Discussiones Mathematicae
Graph Theory 37 (2017) 465–476
doi:10.7151/dmgt.1955

THE CHROMATIC NUMBER OF RANDOM INTERSECTION GRAPHS

Katarzyna Rybarczyk[1]

Adam Mickiewicz University
Faculty of Mathematics and Computer Science
60–769 Poznań, Poland

e-mail: kryba@amu.edu.pl

Abstract

We study problems related to the chromatic number of a random intersection graph $\mathcal{G}(n, m, p)$. We introduce two new algorithms which colour $\mathcal{G}(n, m, p)$ with almost optimum number of colours with probability tending to 1 as $n \to \infty$. Moreover we find a range of parameters for which the chromatic number of $\mathcal{G}(n, m, p)$ asymptotically equals its clique number.

Keywords: random intersection graphs, chromatic number, colouring algorithms.

2010 Mathematics Subject Classification: 05C80, 05C15, 05C85.

1. Introduction

Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a set of vertices and $\mathcal{W} = \{w_1, \ldots, w_m\}$ be an auxiliary set of objects. Each vertex $v \in \mathcal{V}$ is assigned a subset of objects $W_v \subseteq \mathcal{W}$. The intersection graph generated by a family of sets $\{W_v \subseteq \mathcal{W} : v \in \mathcal{V}\}$ is the graph with vertex set $\mathcal{V}$ in which two vertices v and v' are connected by an edge if their sets of objects W_v and $W_{v'}$ intersect (i.e., $W_v \cap W_{v'} \neq \emptyset$). An intersection graph generated by a family of sets $\{W_v \subseteq \mathcal{W} : v \in \mathcal{V}\}$ is a random intersection graph if for each W_v the set $v \in \mathcal{V}$ is chosen at random.

In this paper we will study the asymptotic (as $n \to \infty$) properties of a random intersection graph $\mathcal{G}(n, m, p)$ introduced by Karoński, Scheinerman and Singer-Cohen in [10]. In a random intersection graph $\mathcal{G}(n, m, p)$ for each vertex $v \in \mathcal{V}$ and each object $w \in \mathcal{W}$ we have $w \in W_v$ with probability p independently of all other vertices and objects. In what follows we will use the

[1]Katarzyna Rybarczyk acknowledges a support of NCN (Narodowe Centrum Nauki) grant 2014/13/D/ST1/01175.

asymptotic notation $o(\cdot)$, $\ll$ and $\gg$ consistently with [9]. Moreover we will write that a random graph on n vertices G_n has property $\mathcal{A}$ with high probability if $\Pr\{G_n \text{ has property } \mathcal{A}\} \to 1$ as $n \to \infty$.

In [10], among others, it was shown that for some range of parameters the asymptotic behaviour of $\mathcal{G}(n,m,p)$ differs from this of a classical random graph $G(n,\hat{p})$ (with the vertex set $\mathcal{V}$, in which each edge appears independently with probability $\hat{p}$). The subsequent article by Fill, Scheinerman and Singer-Cohen [6] (see also [14]) determined a range of parameters for which both models $\mathcal{G}(n,m,p)$ and $G(n,\hat{p})$ are equivalent. It appears that for many graph properties there exists a range of parameters for which $\mathcal{G}(n,m,p)$ and $G(n,\hat{p})$ with the same edge density have the property with asymptotically (as $n \to \infty$) the same probability and outside which $\mathcal{G}(n,m,p)$ and $G(n,\hat{p})$ differ. We will be particularly interested in some of those properties related to the chromatic number of $\mathcal{G}(n,m,p)$. We will mainly concentrate on analysing algorithmic aspects of the problem.

The classical results concerning $G(n,\hat{p})$ ([4, 5, 8, 12, 16], see also Chapter 7 in [9]) show that as $1/n \ll \hat{p} \ll 1$ with high probability $\chi(G(n,\hat{p})) = (1+o(1))\frac{n}{\alpha(G(n,\hat{p}))}$, where $\chi(G)$ is the chromatic number of G and $\alpha(G)$ is its independence (stability) number. Therefore for sparse $G(n,\hat{p})$ with $\hat{p}$ above the phase transition threshold, a trivial relation $\chi(G(n,\hat{p})) \geq n/\alpha(G(n,\hat{p}))$ is with high probability asymptotically tight. Moreover, for $1/n \ll \hat{p} \ll 1$, the greedy algorithm (or its slight modification) with high probability uses $(1+o(1))2\chi(G(n,\hat{p}))$ colours to colour properly $G(n,\hat{p})$. Last but not least, no polynomial time algorithm which with high probability uses at most $(2-\delta)G(n,\hat{p})$ colours (for some constant $\delta > 0$) has been found so far. For an extensive discussion on the problem see [7] or Chapter 7 in [9].

