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PRIME FACTORIZATION AND DOMINATION IN THE HIERARCHICAL PRODUCT OF GRAPHS

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Abstract

In 2009, Barrière, Dalfó, Fiol, and Mitjana introduced the generalized hierarchical product of graphs. This operation is a generalization of the Cartesian product of graphs. It is known that every connected graph has a unique prime factor decomposition with respect to the Cartesian product. We generalize this result to show that connected graphs indeed have a unique prime factor decomposition with respect to the generalized hierarchical product. We also give preliminary results on the domination number of generalized hierarchical products.

Keywords: generalized hierarchical product, Cartesian product, prime factor decomposition.

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1. Introduction

The most appealing aspect of the generalized hierarchical product of graphs is that it is a generalization of the well-studied Cartesian product of graphs. In particular, within a Cartesian product graph there exists an interesting set of subgraphs which possess many of the algebraic properties that the Cartesian product enjoys. In the literature for product graphs, Godsil and McKay [6] first defined a rooted product graph in 1978; a class of graphs which is a proper subset of the class of all generalized hierarchical product graphs. Rooted product graphs arise throughout the literature in various contexts, some of which can be seen in [1, 5, 10, 11]. Then in 2009, Barrière, Dalfó, Fiol, and Mitjana [3] introduced the generalized hierarchical product, which can be used to model scale-free networks such as the World Wide Web, some metabolic networks, and telephone graphs [9, 13]. The generalized hierarchical product is actually a generalization of the rooted product graph as well as a generalization of the Cartesian product.

Sabidussi [14] and Vizing [15] showed independently that not only does every graph have a prime factor decomposition with respect to the Cartesian product, but that this prime factorization is unique for connected graphs. An alternate proof of this property can also be found in [7]. As with any meaningful attempt at generalizing a set that exhibits some algebraic structure, we must investigate whether each of the important algebraic properties of the Cartesian product carries over to the generalized hierarchical product. Barrière, Dalfó, Fiol, and Mitjana [3] showed that indeed the generalized hierarchical product is associative, but it is not commutative. We now ask whether this new product retains the very important property from the Cartesian product: unique prime factorization. For if this product does not exhibit unique prime factorization, then the study of a large class of parameters within such a product would become meaningless as we prefer to relate the value of any invariant for a product graph to the values for the invariant in the product's underlying factor graphs.

The remainder of this paper is organized as follows. Section 1.1 is dedicated to notation and preliminary results. In Section 2, we show that although prime factor decomposition with respect to the generalized hierarchical product need not be unique in the case of disconnected graphs, it is in fact unique in the class of connected graphs. In Section 3, we give preliminary results on the domination number of specific generalized hierarchical products.

1.1. Preliminaries

We consider only finite, simple, and undirected graphs. Given a graph G, we let $V(G)$ represent the vertex set of G and $E(G)$ represent the edge set of G. The Cartesian product of graphs G and H, denoted $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$, whereby two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

Given graphs $G_1, \ldots, G_n$ and vertex subsets $U_i \subseteq V(G_i)$ for $1 \leq i \leq n-1$, the *generalized hierarchical product*, denoted $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, is the graph with vertex set $V(G_1) \times \cdots \times V(G_n)$ and adjacencies

$$(x_1,\ldots,x_{n-1},x_n) \sim \begin{cases} (y_1,x_2,\ldots,x_n) & \text{if } x_1y_1 \in E(G_1), \\ (x_1,y_2,x_3,\ldots,x_n) & \text{if } x_2y_2 \in E(G_2) \text{ and } x_1 \in U_1, \\ \quad\vdots & \qquad\vdots \\ (x_1,\ldots,x_{n-1},y_n) & \text{if } x_ny_n \in E(G_n) \text{ and } x_i \in U_i, \\ & \quad 1 \le i \le n-1. \end{cases}$$

In the above definition, the subset $U_i \subseteq V(G_i)$ is referred to as the *root set of* G_i for each $1 \le i \le n-1$, and if each $U_i = \emptyset$, then $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ is the disjoint union of $|V(G_1)| \times \cdots \times |V(G_n)|$ disjoint copies of G_1. We alert the reader's attention to the fact that when $|U_i| = 1$ for $1 \le i \le n-1$, $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ is a rooted product. Moreover, Barrière, Comellas, Dalfó, and Fiol [2] first defined the "hierarchical product" to mean the graph $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ when $|U_i| = 1$ for $1 \le i \le n-1$, i.e., the rooted product. In this paper, we will refer to a "generalized hierarchical product" as simply a "hierarchical product". That is, we say a graph $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ is a hierarchical product, regardless of the cardinality of each U_i.

