{2} > 22$ crossings. This confirms that $\mathrm{cr}_D(T^{v_0}, C_4) = 1$.

Assume first that the crossing between $C_4$ and $T^{v_0}$ is the only crossing on the edges of $C_4$. Then the subdrawing of $C_4 \cup T^{v_0}$ induced by $D$ is unique as shown in Figure 3(a). Clearly, every subgraph $T^{v_i}$, $i = 1, 2, 3, 4, 5$, crosses $T^{v_0}$. Moreover, none of them crosses $T^{v_0}$ twice, otherwise the edges of $C_4 \cup T^{v_0}$ are crossed more than six times. Hence, in $D$, all five vertices $v_i$, $i \neq 0$, are placed in the region $\alpha$ of the subdrawing of $C_4 \cup T^{v_0}$. Now, the boundary of the region $\alpha$ forms the 4-cycle and, in $D$, all five vertices $v_i$ inside $\alpha$ are adjacent with four vertices on its boundary such that the considered edges do not cross the boundary of $\alpha$. Thus, applying Lemma 2 on the subgraph isomorphic to $D_5 + C_4$, the edges of $\bigcup_{i=0}^{5} T^{v_i}$ cross each other at least $2\binom{5}{2}$ times and in $D$ there are at least $2\binom{5}{2} + 5 + 1 > 22$ crossings.

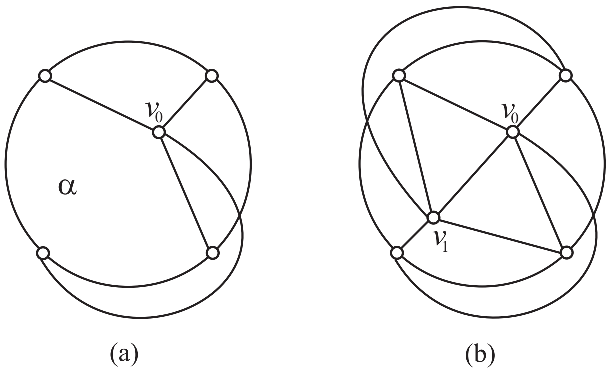


Figure 3. The subdrawings of $C_4 \cup T^{v_0}$ and $C_4 \cup T^{v_0} \cup T^{v_1} \cup \{v_0, v_1\}$.

The last possibility is that $\mathrm{cr}_D(T^{v_0}, C_4) = 1$ and the edges of $C_4$ are crossed twice. This implies that $\mathrm{cr}_D(S_5, C_4 \cup T^{v_0}) = 0$, otherwise, together with the crossings between five subgraphs $T^{v_i}$ and $C_4 \cup T^{v_0}$, more than six crossings appear on the edges of $C_4 \cup T^{v_0}$. Thus, one of the subgraphs $T^{v_i}$, say $T^{v_1}$, crosses $C_4$ once. The subdrawing of $C_4 \cup T^{v_1}$ induced by $D$ is the same as the drawing in Figure 3(a) if we replace the vertex $v_0$ by the vertex $v_1$. Since no other crossing is allowed on $C_4$, every subgraph $T^{v_i}$, $i = 2, 3, 4, 5$, crosses both $T^{v_0}$ and $T^{v_1}$. Moreover, by Corollary 3, the edges of $T^{v_2} \cup T^{v_3} \cup T^{v_4} \cup T^{v_5}$ cross each other at least $2\binom{4}{2}$ times. So, at least 22 crossings appear in $D$ on the edges other than the edges of $S_5$. Now, the unique subdrawing of the subgraph $C_4 \cup T^{v_0} \cup T^{v_1} \cup \{v_0, v_1\}$ induced by $D$ is shown in Figure 3(b). It is easy to verify that every subgraph $T^{v_i}$, $i = 2, 3, 4, 5$, crosses the edges of $T^{v_0} \cup T^{v_1} \cup \{v_0, v_1\}$ at least three times and in $D$ there are at least $4 \cdot 3 + 2\binom{4}{2} + 2 > 22$ crossings. This completes the proof. ■