The problems related to the chromatic number of $\mathcal{G}(n,m,p)$ were first studied by Behrisch, Taraz and Ueckerdt [2] and then by Nikoletseas, Raptopoulos and Spirakis [13]. In [13] the authors analysed the problem of colourability of almost all vertices (i.e., all but at most $o(n)$ vertices) while in [2] two colouring algorithms were studied. Each of the algorithms analysed in [2], for some range of parameters, with high probability colour $\mathcal{G}(n,m,p)$ with $\omega(\mathcal{G}(n,m,p))$ colours. Here by $\omega(G)$ we denote the clique number of a graph G. In particular in [2] the following two theorems were shown.

Theorem 1. *Let $m = n^{\beta}$ with $\beta > 0$ fixed and $mp \ll \sqrt{\frac{m}{n}}$ (i.e., $mp^2 \ll \frac{1}{n}$). Then $\mathcal{G}(n,m,p)$ can with high probability be coloured optimally in linear time and $\chi(\mathcal{G}(n,m,p)) = \omega(\mathcal{G}(n,m,p))$.*

If $mp^2 = o(1)$ then the probability that two vertices are connected by an edge in $\mathcal{G}(n,m,p)$ equals $1-(1-p^2)^m = (1+o(1))mp^2$. Moreover $mp^2 = \frac{1}{n}$ is a threshold function for phase transition in $\mathcal{G}(n,m,p)$ [1]. Therefore Theorem 1 describes the structure of $\mathcal{G}(n,m,p)$ far below phase transition threshold. For

$G(n, \hat{p})$ this case is trivial since then with high probability in $G(n, \hat{p})$ all components are trees. The proof of Theorem 1 basis on the fact that for $mp^2 \ll 1/n$, $\mathcal{G}(n, m, p)$ with high probability has no induced cycles of length greater than 3.

Theorem 2. *Let $m = n^\beta$ with $0 < \beta < 1$ fixed and $mp \ll \frac{1}{m \ln n}$. Then $\mathcal{G}(n, m, p)$ can with high probability be coloured optimally in linear time. Moreover, for $np > \ln^4 n$ we have with high probability*

$$\chi(\mathcal{G}(n, m, p)) = \omega(\mathcal{G}(n, m, p)) = (1 + o(1))np.$$

Theorems 1 and 2 show that if mp is „small" then with high probability $\chi(\mathcal{G}(n, m, p)) = \omega(\mathcal{G}(n, m, p))$, i.e., a trivial relation $\chi(\mathcal{G}(n, m, p)) \geq \omega(\mathcal{G}(n, m, p))$ is tight. Moreover these results give algorithms which, in this range of parameters, use exactly $\chi(\mathcal{G}(n, m, p))$ colours. A natural question arises, for what choice of n, m and p with high probability

$$\chi(\mathcal{G}(n, m, p)) = (1 + o(1))\omega(\mathcal{G}(n, m, p)) \tag{1}$$

and when with high probability

$$\chi(\mathcal{G}(n, m, p)) = (1 + o(1))\frac{n}{\alpha(\mathcal{G}(n, m, p))}.$$

Moreover one could ask whether, in the case where with high probability (1) is fulfilled, there exists a simple algorithm using almost the optimum number of colours. As the chromatic number and the independence number are tightly related, based on results determining $\alpha(\mathcal{G}(n, m, p))$ [15] one could state the following conjecture.

Conjecture 1. *Let $m \geq n^\beta$ for some constant $\beta \in (0, 1)$ and $\frac{1}{n} \ll mp^2 \ll 1$.*

(i) *If $m = o(n)$ and $mp \ll \ln \frac{n}{m}$ then with high probability*

$$\chi(\mathcal{G}(n, m, p)) = (1 + o(1))\omega(\mathcal{G}(n, m, p))$$

and there exists a polynomial time algorithm which with high probability colours $\mathcal{G}(n, m, p)$ with $(1 + o(1))\chi(\mathcal{G}(n, m, p))$ colours.