Figure 1 depicts the hierarchical product $K_3(U_1) \sqcap K_3(U_2) \sqcap K_2$, where $V(K_n) = \{0,\ldots,n-1\}$ for $n \in \{2,3\}$ and $U_1 = U_2 = \{0,2\}$.

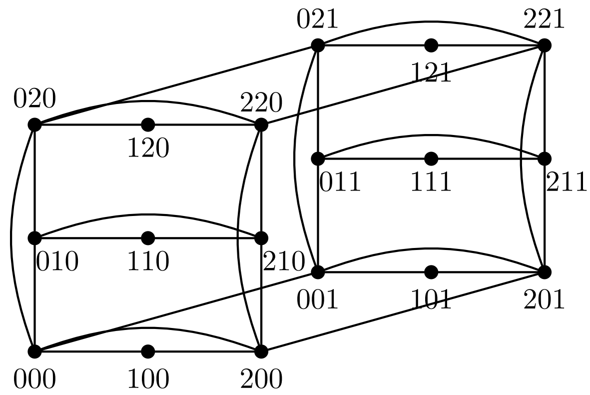


Figure 1. $K_3(U_1) \sqcap K_3(U_2) \sqcap K_2$, where $U_1 = U_2 = \{0,2\}$.

The role of the Cartesian product is quite important in the proof technique used in later sections. For this reason, we make a natural association between a given hierarchical product and a Cartesian product.

Definition 1. Given a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, the *Cartesian product associated with* G is the graph $G^\square = G_1 \square \cdots \square G_n$.

In general, given graphs $G_1, \ldots, G_n$ and graph product operation $*$, for any index $1 \leq i \leq n$ there is a projection map

$$p_i : G_1 * \cdots * G_{n-1} * G_n \to G_i$$

defined as $p_i((x_1, \ldots, x_n)) = x_i$. We call x_i the i^{th} *coordinate* of the vertex $(x_1, \ldots, x_n)$. When $*$ represents the hierarchical product operation, then we must also choose subsets $U_i \subseteq V(G_i)$ for $1 \leq i \leq n-1$ so that $p_i : G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n \to G_i$. We refer to G_i as a *factor* with respect to its graph product $G_1 * \cdots * G_n$.

By definition, when $G = G_1 * \cdots * G_n$, if $(x_1, \ldots, x_n)(y_1, \ldots, y_n)$ is an edge of $E(G)$, then $x_i = y_i$ or $x_i y_i \in E(G_i)$ for each $1 \leq i \leq n$. For this reason, each projection p_i is a weak homomorphism. Fix $j \in \{1, \ldots, n\}$ and let $a = (a_1, \ldots, a_n) \in V(G)$. The G_j-*layer* through a is the induced subgraph

$$G_j^a = \langle \{x \in V(G) \mid \ p_i(x) = a_i \text{ for } i \neq j\} \rangle.$$

Notice that if $a_i \in U_i$ for $1 \leq i < j$, then G_j^a is connected. Moreover, the restriction $p_i : G_j^a \to G_j$ is an isomorphism, so $G_j^a \cong G_j$. On the other hand, if $a_i \notin U_i$ for some $i \in \{1, \ldots, j-1\}$, then G_j^a is a totally disconnected graph.

As mentioned above, in [3], Barrière, Dalfó, Fiol, and Mitjana show that the hierarchical product is associative. Thus, it suffices to study the case of two factors when appropriate. A graph is *prime* with respect to a given graph product if it is nontrivial and cannot be represented as the product of two nontrivial graphs. Letting K_1 represent the graph consisting of a single vertex, then a nontrivial graph G is prime with respect to the hierarchical product if $G = G_1(U_1) \sqcap G_2$ implies that G_1 or G_2 is K_1. The result that every graph has a prime factor decomposition with respect to the Cartesian product was shown in [7, p. 65], and an analogous proof shows that every graph has a prime factor decomposition with respect to the hierarchical product.

Proposition 2. *Every nontrivial graph G has a prime factor decomposition with respect to the hierarchical product. The number of prime factors is at most* $\log_2 |V(G)|$.

2. Prime Factor Decomposition

We first consider prime factorizations with respect to the hierarchical product when at least one of the factor graphs is disconnected. In this case, prime factorization is not unique with respect to the Cartesian product. Consequently, it is not unique with respect to the hierarchical product as the Cartesian product is a specific type of hierarchical product. We choose a graph which cannot be represented as a Cartesian product to exemplify this fact.