**Theorem 7.** $\mathrm{cr}(S_3 + C_n) = 2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2$ *for* $n \geq 3$.

***Proof.*** By Lemma 5, the result is true for $n = 3$ and $n = 4$. The drawing of the graph $S_3 + C_n$ with $2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2 = \frac{1}{2}n(n-1) + 2$ crossings can be obtained by deleting the vertices $v_4$ and $v_5$ from the drawing in Figure 2. So, $\mathrm{cr}(S_3 + C_n) \leq \frac{1}{2}n(n-1) + 2$. To prove the reverse inequality suppose that there is an optimal drawing $D$ with less than $\frac{1}{2}n(n-1) + 2$ crossings for $n \geq 5$. The subgraph $(S_3 + C_n) \setminus E(C_n)$ is isomorphic to the complete tripartite graph $K_{1,3,n}$. Asano in [2] proved that $\mathrm{cr}(K_{1,3,n}) = \frac{1}{2}n(n-1)$. This forces that the edges of $C_n$ are crossed at most once in $D$.

By Lemma 1, the edges of $C_n$ do not cross each other in $D$ and the subdrawing of $C_n$ induced by $D$ divides the plane into two regions. If the edges of $C_n$ are not crossed, then all four vertices $v_0, v_1, v_2$, and $v_3$ are placed in one of these two regions and, by Lemma 2, in $D$ there are at least $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor > \frac{1}{2}n(n-1) + 1$ crossings, a contradiction. So, if all vertices $v_0, v_1, v_2$, and $v_3$ are placed in $D$ in the same region of $C_n$, exactly one subgraph $T^{v_i}$, $i \in \{0, 1, 2, 3\}$, crosses $C_n$ once. But, for $n \geq 4$, it crosses also all three subgraphs $T^{v_j}$, $j \in \{0, 1, 2, 3\}$, $j \neq i$, and Corollaries 3 and 4 imply that in $D$ there are at least $\binom{3}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 1 > \frac{1}{2}n(n-1) + 1$ crossings. This contradicts the assumption of $D$. Hence, the cycle $C_n$ separates the vertices of $S_3$. As $C_n$ does not have more than one crossing, one vertex, say $v_3$, of degree one of $S_3$ is separated from the other three. In this case, by Corollary 3, the edges of $T^{v_0} \cup T^{v_1} \cup T^{v_2}$ cross each other at least $\binom{3}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times and in $D$ there are at least $3\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 1$ crossings. This number exceeds the considered number of crossings in $D$ for $n \geq 5$ and the proof is done. ∎

**Theorem 8.** $\mathrm{cr}(S_4 + C_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor + 2$ *for* $n \geq 3$.

***Proof.*** By Lemma 5, the result is true for $n = 3$ and $n = 4$. The drawing of the graph $S_4 + C_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor + 2 = n(n-1) + 2$ crossings can be obtained by deleting the vertex $v_5$ from the drawing in Figure 2. Hence, $\mathrm{cr}(S_4 + C_n) \leq n(n-1) + 2$. To prove the reverse inequality assume that, for $n \geq 5$, there is an optimal drawing $D$ of the graph $S_4 + C_n$ with fewer than $n(n-1) + 2$ crossings.

By Lemma 1, the edges of $C_n$ do not cross each other in $D$. The subgraph $(S_4 + C_n) \setminus E(C_n)$ of the graph $S_4 + C_n$ is isomorphic to the complete tripartite graph $K_{1,4,n}$. It was shown in [12] that $\mathrm{cr}(K_{1,4,n}) = n(n-1)$. This implies that the edges of $C_n$ are crossed at most once in $D$. On the other hand, if no edge of $C_n$ is crossed, then all five vertices of $S_4$ are placed in the same region in the view of the subdrawing of $C_n$ induced by $D$ and, by Lemma 2, in $D$ there are at least $\binom{5}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ crossings. This number is greater than the considered number of crossings in the drawing $D$. Thus, the edges of $C_n$ are crossed exactly once in $D$.

