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SATURATION SPECTRUM OF PATHS AND STARS

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Abstract

A graph G is H-saturated if H is not a subgraph of G but the addition of any edge from $\overline{G}$ to G results in a copy of H. The minimum size of an H-saturated graph on n vertices is denoted $\mathrm{sat}(n, H)$, while the maximum size is the well studied extremal number, $\mathrm{ex}(n, H)$. The saturation spectrum for a graph H is the set of sizes of H saturated graphs between $\mathrm{sat}(n, H)$ and $\mathrm{ex}(n, H)$. In this paper we completely determine the saturation spectrum of stars and we show the saturation spectrum of paths is continuous from $\mathrm{sat}(n, P_k)$ to within a constant of $\mathrm{ex}(n, P_k)$ when n is sufficiently large.

Keywords: saturation spectrum, stars, paths.

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1. Introduction

Given a graph G let the vertex set and edge set of G be denoted by $V(G)$ and $E(G)$ respectively. Let $|G| = |V(G)|$, $e(G) = |E(G)|$ and $\overline{G}$ denote the complement of G. A graph G is called *H-saturated* if H is not a subgraph of G but for every $e \in E(\overline{G})$, H is a subgraph of $G+e$. Let $\mathrm{SAT}(n,H)$ denote the set of H-saturated graphs of order n. The *saturation number* of a graph H, denoted $\mathrm{sat}(n,H)$, is the minimum number of edges in an H-saturated graph on n vertices and $\underline{\mathrm{SAT}}(n,H)$ is the set of H-saturated graphs of order n with size $\mathrm{sat}(n,H)$. The *extremal number* of a graph H, denoted $\mathrm{ex}(n,H)$ (also called the *Turán number*) is the maximum number of edges in an H-saturated graph on n vertices and $\overline{\mathrm{SAT}}(n,H)$ is the set of H-saturated graphs of order n with size $\mathrm{ex}(n,H)$.

The *saturation spectrum* of a graph H, denoted $\mathrm{spec}(n,H)$, is the set of sizes of H-saturated graphs of order n, $\mathrm{spec}(n,H) = \{e(G) : G \in \mathrm{SAT}(n,H)\}$.

In this paper we investigate the saturation spectrum for P_k- and $K_{1,t}$-saturation, where P_k is a path on k vertices. In particular, in Section 3 we show that the saturation spectrum of $K_{1,t}$ contains all values from $\mathrm{sat}(n,K_{1,t})$ to $\mathrm{ex}(n,K_{1,t})$ for fixed n such that $n \geq t+1$. Finally, in Section 4 we show when n is sufficiently large, the saturation spectrum of P_k contains all values from $\mathrm{sat}(n,P_k)$ to $\mathrm{ex}(n,P_k) - c(k)$ for some constant $c(k)$.

2. Known Results

The saturation spectrum of K_3 was studied in [3]. Later the saturation spectrum of K_4 was studied in [1]. Shortly after, the saturation spectrum for larger complete graphs was studied in [2]. In this section we will describe the known results relating to the saturation spectrum of stars and paths.

Theorem 1 [7]**.** *Saturation numbers for paths and stars.*

(a) $\mathrm{sat}(n,K_{1,t}) = \begin{cases} \binom{t}{2} + \binom{n-t}{2} & \text{if } t+1 \leq n \leq t+\frac{t}{2}, \\ \left\lceil \frac{t-1}{2}n - \frac{t^2}{8} \right\rceil & \text{if } t+\frac{t}{2} \leq n. \end{cases}$

(b) *For* $n \geq 3, \mathrm{sat}(n,P_3) = \left\lfloor \frac{n}{2} \right\rfloor$.

(c) *For* $n \geq 4, \mathrm{sat}(n,P_4) = \begin{cases} \frac{n}{2} & n \text{ even}, \\ \frac{n+3}{2} & n \text{ odd}. \end{cases}$

(d) *For* $n \geq 5, \mathrm{sat}(n,P_5) = \left\lceil \frac{5n-4}{6} \right\rceil$.

In order to prove the main theorems in Sections 3 and 4 it is helpful to understand the structure of graphs in $\underline{\mathrm{SAT}}(n,K_{1,t})$ and $\underline{\mathrm{SAT}}(n,P_k)$. In 1986, Kászonyi and Tuza characterized the $K_{1,t}$-saturated graphs of minimum size. The characterization depends on the order of the host graph and is not in general unique.