Theorem 3. *Prime factorization is not unique for the hierarchical product in the class of possibly disconnected simple graphs.*

Proof. First, note that

$$(K_1 + K_2)(U_1) \sqcap K_2 \cong K_2(W_1) \sqcap (K_1 + K_1 + K_1),$$

where U_1 is the isolated vertex of $K_1 + K_2$ and $W_1 = \emptyset$. Figure 2 illustrates the two factorizations given, where $V(K_1 + K_2) = V(K_1 + K_1 + K_1) = \{0, 1, 2\}$ and $V(K_2) = \{0, 1\}$. Notice that the cardinality of each of the factors K_2, $K_1 + K_2$, and $K_1 + K_1 + K_1$ is prime. It follows that each of the factors in both prime factorizations is prime. ■

(a) $(K_1 + K_2)(U_1) \sqcap K_2$, where $U_1 = \{0\}$ (b) $K_2(\emptyset) \sqcap (K_1 + K_1 + K_1)$

Figure 2. Two different prime factorizations of $3K_2$.

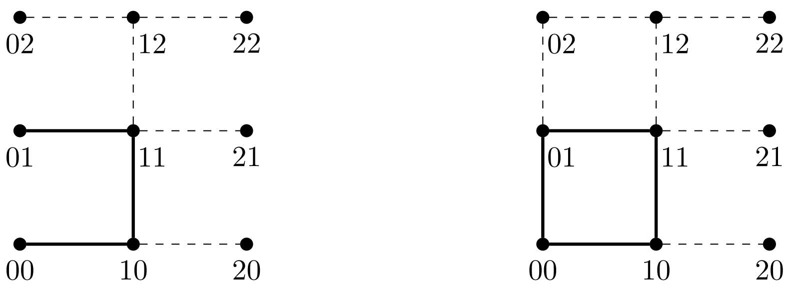
Next we show the uniqueness of prime factorization with respect to the hierarchical product for connected graphs. Throughout the remainder of this section, we assume that all graphs are connected. Note that a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ is connected if and only if each G_i is connected for all $1 \leq i \leq n$ and each $U_j \neq \emptyset$ for all $1 \leq j \leq n-1$. We generalize the method of the proof given in [7, pp. 66–68], which was used to prove the uniqueness of prime factorization with respect to the Cartesian product.

Given a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ and any two incident edges e and f in $E(G)$, we say that e and f are in *different layers* if for some $1 \leq j < k \leq n$ we have $p_j(e) \in E(G_j)$ and $p_k(f) \in E(G_k)$. Next we define two different structures in G, which we call semi-square and square. Lemma 5 shows that given any two incident edges e and f of G that are contained in different layers, either e and f belong to a unique semi-square in G or they belong to a unique square in G.

Definition 4. Let $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, where $U_i \subseteq V(G_i)$ for $1 \leq i \leq n-1$. Given some vertex $x_i \in V(G_i)$ for each $1 \leq i \leq n$ and vertices $y_j \in V(G_j) \setminus \{x_j\}$ and $y_k \in V(G_k) \setminus \{x_k\}$ for some $1 \leq j < k \leq n$, consider the subgraph H of G induced by the vertices

$$u_1 = (x_1, \dots, x_j \dots, x_k, \dots, x_n),$$
$$u_2 = (x_1, \dots, y_j, \dots, x_k, \dots, x_n),$$
$$u_3 = (x_1, \dots, y_j, \dots, y_k, \dots, x_n),$$
$$u_4 = (x_1, \dots, x_j, \dots, y_k, \dots, x_n).$$

We say that H is a *semi-square* if $E(H) = \{u_1u_2, u_2u_3, u_3u_4\}$. In this case, one can easily verify that $y_j \in U_j$, $x_jy_j \in E(G_j)$, and $x_ky_k \in E(G_k)$. We say that H is a *square* if $E(H) = \{u_1u_2, u_2u_3, u_3u_4, u_1u_4\}$. In this case, it follows that $x_j, y_j \in U_j$, $x_jy_j \in E(G_j)$, and $x_ky_k \in E(G_k)$.



(a) $U_1 = \{1\}$ (b) $U_1 = \{0, 1\}$

Figure 3. $P_3(U_1) \sqcap P_3$, where $V(P_3) = \{0, 1, 2\}$.

In Figure 3 we illustrate both a semi-square, in Figure 3(a), and a square, in Figure 3(b), for the graph $P_3(U_1) \sqcap P_3$. In both figures, the edges of the square or semi-square are in bold. Note that the following result is an immediate consequence of Definition 4.