Regardless of $C_n$ is crossed by $S_4$ or by some subgraph $T^{v_i}$, $i \in \{0,1,2,3,4\}$, at least four subgraphs $T^{v_i}$ are placed in the same region of the subdrawing of $C_n$ and none of them crosses $C_n$. Thus, by Corollary 3, in $D$ there are at least $\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 1$ crossings. This exceeds the considered number of crossings in $D$ for $n \geq 5$, and the proof is done. ■

**Theorem 9.** $\mathrm{cr}(S_5 + C_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor + 3$ *for* $n \geq 3$.

***Proof.*** By Lemmas 5 and 6, Theorem 9 is true when $n = 3$ and $n = 4$. The drawing in Figure 2 shows that $\mathrm{cr}(S_5 + C_n) \leq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor + 3$. For $n \geq 5$, assume that there is an optimal drawing $D$ of the graph $S_5 + C_n$ with at most $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor + 2$ crossings. By Lemma 1, the edges of $C_n$ do not cross each other in $D$. The subgraph $(S_5 + C_n) \setminus E(C_n)$ is isomorphic to the complete tripartite graph $K_{1,5,n}$. It was proved in [13] that $\mathrm{cr}(K_{1,5,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor$. Thus, $C_n$ is crossed at most twice in $D$. At most four vertices $v_i$, $i \in \{0,1,2,3,4,5\}$, are placed in the same region of $C_n$ such that the corresponding subgraphs $T^{v_i}$ do not cross $C_n$. Otherwise, by Corollary 3, in $D$ there are at least $\binom{5}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor > 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n}{2}\right\rfloor + 2$ crossings. This forces that the edges of $C_n$ are crossed exactly twice and that every subgraph $T^{v_i}$ crosses $C_n$ at most once. In addition, as the deleting of all edges of the subgraph $C_n \cup T^{v_0}$ results in the graph $K_{5,n+1}$ with crossing number $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor$, at most $2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2$ crossings can appear on the edges of $C_n \cup T^{v_0}$ in $D$.

We show first that $T^{v_0}$ crosses $C_n$ in $D$. If $\mathrm{cr}_D(T^{v_0}, C_n) = 0$, at least three subgraphs $T^{v_i}$, $i \in \{1,2,3,4,5\}$, are placed in the same region of $C_n$ as the vertex $v_0$ with $\mathrm{cr}_D(T^{v_i}, C_n) = 0$. By Corollary 3, each of these three considered subgraphs crosses $T^{v_0}$ at least $\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ times. Hence, the edges of $C_n \cup T^{v_0}$ are crossed at least $3\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 1$ times. This is greater than $2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2$ for $n \geq 5$. Thus, $\mathrm{cr}_D(T^{v_0}, C_n) = 1$ and there is exactly one another crossing on the edges of $C_n$. For the case $\mathrm{cr}(S_5, C_n) = 1$, one subgraph $T^{v_i}$, say $T^{v_1}$, is placed in the other region of $C_n$ as $v_0$ with $\mathrm{cr}_D(T^{v_1}, C_n) = 0$. The drawing in Figure 3(a) can be generalized such that it confirms that, in this case, $T^{v_1}$ crosses $T^{v_0}$ at least once. If $\mathrm{cr}(S_5, C_n) = 0$, one subgraph $T^{v_i}$, say again $T^{v_1}$, crosses $C_n$ once. But it crosses also $T^{v_0}$. By Corollary 4, each of the remaining four subgraphs $T^{v_i}$, $i = 2,3,4,5$, crosses $T^{v_0}$ at least $\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$ times. Hence, on the edges of $C_n \cup T^{v_0}$ there are at least $4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3$ crossings. This contradiction with at most $2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2$ crossings on the edges of $C_n \cup T^{v_0}$ completes the proof. ■