Theorem 2 [7]. $\underline{\mathrm{SAT}}(n, K_{1,t}) = \begin{cases} K_t \cup K_{n-t} & \textit{if } t+1 \leq n \leq \frac{3t}{2}, \\ G' \cup K_p & \textit{if } \frac{3t}{2} \leq n, \end{cases}$
where $p = \left\lfloor \frac{t+1}{2} \right\rfloor$ *and* G' *is a* $(t-1)$*-regular graph on* $n-p$ *vertices. Note that in the case when* $n \geq \frac{3t}{2}$*, there is a single edge connecting* G' *and* K_p *if* $t-1$ *and* $n-p$ *are both odd.*

Kászonyi and Tuza also described graphs in $\underline{\mathrm{SAT}}(n, P_k)$. In particular they give a tree that is a subgraph of all P_k-saturated trees. We begin by describing this tree. A *perfect* 3*-ary tree* is a tree such that every vertex has degree 3 or degree 1 and all degree 1 vertices are the same distance from the center. We let T_{k-1} denote the perfect 3-ary tree with longest path on exactly $k-1$ vertices (see Figure 1).

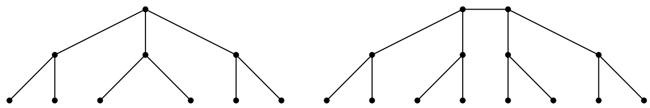

Figure 1. T_5 and T_6.

Theorem 3 [7]. *Let* P_k *be a path on* $k \geq 3$ *vertices and let* T_{k-1} *be the perfect* 3*-ary tree defined above. Further let*

$$a_k = \begin{cases} 3 \cdot 2^{m-1} - 2 & \textit{if } k = 2m, \\ 4 \cdot 2^{m-1} - 2 & \textit{if } k = 2m+1. \end{cases}$$

Then, for $n \geq a_k$*,* $\underline{\mathrm{SAT}}(n, P_k)$ *consists of a forest with* $\lfloor n/a_k \rfloor$ *components. Furthermore, if* T *is a* P_k*-saturated tree, then* $T_{k-1} \subseteq T$*.*

It is also helpful to understand the structure of graphs in $\overline{\mathrm{SAT}}(n, K_{1,t})$ and $\overline{\mathrm{SAT}}(n, P_k)$. It is well known that $\mathrm{ex}(n, K_{1,t}) = \left\lfloor \frac{n(t-1)}{2} \right\rfloor$ and that $\overline{\mathrm{SAT}}(n, K_{1,t})$ consists of $(t-1)$-regular graphs unless n and $t-1$ are both odd, in which case there is a single vertex of degree $t-2$.

The structure of graphs in $\overline{\mathrm{SAT}}(n, P_k)$ was studied by Erdős and Gallai in 1959.

Theorem 4 [5]. *Let* G *be a graph of order* n *which contains no path with more than* $k-1$ *vertices. Then* $|E(G)| \leq \frac{(k-2)n}{2}$ *and equality holds if and only if each component of* G *is a complete graph of order* $k-1$*.*

In [6], the saturation spectrum of small paths was studied. In particular, $\mathrm{spec}(n, P_5)$ and $\mathrm{spec}(n, P_6)$ were determined.

Theorem 5 [6]. *Let* $n \geq 5$ *and* $\mathrm{sat}(n, P_5) \leq m \leq \mathrm{ex}(n, P_5)$ *be integers,* $m \in \mathrm{spec}(n, P_5)$ *if and only if* $n = 1, 2 \pmod 4$*, or*

$$m \notin \begin{cases} \{\frac{3n-5}{2}\} & \text{if } n \equiv 3 \pmod 4, \\ \{\frac{3n}{2} - 3, \frac{3n}{2} - 2, \frac{3n}{2} - 1\} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Theorem 6 [6]. *Let* $n \geq 10$ *and* $\mathrm{sat}(n, P_6) \leq m \leq \mathrm{ex}(n, P_6)$ *be integers,* $m \in \mathrm{spec}(n, P_6)$ *if and only if* $(n, m) \notin \{(10, 10), (11, 11), (12, 12), (13, 13), (14, 14), (11, 14)\}$ *and*

$$m \notin \begin{cases} \{2n-4, 2n-3, 2n-1\} & \text{if } n \equiv 0 \pmod 5, \\ \{2n-4\} & \text{if } n \equiv 2 \pmod 5, \\ \{2n-4\} & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

This is the starting point for this paper. Following the same lines of investigation we completely determine the edge spectrum for saturation of stars and we study the edge spectrum for saturation of long paths when n is sufficiently large.

3. Stars

In this section we will show that the saturation spectrum of $K_{1,t}$ contains all values from the saturation number to the extremal number. The following theorem is the main result of this section.