(ii) *If $n = O(m)$ or $m = o(n)$ and $mp \gg \ln \frac{n}{m}$ then with high probability*

$$\chi(\mathcal{G}(n, m, p)) = (1 + o(1))\frac{n}{\alpha(\mathcal{G}(n, m, p))}.$$

We will propose two new algorithms which colour intersection graphs. The analysis of them shows that Conjecture 1 is true in two cases. The first one is $np \geq 2 \ln n$ and $mp = o\left(\frac{\ln \ln n}{\ln n}\right)$. The second case is $\ln \ln n \ll mp \ll \ln \frac{n}{m \ln m}$ and $np \gg \ln n$. Therefore we considerably sharpen Theorem 2 and propose new ideas which possibly might be used for other values of n, m and p.

Recall that $\mathcal{V} = \{v_1, \ldots, v_n\}$ and $\mathcal{W} = \{w_1, \ldots, w_m\}$. In what follows we use the following notation.

$$\begin{aligned} V_w &= \{v \in \mathcal{V} : w \in W_v\}, \text{ for } w \in \mathcal{W}; \\ X_i &= |V_{v_i}|, 1 \le i \le m; \\ Y_i &= |W_{w_i}|, 1 \le i \le n. \end{aligned}$$

Note that in $\mathcal{G}(n, m, p)$ sets W_{v_i} $(1 \le i \le n)$ are independent and Y_i $(1 \le i \le n)$ are independent with the binomial distribution $\mathrm{Bin}(m, p)$. Moreover for all $w \in \mathcal{W}$ sets $V_w = \{v \in \mathcal{V} : w \in W_v\}$ are independent and their sizes X_i $(1 \le i \le m)$ are independent with the binomial distribution $\mathrm{Bin}(n, p)$. By colouring greedily we will mean using the algorithm which, given a palette of colours $\{k_1, k_2, \ldots\}$ and a graph with an ordered set of vertices $\{v_1, v_2, \ldots, v_n\}$, colours properly vertices one by one, using each time the available colour with the smallest index (i.e., for each vertex, the algorithm uses the colour with the smallest index chosen from the set of colours which have not been used to colour the neighbours of the considered vertex).

2. Result for $mp \ll \ln\ln n / \ln n$.

ALGORITHM 1

INPUT: an intersection graph G with an ordered vertex set $\mathcal{V}$ generated by a family of sets $\{W_v : v \in \mathcal{V}\}$
and
an infinite palette of colours;

OUTPUT: a proper vertex colouring of G;

1. $\mathcal{V}_2 := \{v \in \mathcal{V} : |W_v| \ge 2\}$;
2. Colour greedily the vertices of $G[\mathcal{V}_2]$, the subgraph of G induced on $\mathcal{V}_2$;
3. Colour greedily the remaining vertices of G.

Theorem 3. *Let $p \in (0,1)$ and let n, m be positive integers. For any $\varepsilon \in (0,1)$ and any $C > 1 + \varepsilon$ there exists $c \in (0,1)$ such that if*

$$np \ge C \ln n \quad \text{and} \quad mp \le c\frac{\ln\ln n}{\ln n}, \tag{2}$$

then for large n with probability at least $1 - 3n^{-\varepsilon}$ ALGORITHM 1 *colours $\mathcal{G}(n, m, p)$ with $\omega(\mathcal{G}(n, m, p))$ colours.*

Proof. Let $\{k_1, k_2, \ldots\}$ be a palette of colours. Let G be an intersection graph generated by a family of sets $\{W_v \subseteq \mathcal{W}, v \in \mathcal{V}\}$. The greedy colouring in Step 2 of ALGORITHM 1 uses at most $\Delta(G[\mathcal{V}_2]) + 1$ colours. Moreover for any i and j, $i \neq j$, in G there is no edge between $V_{w_i} \setminus \mathcal{V}_2$ and $V_{w_j} \setminus \mathcal{V}_2$. Therefore, after Step 2, the largest index of the colours from $\{k_1, k_2, \ldots\}$ used to colour vertices from V_{w_i} ($w_i \in \mathcal{W}$) is at most the maximum of $\Delta(G[\mathcal{V}_2]) + 1$ and X_i. Thus ALGORITHM 1 uses at most $\max\{\Delta(G[\mathcal{V}_2]) + 1, X_1, \ldots, X_m\}$ colours. On the other hand $\max_{1 \leq i \leq m} X_i \leq \omega(G)$. Therefore to prove the theorem it is enough to prove that under conditions of the theorem in $\mathcal{G}(n, m, p)$ with probability at least $1 - 3n^{-\varepsilon}$ we have

$$\Delta(\mathcal{G}(n, m, p)[\mathcal{V}_2]) < \max_{1 \leq i \leq m} X_i. \tag{3}$$