## 4. Cartesian Products of Wheels and Trees

In this section, using the results obtained in Section 3, we give the crossing number of the Cartesian products of wheels and trees. Let $W_n$ denote the wheel of $n + 1$

vertices. In Figure 2 it is easy to verify that the graph $S_5 + C_n$ is isomorphic to the graph $W_n + D_5$. In general, the graph $S_m + C_n$ is isomorphic to the graph $W_n + D_m$ for all integers $m \geq 1$ and $n \geq 3$. So, we know the crossing numbers of the graph $W_n + D_m$ for all $m \leq 5$. Now, using zip product operation introduced in [4], we establish the crossing number of the Cartesian product $W_n \Box T$ for all trees with maximum degree five. For better reading, we repeat the related terms, notations and results introduced by Bokal in [5].

For a graph $G$ and a discrete graph $D_m$, let the vertices of $D_m$ be called *apices* of the graph $G+D_m$. Consider now two graphs $G_1$ and $G_2$. For a multiset $L \subseteq V(G_2)$, we denote with $G_1 \Box_L G_2$ the *capped Cartesian product* of graphs $G_1$ and $G_2$, that is, the graph obtained by adding a distinct vertex $v'$ to $G_1 \Box G_2$ for each copy of a vertex $v \in L$ and joining $v'$ to all the vertices of $G_1 \Box \{v\}$. We call each $v'$ a *cap* of $v$. Let $\chi_L(v)$ denote the multiplicity of $v$ in $L$ and $\ell(v) := \deg_{G_2}(v) + \chi_L(v)$. An edge $\{u, v\} \in E(G_2)$ is *unbalanced* if $\ell(u) \neq \ell(v)$. Let $\beta(G_2)$ be the number of unbalanced edges of $G_2$.

A drawing $D$ of $G+D_m$ is *apex-homogeneous* if there exists a permutation $\rho$ of the vertices of $G$ such that the vertex rotation around every apex in $D$ is $\rho$ or $\rho^{-1}$. Two drawings $D^{(i)}$ of $G + D_i$ and $D^{(j)}$ of $G + D_j$ are *pairwise apex-homogeneous*, if they are apex-homogeneous with respect to the same permutation $\rho$. A graph $G$ has *all apex-homogeneous drawings* if there exist drawings $D^{(m)}$ of $G + D_m$ for all $m \geq 1$, such that every two of them are pairwise apex-homogeneous. The next result given by Bokal enables us to establish the crossing numbers of $W_n \Box T$.

**Theorem 10** ([5])**.** *Let $G$ be a graph of order $n$, let $T$ be a tree, and let $L \subseteq V(T)$ be a multiset with either $\ell(v) \geq 3$ or, if $G$ has a dominating vertex, $\ell(v) \geq 2$ for every $v \in V(T)$. Define*

$$B = \sum_{v \in V(T)} \operatorname{cr}\left(G + D_{l(v)}\right).$$

*Then, $B \leq \operatorname{cr}(G \Box_L T) \leq B + \frac{\beta(T)}{2}\binom{n}{2}$. Also, $\operatorname{cr}(G \Box_L T) = B$ whenever $G$ has all apex-homogeneous drawings such that each of them is optimal.*

Let $v \in V(G)$ be a vertex of degree $d$ in $G$. A *bundle* $B_v$ of the vertex $v$ is a set of $d$ edge disjoint paths from $v$ to some other vertex $u \in V(G)$. Let $F \subset E(G)$ be a subset of edges of $G$ and $\pi$ a permutation of $F$. A $\pi$-subdivision $G^\pi$ of $G$ is the graph obtained from $G$ by subdividing every edge $e \in F$ with the vertex $v_e$ and adding the edges $\{\{v_e, v_{\pi(e)}\} | e \in F\}$.

**Theorem 11** ([5])**.** *Let $v$ be a vertex that has a bundle $B_v$ in a graph $G$ and let $\pi$ be cyclic permutation of a subset of $F$ of all but one of the edges incident with $v$, $|F| \geq 3$. Then*

$$\operatorname{cr}(G^\pi) \geq \operatorname{cr}(G) + 1,$$

*with equality if $\pi$ respects the edge rotation around $v$ in some optimal drawing of $G$.*

In the rest of the section, we give the crossing numbers of the Cartesian products of the wheel $W_n$ and all trees $T$ with maximum degree at most five.