Theorem 7. *Let* $S = K_{1,t}$ *for* $t \geq 3$*. If* $n \geq t + 1$*, then* $\mathrm{spec}(n, S)$ *is continuous from* $\mathrm{sat}(n, S)$ *to* $\mathrm{ex}(n, S)$*.*

Before proving Theorem 7 we give two lemmas that describe edge exchanges that can be used to transform a $K_{1,t}$-saturated graph G into a $K_{1,t}$-saturated graph with one more edge. We will refer to the exchange in Lemma 8 as a type I exchange and the exchange in Lemma 9 as a type II exchange.

Lemma 8. *In a* $K_{1,t}$*-saturated graph* G*, if there is vertex* v *of degree at most* $t - 3$ *that is nonadjacent to* u *or* w *where* $uw \in E(G)$ *and* $d(u) = d(w) = t - 1$*, then* $G' = G - uw + \{vw, vu\}$ *is* $K_{1,t}$*-saturated with* $e(G') = e(G) + 1$*.*

Proof. First note that the degrees of $d_G(u) = d_{G'}(u)$, $d_G(w) = d_{G'}(w)$ and $d_G(v) + 2 = d_{G'}(v)$. Since $d_G(v) \leq t - 3$, it is easy to see that no vertex of degree t is created and hence $K_{1,t}$ is not a subgraph of G'. Now consider $e \in E(\overline{G'})$. If

e is incident to u or w then $G' + e$ contains $K_{1,t}$ since u and w are both of degree $t-1$. If e is incident to v then $G' + e$ contains $K_{1,t}$ otherwise G would not be $K_{1,t}$-saturated. Similarly, if e is is not incident to u, v or w then $G' + e$ contains $K_{1,t}$; otherwise G would not be $K_{1,t}$-saturated. ■

Lemma 9. *In a $K_{1,t}$-saturated graph G, if there are vertices v_1 and v_2 of degree at most $t-2$ and an edge uw such that u and w are of degree $t-1$ where $v_1w, v_2u \notin E(G)$, then $G' = G - uw + \{v_1w, v_2u\}$ is $K_{1,t}$-saturated with $e(G') = e(G) + 1$.*

Proof. First note that the degrees of $d_G(u) = d_{G'}(u)$, $d_G(w) = d_{G'}(w)$, $d_G(v_1) + 1 = d_{G'}(v_1)$, and $d_G(v_2) + 1 = d_{G'}(v_2)$. Since $d_G(v_1) \leq t-2$ and $d_G(v_2) \leq t-2$, no vertex of degree t is created centered at v_1, v_2, u or w. Hence $K_{1,t}$ is not a subgraph of G'. Now consider $e \in E(\overline{G'})$. If e is incident to u or w then $G' + e$ contains $K_{1,t}$ since u and w are both of degree $t-1$. If e is incident to v_1 or v_2 then $G' + e$ contains $K_{1,t}$ otherwise G would not be $K_{1,t}$-saturated. Similarly, if e is is not incident to v_1, v_2, u or w then $G' + e$ contains $K_{1,t}$; otherwise G would not be $K_{1,t}$-saturated. ■

The proof for Theorem 7 is split into cases according to the number of vertices in the host graph G relative to t. To ease reading, cases are listed as lemmas.

Lemma 10. *Let $n = t + 1$. For each $t \geq 3$ and m such that* $\text{sat}(n, K_{1,t}) \leq m \leq \text{ex}(n, K_{1,t})$ *there exists a $K_{1,t}$-saturated graph G with $e(G) = m$.*

Proof. We construct a sequence of $K_{1,t}$-saturated graphs, $G_1, \ldots, G_s$ where $e(G_i) + 1 = e(G_{i+1})$, and this sequence contains a graph of each size from $\text{sat}(n, K_{1,t})$ to $\text{ex}(n, K_{1,t})$. Let $G_1 = K_t \cup \{v\}$, by Theorem 2 we see that $G_1 \in \underline{\text{SAT}}(n, K_{1,t})$. In order to construct the sequence of graphs we will need a large matching from K_t so that we may use type I exchanges. Let M be a maximum matching of K_t; clearly M contains $\lfloor t/2 \rfloor$ edges. Now to create G_{i+1} from G_i we use an edge of M and v to perform a type I exchange. Lemma 8 implies that G_{i+1} is a $K_{1,t}$-saturated graph with $e(G_{i+1}) = e(G_i) + 1$. We note that we can perform $\lfloor t/2 \rfloor$ type I exchanges when t is odd so that $G_s = G_{\lfloor t/2 \rfloor}$ is a $(t-1)$-regular graph and when t is even we can perform $t/2 - 1$ type I exchanges so that $d_{G_s}(v) = t-2$ and all other vertices in G_s are degree $t-1$. Notice that in either case, G_s is the extremal graph. ■