Let

$$\psi(x) = x \ln x - x + 1. \tag{4}$$

By Chernoff's inequality (see for example Theorem 2.1 in [9]) for any random variable X with the binomial distribution with mean μ we have

$$\Pr\{X \geq x\mu\} \leq \exp(-\psi(x)\mu) \qquad \text{for } x > 1; \tag{5}$$

$$\Pr\{X \leq x\mu\} \leq \exp(-\psi(x)\mu) \qquad \text{for } x \in (0, 1). \tag{6}$$

Let $\varepsilon > 0$, $C > (1 + \varepsilon)$, and $np \geq C \ln n$. Moreover, define $A \in (0, 1)$ and $c \in (0, 1)$ to be such that

$$\psi(A) = \frac{(1 + \varepsilon) \ln n}{np} \quad \text{and} \quad A \ln c - c + 1 = 0. \tag{7}$$

Note that $\frac{(1+\varepsilon)\ln n}{np} < 1$, i.e., A is well defined.

Moreover (2) implies $m \leq n$. Therefore by (6) and (7)

$$\begin{aligned} \Pr\left\{ \underset{1 \leq i \leq m}{\forall} X_i > Anp \right\} &\geq 1 - \Pr\left\{ \underset{1 \leq i \leq m}{\exists} X_i \leq Anp \right\} \\ &\geq 1 - m \exp(-(1 + \varepsilon) \ln n) = 1 - n^{-\varepsilon}. \end{aligned}$$

Thus

$$\Pr\left\{ \max_{1 \leq i \leq m} X_i > Anp \right\} \geq 1 - n^{-\varepsilon}. \tag{8}$$

Now we will find the upper bound on $\Delta(\mathcal{G}(n, m, p)[\mathcal{V}_2])$. For $v_i \in \mathcal{V}_2$, let Z_i ($1 \leq i \leq n$) be the number of neighbours of vertex $v_i \in \mathcal{V}$ which are contained in $\mathcal{V}_2$. Moreover let $Z_i = 0$ for $v_i \notin \mathcal{V}_2$. Given $Y_i = d$ $(d \geq 2)$ Z_i has the binomial distribution with parameters $n-1$ and

$$\left(1-(1-p)^d\right) - dp(1-p)^{m-1} \leq dmp^2.$$

Therefore, under condition $Y_i = d$ $(d \geq 2)$ Z_i is stochastically dominated by random variable $Z^*_{i,d}$ with the binomial distribution $\mathrm{Bin}\left(n, dmp^2\right)$.

By (5), for large n,

$$\begin{aligned}
\Pr\left\{Y_i \geq \left\lceil \frac{\ln n}{\ln\ln n}\right\rceil\right\} &\leq \exp\left(-\frac{\ln n}{\ln\ln n}\cdot \ln\left(\frac{\frac{\ln n}{\ln\ln n}}{mp}\right) + \frac{\ln n}{\ln\ln n} - mp\right)\\
&\leq \exp\left(-\frac{\ln n}{\ln\ln n}\cdot \ln\left(\frac{\frac{\ln n}{\ln\ln n}}{\frac{\ln\ln n}{\ln n}}\right) + \frac{\ln n}{\ln\ln n} - mp\right)\\
&\leq \exp\left(-\frac{\ln n}{\ln\ln n}\left(2\ln\ln n - 2\ln\ln\ln n\right) + \frac{\ln n}{\ln\ln n} - mp\right)\\
&\leq \exp\left(-2\ln n\left(1-\frac{\ln\ln\ln n}{\ln\ln n} - \frac{1}{2\ln\ln n} + \frac{mp}{2\ln n}\right)\right)\\
&\leq \exp(-(1+\varepsilon)\ln n) = n^{-1-\varepsilon}.
\end{aligned}$$

In the second inequality we have used the fact that by definition we have $c < 1$, i.e., $mp \leq c\frac{\ln\ln n}{\ln n} \leq \frac{\ln\ln n}{\ln n}$.