**Theorem 12.** *Let $T$ be a tree with maximum degree $\Delta(T) \leq 5$. Let $d_i$ be the number of vertices of degree $i$ in $T$. Then, for $n \geq 3$,*

$$\begin{aligned}\mathrm{cr}(W_n \Box T) = d_1 + d_2 + 2d_3 + 2d_4 + 3d_5 + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \Big( d_2 + 2d_3 + 4d_4 + 6d_5 \Big) \\ + \left\lfloor \frac{n}{2} \right\rfloor \Big( d_3 + 2d_4 + 4d_5 \Big) .\end{aligned}$$

***Proof.*** Let $T'$ be the tree obtained from $T$ by removing all vertices of degree one in $T$. For a vertex $v$ of $T'$, let $r_v$ be the number of $T$-leaves adjacent to $v$ in $T$, and let $L$ be the set of vertices in $T'$, each with multiplicity $r_v$. Thus, $\ell(v) = d_{T'}(v) + r_v = d_T(v) \geq 2$ for all $v \in V(T')$. Note that the central vertex of $W_n$ is a dominating vertex, i.e., a vertex adjacent to all other vertices of the graph. Let us restrict on the graph $W_n + D_m$ for $m \leq 5$. The drawing in Figure 2 shows that, by this restriction, the wheel $W_n$ has all apex-homogeneous drawings such that each of them is optimal. Thus, by Theorem 10, $\mathrm{cr}(W_n \Box_L T') = \sum_{i=2}^{\Delta(T)} (d_i \cdot cr(W_n + D_i))$.

The graph $W_n \Box_L T'$ is obtained from the Cartesian product $W_n \Box T'$ by adding $r_v$ caps to $W_n \Box \{v\}$ for every vertex $v$ of $T$ with $r_v$ $T$-leaves adjacent to $v$. This consistency in combination with Theorem 11 also implies that a properly chosen $\pi$-subdivision of edges connecting a cap of $W_n \Box_L T'$ with the corresponding rim increases the crossing number by precisely one. Thus, after such $\pi$-subdivision for each of $d_1$ leafs of $T$, the graph $W_n \Box T$ is obtained. This, together with the fact that $\mathrm{cr}(W_n + D_1) = 1$, proves that $\mathrm{cr}(W_n \Box T) = \sum_{i=1}^{\Delta(T)} (d_i \cdot \mathrm{cr}(W_n + D_i))$.

The values of crossing numbers of $W_n + D_m$ are known for all $m \leq 5$. In [5] Bokal proved that $\mathrm{cr}(W_n + D_2) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1$. As the graph $W_n + D_m$ is isomorphic to the graph $S_m + C_n$, by Theorems 7, 8, and 9 we have that $\mathrm{cr}(W_n + D_3) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2$, $\mathrm{cr}(W_n + D_4) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$, and $\mathrm{cr}(W_n + D_5) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4 \left\lfloor \frac{n}{2} \right\rfloor + 3$ for $n \geq 3$. Applying this, together with the formula $\mathrm{cr}(W_n \Box T) = \sum_{i=1}^{\Delta(T)} (d_i \cdot \mathrm{cr}(W_n + D_i))$ for $\Delta(T) \leq 5$, the result is done. ■

At present, the crossing numbers of very few join products of graphs with discrete graphs are known. In [21] there are collected the crossing numbers of $G + D_m$ for all graphs $G$ of order at most four. Only some few results on crossing numbers of $G + D_m$ are known for graphs $G$ on five and six vertices. Note that Theorem 10 cannot be simply used for estimating the crossing number of the Cartesian product $G \Box T$ for all graphs $G$, for which the crossing number of the join product $G + D_m$ is known. The reason is that not every graph contains a dominating vertex and, in this case, the vertices of degree two in $T$ are not treatable using zip product.

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