Lemma 11. *For each $t \geq 3$, $t + 2 \leq n \leq \frac{3t}{2}$ and m such that* $\text{sat}(n, K_{1,t}) \leq m \leq \text{ex}(n, K_{1,t})$ *there exists a $K_{1,t}$-saturated graph of size m.*

Proof. To show this, we will construct a sequence of $K_{1,t}$-saturated graphs, $G_1, \ldots, G_s$, that contains a graph of each size from $\text{sat}(n, K_{1,t})$ to $\text{ex}(n, K_{1,t})$. Let $G_1 = K_t \cup K_{n-t}$. By Theorem 2 we see that $G_1 \in \underline{\text{SAT}}(n, K_{1,t})$. In order to construct the sequence of graphs we use large disjoint matchings from K_t so that we may use type I and type II exchanges. It is well known (cf. [4]) that K_t contains $t-1$ matchings, $M_1,\dots,M_{t-1}$, each of size $\lfloor \frac{t}{2} \rfloor$. Since $n \le 3t/2$ implies $n-t \le t/2$, each one of the the $t-1$ matchings can be associated with a vertex of K_{n-t}. For convenience, let $V(K_{n-t}) = \{v_1,\dots,v_{n-t}\}$ and say that v_i is associated with M_i for $1 \le i \le n-t$.

Starting with G_1, iteratively change the degree of each vertex in K_{n-t} from $n-t-1$ to $t-1$. In order to do this each vertex in $V(K_{n-t})$ needs $2t-n$ more incident edges. Proceed based on the parity of $2t-n$. If $2t-n$ is odd, pair the vertices in K_{n-t} so that v_i is paired with v_{i+1} for each odd $i < n-t$. Note that when $n-t$ is odd, v_{n-t} is unpaired. Associate each of the pairs with an edge from M_{n-t+1}. Then, iteratively use each pair and associated edge to preform a type II exchange to create $G_2,\dots,G_{\lfloor \frac{n-t}{2}+1 \rfloor}$.

Notice that in $G_{\lfloor \frac{n-t}{2}+1 \rfloor}$ it is possible that v_i is adjacent to some vertex in M_i. Thus there are at least $\lfloor t/2 \rfloor - 1$ edges in M_i that are not incident to v_i. Create the remaining graphs in the sequence by preforming $(2t-n-1)/2$ type I exchanges with each v_i and M_i. In order to preform $(2t-n-1)/2$ type I exchanges, it must be verified that $(2t-n-1)/2 \le \lfloor t/2 \rfloor - 1$, otherwise M_i has too few edges to preform the type I exchanges with v_i. Since $n \ge t+2$, it follows that:

$$\begin{aligned} n &\ge t+2 \\ t-3 &\ge 2t-n-1 \\ \frac{t-1}{2}-1 &\ge \frac{2t-n-1}{2} \\ \left\lfloor \frac{t}{2} \right\rfloor - 1 &\ge \frac{2t-n-1}{2}. \end{aligned}$$

Lemmas 8 and 9 imply that after completing the $(2t-n-1)/2$ type I exchanges and a type II with each v_i we have $d(v_i) = t-1$ for $1 \le i \le n-t-1$. Further, if $n-t$ is odd then $d(v_{n-t}) = t-2$ and if $n-t$ is even then $d(v_{n-t}) = t-1$. In either case, it follows that G_s is the extremal graph.

Now consider the case when $2t-n$ is even. In this case, only type I exchanges will be used. Construct $G_2,\dots,G_s$ by preforming $(2t-n)/2$ type I exchanges using each v_i and associated M_i. It remains to verify that $(2t-n)/2 \le \lfloor t/2 \rfloor$ so that $(2t-n)/2$ type I exchanges can be completed. Again, since $n \ge t+2$, it follows that:

$$\begin{aligned} n &\ge t+2 \\ t-2 &\ge 2t-n \\ \frac{t-2}{2} &\ge \frac{2t-n}{2} \\ \left\lfloor \frac{t}{2} \right\rfloor &\ge \frac{2t-n}{2}. \end{aligned}$$

Finally Lemma 8 implies that after completing the $(2t-n-1)/2$ type I exchanges to each v_i that $d(v_i) = t-1$. So, it follows that G_s is the extremal graph. ■

Lemma 12. *For each* $t \geq 3$, $n > \frac{3t}{2}$ *and* m *such that* $\mathrm{sat}(n, K_{1,t}) \leq m \leq \mathrm{ex}(n, K_{1,t})$ *there exists a* $K_{1,t}$*-saturated graph of size* m.