Moreover for $d \leq \left\lfloor\frac{\ln n}{\ln\ln n}\right\rfloor$ by (4), (5), and (7) we get

$$\begin{aligned}
\Pr\left\{Z^*_{i,d} \geq Anp\right\} &\leq \exp\left(-dnmp^2\psi\left(\frac{A}{dmp}\right)\right)\\
&= \exp\left(-dnmp^2\left(\frac{A}{dmp}\ln\frac{A}{dmp} - \frac{A}{dmp} + 1\right)\right)\\
&= \exp\left(-np\left(A\ln A - A\ln dmp - A + dmp\right)\right)\\
&\leq \exp\left(-np\psi\left(A\right) + np\left(A\ln(dmp) - dmp + 1\right)\right)\\
&\leq \exp\left(-(1+\varepsilon)\ln n + np\left(A\ln c - c + 1\right)\right)\\
&= \exp\left(-(1+\varepsilon)\ln n\right) = n^{-1-\varepsilon}.
\end{aligned}$$

In the first inequality we used (5). The second last line follows by (7) and by the fact that function $f(x) = A\ln x - x + 1$ is increasing for $x \leq A$ (recall that $dmp \leq c \leq A$). In order to obtain the last line, (7) was used.

Thus

$$\Pr\left\{\Delta(\mathcal{G}\left(n,m,p\right)[\mathcal{V}_2]) \geq Anp\right\} = \Pr\left\{\underset{1\leq i\leq n}{\exists} Z_i \geq Anp\right\} \leq n\Pr\left\{Z_1 \geq Anp\right\}$$

$$(9) \quad \leq n \sum_{d=2}^{\left\lfloor \frac{\ln n}{\ln \ln n} \right\rfloor} \Pr\left\{Z^*_{1,d} \geq Anp\right\} \Pr\left\{Y_1 = d\right\} + n \Pr\left\{Y_1 \geq \left\lceil \frac{\ln n}{\ln \ln n} \right\rceil\right\} \leq 2n^{-\varepsilon}.$$

Therefore by (8) and (9) with probability at least $1 - 3n^{-\varepsilon}$

$$\Delta(\mathcal{G}\,(n,m,p)\,[\mathcal{V}_2]) < Anp < \max_{1 \leq i \leq m} X_i \leq \omega(\mathcal{G}\,(n,m,p)),$$

i.e., (3) is fulfilled. ∎

3. RESULT FOR $mp \gg \ln \ln n$

The result presented in this section will partly rely on Theorem 1.2 from [11]. The result from [11] concerns a uniform random intersection graph in which, for all $v \in \mathcal{V}$, W_v is chosen independently and uniformly at random from all D-element subsets of $\mathcal{W}$. In [11] it was shown that, under some conditions on D, n and m, a greedy random algorithm colours a uniform random intersection graph with almost the optimum number of colours.

We will use the fact that the algorithm analysed in [11] may be easily rewritten to work on any random intersection graph in which, for each $v \in \mathcal{V}$, $|W_v| \leq D$ and W_v is chosen independently and uniformly at random from all subsets of $\mathcal{W}$ of size $|W_v|$. Namely each vertex $v \in \mathcal{V}$ would independently choose additional $D - |W_v|$ vertices from $\mathcal{W} \setminus W_v$ to obtain a uniform random intersection graph. Then one would run the algorithm described in [11] on the constructed uniform random intersection graph. We will call the above described algorithm ALGORITHM[D].

Lemma 4 (Corollary of Theorem 1.2 from [11])**.** *Let $D, m, n \geq 2$, $\bar{D} = \frac{Dn}{m}$, and $\mathcal{V}_{\leq D} := \{v \in \mathcal{V} : |W_v| \leq D\}$. For any $\delta > 0$ there is a constant $c_\delta > 0$ such that the following holds. Suppose that*

$$(10) \qquad D \leq c_\delta \ln\left(\frac{m}{\ln \bar{D}}\right) \quad \text{and} \quad D \leq c_\delta \ln\left(\frac{\bar{D}}{\ln m}\right).$$

Then ALGORITHM[D] *properly colours all vertices of $\mathcal{G}(n,m,p)[\mathcal{V}_{\leq D}]$ with at most $\lceil \bar{D}(1+\delta) \rceil$ colours with probability at least $1 - 2m^{-1} - 2\bar{D}^{-1}$.*

Remark 5. Let $D \geq 2$ and $m \geq n^\beta$ for some $\beta \in (0,1)$. Then by a simple calculation we get that for large n

$$\ln\left(\frac{\bar{D}}{\ln m}\right) \geq \ln\left(\frac{n}{m \ln m}\right)$$

and

$$\ln\left(\frac{m}{\ln \bar{D}}\right) = \ln\left(\left(\frac{n}{m\ln m}\right)^{\beta}\right) + \ln\left(\frac{m^{1+\beta}(\ln m)^{\beta}}{n^{\beta}\ln \bar{D}}\right) \geq \beta \ln\left(\frac{n}{m\ln m}\right).$$

ALGORITHM 2

INPUT: an intersection graph G with an ordered vertex set $\mathcal{V}$ generated by a family of sets $\{W_v : v \in \mathcal{V}\}$;
an infinite palette of colours;
integer D;

OUTPUT: a proper vertex colouring of G;

1. $\mathcal{V}_{\leq D} := \{v \in \mathcal{V} : |W_v| \leq D\}$;
2. Colour $G[\mathcal{V}_{\leq D}]$ using ALGORITHM[D];
3. Colour greedily remaining vertices of G.