Proof. Proceed in a fashion similar to the proof of Lemma 11. Construct a sequence of $K_{1,t}$-saturated graphs, $G_1, \ldots, G_s$, that contains a graph of each size from $\mathrm{sat}(n, K_{1,t})$ to $\mathrm{ex}(n, K_{1,t})$. Begin by constructing a $(t-1)$-regular (or nearly regular depending on the parity of n and t) graph, G', on r vertices where $r = n - \left\lfloor \frac{t+1}{2} \right\rfloor$ such that G' has a sufficient number of large matchings for the algorithm. A well known result (cf. [4]) shows that a complete graph K_r decomposes into $r-1$ matchings of size $r/2$ when r is even or $\frac{r-1}{2}$ hamilton cycles when r is odd will be used.

First suppose that r is even. To form G', begin with a matching decomposition of $K_r = M_1 \cup \cdots \cup M_{r-1}$. Let $G' = M_1 \cup \cdots \cup M_{t-1}$. Clearly G' is $(t-1)$-regular and contains $t-1$ disjoint matchings, $M_1, \ldots, M_{t-1}$, of size $r/2$.

When r is odd begin with a hamiltonian cycle decomposition of $K_r = C_1 \cup \cdots \cup C_{(r-1)/2}$. If $t-1$ is even then let $G' = C_1 \cup \cdots \cup C_{(t-1)/2}$. If $t-1$ is odd then let $G' = C_1 \cup \cdots \cup C_{(t-2)/2} \cup M$ where M is a maximum matching of $C_{t/2}$; in this case there is a single vertex of degree $t-2$ all other vertices are of degree $t-1$. Further since each hamiltonian cycle of K_r contains two disjoint matchings of size $(r-1)/2$, G' contains $t-1$ disjoint matchings, $M_1, \ldots, M_{t-1}$, of size at least $(r-1)/2$.

Let $G_1 = G' \cup K_{\lfloor \frac{t+1}{2} \rfloor}$ and label the vertices in $V(G') = \{u_1, \ldots, u_{n-\lfloor \frac{t+1}{2} \rfloor}\}$ and $V(K_{\lfloor \frac{t+1}{2} \rfloor}) = \{v_1, \ldots, v_{\lfloor \frac{t+1}{2} \rfloor}\}$. If r and $t-1$ are both odd then a single edge from the vertex of degree $t-2$ in G' is added to a vertex in $K_{\lfloor \frac{t+1}{2} \rfloor}$, without loss of generality let this edge be $u_1 v_{\lfloor \frac{t+1}{2} \rfloor}$. Theorem 2 implies that G_1 is a minimally $K_{1,t}$-saturated graph. Associate each vertex v_i with a matching M_i in G'.

Starting with G_1, iteratively change the degree of each vertex in $K_{\lfloor \frac{t+1}{2} \rfloor}$ from $\left\lfloor \frac{t+1}{2} \right\rfloor - 1$ to $t-1$. Each vertex, v_i, needs $\left\lfloor \frac{t}{2} \right\rfloor$ more incident edges. Notice that when r and $t-1$ are both odd that only $\left\lfloor \frac{t}{2} \right\rfloor - 1$ incident edges need to be added to $v_{\lfloor \frac{t+1}{2} \rfloor}$. Proceed based on the parity of $\left\lfloor \frac{t}{2} \right\rfloor$. If $\left\lfloor \frac{t}{2} \right\rfloor$ is odd, then pair the vertices in $K_{\lfloor \frac{t+1}{2} \rfloor}$ so that v_i is paired with v_{i+1} for each odd $i < \left\lfloor \frac{t+1}{2} \right\rfloor$. Note that if $\left\lfloor \frac{t+1}{2} \right\rfloor$ is odd then $v_{\lfloor \frac{t+1}{2} \rfloor}$ is unpaired. Associate each of the pairs with an edge from $M_{\lfloor \frac{t+1}{2} \rfloor + 1}$. Then, iteratively use each pair and associated edge to preform a type II exchange to create $G_2, \ldots, G_{\frac{\lfloor \frac{t}{2} \rfloor + 1}{2}}$.