Theorem 6. *Let $p \in (0,1)$ and $m \geq n^{\beta}$ for some constant $\beta \in (0,1)$. For all $\delta \in (0,1)$ there exists c and C such that if*

$$\frac{1}{c}\ln\ln n \leq mp \leq c\ln\left(\frac{n}{m\ln m}\right) \quad \text{and} \quad np \geq C\ln n$$

and

$$D = \left\lceil \left(1+\frac{\delta}{5}\right)mp \right\rceil + 1,$$

then for large n with probability at least $1-2m^{-1}-2(np)^{-1}-2n^{-0.2}$, ALGORITHM 2 *colours $\mathcal{G}(n,m,p)$ with at most $(1+\delta)np$ colours and*

$$\chi(\mathcal{G}(n,m,p)) \leq (1+\delta)np\,.$$

In the proof we will use the following lemma.

Lemma 7 (Lemma 1 in [3])**.** *Let d_i and d_j be positive integers and $0<\xi<1$ be such that $d_i+d_j \leq \xi m$. Then the probability that v_i and v_j are neighbours in $\mathcal{G}(n,m,p)$ under the condition that $|W_{v_i}| = d_i$ and $|W_{v_j}| = d_j$ is*

$$\frac{d_i d_j}{m}\left(1-\frac{d_i d_j}{m}\right) \leq \Pr\left\{W_{v_i}\cap W_{v_j} \neq \emptyset \mid |W_{v_i}| = d_i, |W_{v_j}| = d_j\right\}$$
$$\leq \frac{d_i d_j}{m}\left(1+\frac{2}{(1-\xi)}\frac{d_i d_j}{m}\right).$$

Proof of Theorem 6. Let

$$c = \min\left\{\frac{1}{8}, \frac{\psi(1+\delta/5)}{2}, \frac{\beta c_{\delta/4}}{(1+\delta/4)}\right\}, \tag{11}$$

where $\psi(\cdot)$ is defined in (4) and $c_{\delta/4}$ is as in Lemma 4.

Let $D = \lceil (1+\delta/5)mp \rceil + 1$. By assumption $c^{-1} \ln\ln n \le mp$, therefore $D \le (1+\delta/4)mp$ for large n. Moreover, let

$$\mathcal{V}_{\le D} = \{v_i \in \mathcal{V} : Y_i \le D\},$$
$$\mathcal{V}_{> D} = \{v_i \in \mathcal{V} : D < Y_i \le \ln n\}.$$

In what follows we will prove that

$$\Pr\{(\mathcal{V}_{\le D} \cup \mathcal{V}_{>D}) = \mathcal{V}\} \ge 1 - n^{-0.2} \tag{12}$$

and

$$\Pr\left\{\Delta(\mathcal{G}(n,m,p)[\mathcal{V}_{>D}]) \le \frac{\delta}{4} np\right\} \ge 1 - n^{-0.2}. \tag{13}$$

By the definition of D and (11), for large n, we have

$$D \le \left(1+\frac{\delta}{4}\right) mp \le \left(1+\frac{\delta}{4}\right) c \ln \frac{n}{m \ln m} \le \beta c_{\delta/4} \ln \frac{n}{m \ln m}.$$

Thus by Remark 5 and by Lemma 4 with probability at least $1-2m^{-1}-2(np)^{-1}$ Algorithm[D] colours $\mathcal{G}(n,m,p)[\mathcal{V}_{\le D}]$ with at most

$$\left(1+\frac{\delta}{4}\right)\frac{Dn}{m} \le \left(1+\frac{\delta}{4}\right)^2 np \le \left(1+\frac{3\delta}{4}\right) np$$

colours. This combined with (12) and (13) shows that for large n with probability at least $1-2m^{-1}-2(np)^{-1}-2n^{-0.2}$ Algorithm 2 uses at most $(1+\delta)np$ colours to colour $\mathcal{G}(n,m,p)$. We are left with showing (12) and (13).