Notice that in $G_{\frac{\lfloor \frac{t}{2} \rfloor + 1}{2}}$ it is possible that v_i is adjacent to some vertex in M_i. Thus there are at least $\lfloor r/2 \rfloor - 1$ in M_i that are not incident to v_i. Create the remaining graphs in the sequence by preforming $(1/2)(\lfloor t/2 \rfloor - 1)$ type I exchanges with each v_i and M_i. In order to preform $(1/2)(\lfloor t/2 \rfloor - 1)$ type I exchanges it must be verified that $(1/2)(\lfloor t/2 \rfloor - 1) \leq \lfloor r/2 \rfloor - 1$. Since $n > \frac{3t}{2}$, it follows that

$$r = n + \left\lfloor \frac{t+1}{2} \right\rfloor > \frac{3t}{2} - \left\lfloor \frac{t+1}{2} \right\rfloor \geq t - 1.$$

As r and t are both integers, it follows that $r \geq t$ and hence $(1/2)(\lfloor t/2 \rfloor - 1) \leq \lfloor r/2 \rfloor - 1$. Lemmas 8 and 9 imply that after completing the $(1/2)(\lfloor t/2 \rfloor - 1)$ type I exchanges and a type II with each v_i that $d(v_i) = t - 1$ for $1 \leq i \leq \left\lfloor \frac{t+1}{2} \right\rfloor - 1$. Further, if $\left\lfloor \frac{t+1}{2} \right\rfloor$ and $t-1$ are odd and r is even then $d(v_{\lfloor \frac{t+1}{2} \rfloor}) = t - 2$ otherwise $d(v_{\lfloor \frac{t+1}{2} \rfloor}) = t - 1$. In either case it follows that G_s is the extremal graph.

Now, consider the case when $\left\lfloor \frac{t}{2} \right\rfloor$ is even. In this case only type I exchanges will be used. Create $G_2, \ldots, G_s$ by preforming $(1/2) \lfloor t/2 \rfloor$ type I exchanges using each v_i and associated M_i. Since $r \geq t$, it follows that $(1/2)\lfloor t/2 \rfloor \leq \lfloor r/2 \rfloor$ so that $(1/2) \lfloor t/2 \rfloor$ type I exchanges may be done with each vertex v_i.

Finally, Lemma 8 implies that after completing the $(1/2) \lfloor t/2 \rfloor$ type I exchanges to each v_i that $d(v_i) = t - 1$ for $1 \leq i \leq \left\lfloor \frac{t+1}{2} \right\rfloor - 1$. If r and $t-1$ are odd then then $d(v_{\lfloor \frac{t+1}{2} \rfloor}) = t - 2$ otherwise $d(v_{\lfloor \frac{t+1}{2} \rfloor}) = t - 1$. Again, in either case it follows that G_s is the extremal graph. ∎

Theorem 7 follows directly from Lemmas 10, 11 and 12.

4. Paths

In this section we show that when n is sufficiently large, $\text{spec}(n, P_k)$ contains all values from $\text{sat}(n, P_k)$ to $\text{ex}(n, P_k) - c$ where c is a constant that depends on k. Recall from Theorem 3 that

$$a_k = \begin{cases} 3 \cdot 2^{m-1} - 2 & \text{if } k = 2m, \\ 4 \cdot 2^{m-1} - 2 & \text{if } k = 2m + 1. \end{cases}$$

The following is the main result of the section, the proof is given towards the end of this section.

Theorem 13. *Let $P = P_k$. If $n = r(k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta$, where $0 \leq \beta < k-1$, then $\text{spec}(n, P)$ is continuous from $\text{sat}(n, P)$ to $\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1$.*

As in the previous section we provide two lemmas that transform a P_k-saturated graph G into a P_k-saturated graph with one more edge. Let $a_k = |T_{k-1}|$. An immediate consequence of the proof of Theorem 3 in [7] is that there exists P_k-saturated trees of every order p such that $p \geq a_k$. Let v be a vertex with pendant neighbors in T_{k-1}. The graph obtained by adding additional pendant vertices to v in T_{k-1} so that the order of the new graph is p will be denoted T_{k-1}^{p} (see Figure 2). Clearly, $T_{k-1}^{p} \in \mathrm{SAT}(n, P_k)$. Let T_{k-1}^{*} denote a P_k-saturated tree of arbitrary order.

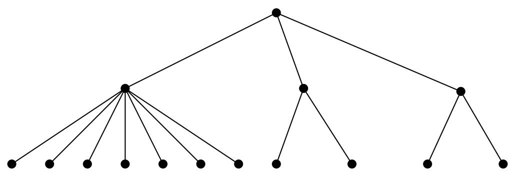

Figure 2. T_5^{15}.