Note that $mp \le c \ln n/(m \ln m) \le c \ln n$ (for large n), i.e., $\ln n/mp \ge c^{-1}$. Moreover $x^{-1}\psi(x)$ is increasing for $x \ge 1$. Therefore by (5) and (11), for large n and any $1 \le i \le n$

$$\begin{aligned}\Pr\{Y_i \ge \ln n\} &\le \exp\left(-\ln n \cdot \frac{mp}{\ln n} \cdot \psi\left(\frac{\ln n}{mp}\right)\right)\\ &\le \exp\left(-\ln n \cdot \frac{\psi(c^{-1})}{c^{-1}}\right) \le \exp\left(-\ln n \cdot \frac{\psi(8)}{8}\right) \le n^{-1.2}.\end{aligned}$$

Thus

$$\Pr\left\{\exists_{1 \le i \le n} Y_i \ge \ln n\right\} \le \sum_{i=1}^{n} \Pr\{Y_i \ge \ln n\} \le n^{-0.2},$$

which implies (12).

Now we focus on showing (13). For $v_i, v_j \in \mathcal{V}$, denote by $v_i \sim v_j$ the event that v_i and v_j are neighbours in $\mathcal{G}(n,m,p)$ (i.e., $W_{v_i} \cap W_{v_j} \neq \emptyset$). Recall that Y_i and Y_j are independent random variables. For $d_i, d_j \leq \ln n$ and large n we have $d_i + d_j \leq m/2$ and $d_i d_j / m \leq \frac{1}{2}$. Therefore by Lemma 7

$$\begin{aligned}
&\Pr\{v_i \sim v_j \cap \{D \leq Y_j \leq \ln n\} \mid Y_i = d_i\} \\
&= \sum_{d_j=D}^{\ln n} \Pr\{v_i \sim v_j \cap \{Y_j = d_j\} \mid Y_i = d_i\} \\
&= \sum_{d_j=D}^{\ln n} \Pr\{v_i \sim v_j \mid \{Y_i = d_i\} \cap \{Y_j = d_j\}\} \Pr\{Y_j = d_j\} \\
&\leq \sum_{d_j=D}^{\ln n} 3\frac{d_i d_j}{m} \binom{m}{d_j} p^{d_j} (1-p)^{m-d_j} \leq 3 d_i p \sum_{d_j=D}^{\ln n} \binom{m-1}{d_j-1} p^{d_j-1} (1-p)^{m-d_j} \\
&\leq 3 d_i p \Pr\{Y_j * \geq D-1\} \leq 3 d_i p \Pr\{Y_j \geq D-1\} \leq 3p \ln n \Pr\{Y_j \geq D-1\},
\end{aligned}$$

where Y_j^* has the binomial distribution $\mathrm{Bin}(m-1,p)$. By (5)

$$\begin{aligned}
\Pr\{Y_j \geq D-1\} &\leq \Pr\left\{Y_j \geq \left(1+\frac{\delta}{5}\right) mp\right\} \\
&\leq \exp\left(-\psi\left(1+\frac{\delta}{5}\right) mp\right) \leq (\ln n)^{-\psi\left(1+\frac{\delta}{5}\right)/c} \leq \ln^{-2} n.
\end{aligned}$$

The last inequality follows by (11).

Therefore finally for $d_i \leq \ln n$

$$\Pr\{v_i \sim v_j \cap \{D \leq Y_j \leq \ln n\} \mid Y_i = d_i\} \leq \frac{3p}{\ln n}.$$

Recall that $Y_1, Y_2, \ldots, Y_n$ are independent random variables and one may choose W_{v_i} by first choosing the value Y_i and given $Y_i = d_i$, choosing W_{v_i} uniformly at random from all d_i–element subsets of $\mathcal{W}$ (each independently of all other vertices in $\mathcal{V}$). Therefore, given $Y_i = d_i$, $D \leq d_i \leq \ln n$, degree of vertex v_i in $\mathcal{G}(n,m,p)[\mathcal{V}_{\geq D}]$ has the binomial distribution

$$\mathrm{Bin}\left(n-1, \Pr\{v_i \sim v_j \cap \{D \leq Y_j \leq \ln n\} \mid Y_i = d_i\}\right),$$

which is stochastically dominated by a random variable Z^* with the binomial distribution $\mathrm{Bin}(n, 3p/\ln n)$.

Let n be large enough to have $\delta \geq 24/\ln n$. If $C \geq 25/\delta$ and $D \leq d_i \leq \ln n$ then by (5) and the monotonicity of $\psi(x)/x$

$$\begin{aligned} \Pr\left\{Z^* \geq \frac{\delta}{4} np\right\} &\leq \exp\left(-\frac{3np}{\ln n} \cdot \frac{\delta \ln n}{12} \cdot \frac{\psi\left(\frac{\delta \ln n}{12}\right)}{\frac{\delta \ln n}{12}}\right) \\ &\leq \exp\left(-\frac{C\delta \ln n}{4} \cdot \frac{\psi(2)}{2}\right) \leq n^{-1.2}. \end{aligned}$$

Therefore

$$\Pr\left\{\Delta(\mathcal{G}(n,m,p)[\mathcal{V}_{\geq D}]) \geq \frac{\delta}{4} np\right\} \leq \sum_{1 \leq i \leq n} \Pr\left\{Z^* \geq \frac{\delta}{4} np\right\} \leq n^{-0.2},$$

which implies (13). ∎

References

[1] M. Behrisch, *Component evolution in random intersection graphs*, Electron. J. Combin. **14** (2007) R17.

[2] M. Behrisch, A. Taraz and M. Ueckerdt, *Coloring random intersection graphs and complex networks*, SIAM J. Discrete Math. **23** (2009) 288–299. doi:10.1137/050647153

[3] M. Bloznelis, *Component evolution in general random intersection graphs*, SIAM J. Discrete Math. **24** (2010) 639–654. doi:10.1137/080713756

[4] B. Bollobás, *The chromatic number of random graphs*, Combinatorica **8** (1988) 49–55. doi:10.1007/BF02122551

[5] B. Bollobas and P. Erdős, *Cliques in random graphs*, Math. Proc. Cambridge Philos. Soc. **80** (1976) 419–427. doi:10.1017/S0305004100053056

[6] J.A. Fill, E.R. Scheinerman and K.B. Singer-Cohen, *Random intersection graphs when $m = \omega(n)$: An equivalence theorem relating the evolution of the $G(n,m,p)$ and $G(n,p)$ models*, Random Structures Algorithms **16** (2000) 156–176. doi:10.1002/(SICI)1098-2418(200003)16:2⟨156::AID-RSA3⟩3.0.CO;2-H

[7] A. Frieze and C. McDiarmid, *Algorithmic theory of random graphs*, Random Structures Algorithms **10** (1997) 5–42. doi:10.1002/(SICI)1098-2418(199701/03)10:1/2⟨5::AID-RSA2⟩3.0.CO;2-Z

[8] G.R. Grimmett and C.J.H. McDiarmid, *On colouring random graphs*, Math. Proc. Cambridge Philos. Soc. **77** (1975) 313–324. doi:10.1017/S0305004100051124

[9] S. Janson, T. Łuczak and A. Ruciński, Random Graphs (Wiley, 2001).

[10] M. Karoński, E.R. Scheinerman and K.B. Singer-Cohen, *On random intersection graphs: The subgraph problem*, Combin. Probab. Comput. **8** (1999) 131–159. doi:10.1017/S0963548398003459

[11] V. Kurauskas and K. Rybarczyk, *On the chromatic index of random uniform hypergraphs*, SIAM J. Discrete Math. **29** (2015) 541–558. doi:10.1137/130942292

[12] T. Łuczak, *The chromatic number of random graphs*, Combinatorica **11** (1991) 45–54. doi:10.1007/BF01375472

[13] S. Nikoletseas, C. Raptopoulos and P.G. Spirakis, *Colouring non-sparse random intersection graphs*, in: Mathematical Foundations of Computer Science 2009, Lecture Notes in Comput. Sci. **5734**, R. Královič, D. Niwiński (Ed(s)), (Springer Berlin Heidelberg, 2009) 600–611. doi:10.1007/978-3-642-03816-7_51

[14] K. Rybarczyk, *Equivalence of the random intersection graph and $G(n, p)$*, Random Structures Algorithms **38** (2011) 205–234. doi:10.1002/rsa.20356

[15] K. Rybarczyk, *Constructions of independent sets in random intersection graphs*, Theoret. Comput. Sci. **524** (2014) 103–125. doi:10.1016/j.tcs.2014.01.006

[16] E. Shamir and E. Upfal, *Sequential and distributed graph coloring algorithms with performance analysis in random graph spaces*, J. Algorithms **5** (1984) 488–501. doi:10.1016/0196-6774(84)90003-8

Received 16 November 2015
Revised 27 September 2016
Accepted 27 September 2016