Lemma 14. *Let G be a P_k-saturated graph that contains two components $T_{k-1}^{p_1}$ and $T_{k-1}^{p_2}$. If $H = G - \{T_{k-1}^{p_1}, T_{k-1}^{p_2}\}$ then $G' = H \cup T_{k-1}^{p_1+p_2}$ is a P_k-saturated graph where $e(G') = e(G) + 1$.*

Proof. Let $p = p_1 + p_2$. Since T_{k-1}^{p} and G are P_k-saturated it follows that G' does not contain P_k. Let $e \in E(\overline{G'})$. In order to show that $G' + e$ contains P_k we will consider several cases. First, if $e \in E(\overline{T_{k-1}^{p}})$, then $T_{k-1}^{p} + e$ contains P_k since T_{k-1}^{p} is P_k-saturated, hence $G' + e$ contains P_k. Now since G is P_k-saturated, if $e \in E(H)$ then $G' + e$ contains P_k. Now suppose that e has at least one endpoint in $V(H)$ and one in $V(T_{k-1}^{p})$. Notice that $H \cup T_{k-1}^{p_1}$ is an induced subgraph of G'. If $G' + e$ does not contain P_k then by deleting pendant vertices not incident to e it can be seen that $H \cup T_{k-1}^{p_1} + e$ does not contain P_k, since deleting vertices can not create a copy of P_k. This implies that G is not P_k-saturated, a contradiction. Therefore G' is P_k-saturated. Finally, note that $e(G) = e(H) + (p_1 - 1) + (p_2 - 1)$ and $e(G') = e(H) + (p_1 + p_2 - 1)$, thus $e(G') = e(G) + 1$. ■

Lemma 15. *Let $k \geq 5$ and G be a P_k-saturated graph. Let p be an integer such that $p \geq (k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right]$. Let T_{k-1}^{p} be a component of G and $F = \left[\binom{k-1}{2} - (k-1)\right]T_{k-1}^{*}$ such that $|F| = p - k + 1$. If $H = G - T_{k-1}^{p}$ then $G' = H \cup K_{k-1} \cup F$ is a P_k-saturated graph where $e(G') = e(G) + 1$.*

Proof. Notice F is a forest of P_k-saturated trees. By Theorem 3 each component of F must have order at least a_k. Since $p \geq (k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right]$, it follows that $|F| \geq a_k\left[\binom{k-1}{2} - (k-1)\right]$. Hence, $|F|$ is large enough for each component to be a P_k-saturated tree.

Note that $e(G) = e(H) + p - 1$ and $e(G') = e(H) + \binom{k-1}{2} + e(F)$. Since F is a forest on $p - k + 1$ vertices and $\left[\binom{k-1}{2} - (k-1)\right]$ components it follows that $e(F) = p - k + 1 - \left[\binom{k-1}{2} - (k-1)\right]$. Thus $e(G') = e(H) + p = e(G) + 1$.

It now remains to show that G' is P_k-saturated. Let $e \in E(\overline{G'})$. First suppose that $e \in E(\overline{F})$, it follows that $G' + e$ contains P_k since F is P_k-saturated by Theorem 3. Now suppose that has both endpoints in $V(H)$. Clearly since G is P_k-saturated $G + e$ contains a copy of P_k such that $V(P_k) \subseteq V(H)$. Thus $G' + e$ contains a copy of P_k. Finally suppose that e has one endpoint in H and one in F. Assume that $G' + e$ does not contain P_k. Let T be the component of F incident to e. Let $\hat{G} = G'[H \cup T]$. Notice $\hat{G} + e$ does not contain P_k. Further since $G = H \cup T^p_{k-1}$ and $\hat{G} = H \cup T$ differ only in the number of pendants adjacent to the vertex of highest degree in T and T^p_{k-1}, it is easy to see that $G + e$ does not contain P_k. Thus G' is P_k-saturated. ∎

The transformation in Lemma 14 will be referred to as a tree exchange and the transformation in Lemma 15 will be referred to as clique exchange. We are now ready to prove our main result of this section.

***Proof of Theorem* 13.** Beginning with a minimally P_k-saturated graph, we will build a sequence of P_k-saturated graphs, $G_1, \ldots, G_f$, of size $\mathrm{sat}(n, P)$ to $\binom{k-1}{2} r + a_k \left[\binom{k-1}{2} - (k-1)\right] + \beta - 1$ where $e(G_{i+1}) = e(G_i) + 1$ for $1 \le i \le f - 1$. Let $G_1 = qT_k \cup T^*_k$ where $q = \left\lfloor \frac{n}{a_k} \right\rfloor - 1$. Theorem 3 implies that $G_1 \in \underline{\mathrm{SAT}}(n, P_k)$. Once G_i is built use one of the following exchanges to build G_{i+1}.

1. If G_i contains two components $T^{p_1}_{k-1}$ and $T^{p_2}_{k-1}$, then use a tree exchange to create G_{i+1}.
2. If G_i contains exactly one tree component and the tree has at least $a_k \left[\binom{k-1}{2} - (k-1)\right] + (k-1)$ vertices, then use a clique exchange to obtain G_{i+1}.

Lemmas 14 and 15 imply that when either a tree exchange or a clique exchange is used, G_{i+1} is a P_k-saturated graph with $e(G_{i+1}) = e(G_i) + 1$. As long as there are at least two tree components or there is a single tree component T in G_i such that $|T| \ge a_k \left[\binom{k-1}{2} - (k-1)\right] + (k-1)$, one of the exchanges can be used to build G_{i+1}. So the final graph built by the algorithm will have one tree component of order less than $a_k \left[\binom{k-1}{2} - (k-1)\right] + (k-1)$.

Since $n = r(k-1) + a_k \left[\binom{k-1}{2} - (k-1)\right] + \beta$, it follows that upon constructing $G_i = (r-1)K_{k-1} \cup T^*_{k-1}$ that $|T^*_{k-1}| = a_k \left[\binom{k-1}{2} - (k-1)\right] + (k-1) + \beta$. Thus another clique exchange can be used followed by tree exchanges to produce $rK_{k-1} \cup T^q_{k-1}$. At this point it is easy to calculate $q = a_k \left[\binom{k-1}{2} - (k-1)\right] + \beta < a_k \left[\binom{k-1}{2} - (k-1)\right] + (k-1)$. So the algorithm will terminate with $G_f = rK_{k-1} \cup T^q_{k-1}$. Thus

$$\begin{aligned} e(G_f) &= \binom{k-1}{2} r + [n - r(k-1) - 1] \\ &= \binom{k-1}{2} r + a_k \left[\binom{k-1}{2} - (k-1)\right] + \beta - 1. \end{aligned}$$

∎

Note that the algorithm in Theorem 13 could be altered to include exchanges with P_k-saturated graphs other than T^p_{k-1} and K_{k-1}. However, the following theorem will show when n is large that the algorithm gives P_k-saturated graphs to within a constant of the extremal number.

Theorem 16. *For n sufficiently large and $k \geq 5$, $\mathrm{spec}(n, P_k)$ contains all values from $\mathrm{sat}(n, P_k)$ to $\mathrm{ex}(n, P_k) - c$ where $c = c(k)$.*

Proof. Let n be expressed as $n = r(k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta$, where β is a constant such that $0 \leq \beta < k-1$. The algorithm in the proof of Theorem 13 gives a sequence of P_k-saturated graphs that contains graphs of each size from $\mathrm{sat}(n, P_k)$ to $\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1$. Let G be a P_k-saturated graph of size $\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1$. Theorem 4 implies that $\mathrm{ex}(n, P_k) \leq \frac{(k-2)n}{2}$. Now it is possible to estimate $\mathrm{ex}(n, P_k) - e(G)$ as follows:

$$\begin{aligned}
\mathrm{ex}(n, P_k) - e(G) &\leq \frac{(k-2)n}{2} - \left[\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1\right] \\
&= \frac{(k-2)\left(r(k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta\right)}{2} \\
&\quad - \left[\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1\right] \\
&= \binom{k-1}{2}r + \frac{(k-2)\left[a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta\right]}{2} \\
&\quad - \left[\binom{k-1}{2}r + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1\right] \\
&= \frac{(k-4)a_k\binom{k-1}{2} - a_k(k-1)(k-4) + (k-4)\beta}{2} + 1 \\
&\leq (k-4)\frac{a_k\binom{k-1}{2} - a_k(k-1) + (k-1)}{2} + 1.
\end{aligned}$$

Thus, for a fixed k the difference between $\mathrm{ex}(n, P_k)$ and $e(G)$ is a constant for all n sufficiently large. ∎

Although Theorems 13 and 16 show that $\mathrm{spec}(n, P_k)$ is continuous between $\mathrm{sat}(n, P_k)$ and $\mathrm{ex}(n, P_k) - c$, where c is a constant dependent on k — although exponential in k, additional techniques would need to be developed to better understand the exact form of the saturation spectrum.

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