Lemma 5. *Given a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, where $U_i \subseteq V(G_i)$ for $1 \le i \le n-1$, let e and f be two incident edges that are in different layers; that is, e is in a G_j-layer and f is in a G_k-layer for some $1 \le j < k \le n$. Then there exists exactly one semi-square or square in G containing e and f, which has no diagonals.*

Proof. We may write

$$e = uw = (x_1, \dots, x_j, \dots, x_k, \dots, x_n)(x_1, \dots, y_j, \dots, x_k, \dots, x_n),$$

where $x_i \in V(G_i)$ for $1 \le i \le n$ and $y_j \in V(G_j)$. Clearly, $x_jy_j \in E(G_j)$ and $x_i \in U_i$ for $1 \le i \le j-1$. Similarly, we may write

$$f = wv = (x_1, \dots, y_j, \dots, x_k, \dots, x_n)(x_1, \dots, y_j, \dots, y_k, \dots, x_n),$$

where $y_k \in V(G_k)$ and $x_k y_k \in E(G_k)$. Moreover, $y_j \in U_j$ and $x_i \in U_i$ for $j+1 \leq i \leq k-1$. Note that by Definition 4, any square or semi-square containing e and f must be unique. So we need only to show that e and f are indeed contained in a square or semi-square. We know

$$g = zv = (x_1, \ldots, x_j, \ldots, y_k, \ldots, x_n)(x_1, \ldots, y_j, \ldots, y_k, \ldots, x_n) \in E(G)$$

since $x_i \in U_i$ for $1 \leq i \leq j-1$ and $x_j y_j \in E(G_j)$. If $x_j \in U_j$, then $uz \in E(G)$, and we may conclude that there exists a unique square H containing edges e, f, g, and uz. Otherwise, there exists a unique semi-square H containing edges e, f, and g. In either case, H does not contain a diagonal by definition of the edge set for this product. ■

Next we focus on connected subgraphs of a given hierarchical product. Of particular importance are those subgraphs that are also representable as a hierarchical product. Analogous to the notion of subproduct with respect to the Cartesian product given in [7], we define a subproduct with respect to the hierarchical product.

Definition 6. Let $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, where $U_i \subseteq V(G_i)$ for $1 \leq i \leq n-1$. A *subproduct* in G is a subgraph of the form

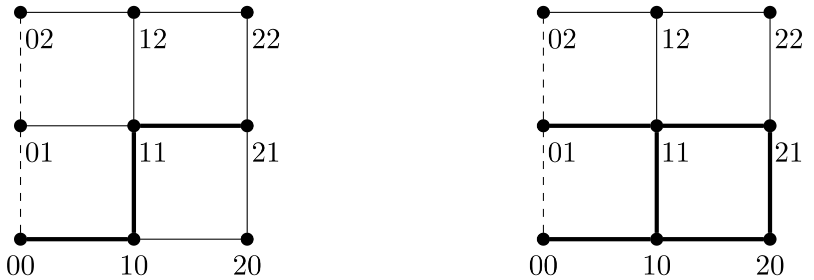
$$H_1(W_1) \sqcap \cdots \sqcap H_{n-1}(W_{n-1}) \sqcap H_n,$$

where $H_i \subseteq G_i$ for $1 \leq i \leq n$ and $W_j \subseteq U_j$ for $1 \leq j \leq n-1$.

We now wish to determine when a subgraph of a hierarchical product can be written as a subproduct. Using the notion of unique squares and semi-squares, we define the following property that will identify such subgraphs. The following terminology will be used throughout the remainder of this section. Given a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, let H be a subgraph of G and $G^\square$ be the Cartesian product associated with G. Let a and b be two distinct vertices of $V(H)$. Since a and b are vertices of $V(G^\square)$ and $G^\square$ is connected, we let $P^\square$ represent a $G^\square$-path from a to b.

Definition 7. Let H be a subgraph of $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$, and let a and b be two distinct vertices of $V(H)$. We say H is *hierarchically convex* if for any shortest $G^\square$-path $P^\square$ between a and b, each G_i-edge of $P^\square$ contained in G is also contained in H for $1 \leq i \leq n$.

For example, consider the hierarchical product $G = P_3(U_1) \sqcap P_3$, where $V(P_3) = \{0, 1, 2\}$ and $U_1 = \{1, 2\}$. In Figure 4, we illustrate two subgraphs of G, H_1 and H_2, where the bold edges represent edges of H_i for $i \in \{1, 2\}$ and the dashed edges represent the missing edges of G contained in $G^\square$. H_1, depicted



(a) H_1 is not hierarchically convex (b) H_2 is hierarchically convex

Figure 4. $P_3(U_1) \sqcap P_3$, where $V(P_3) = \{0, 1, 2\}$ and $U_1 = \{1, 2\}$.

in Figure 4(a), is not hierarchically convex, while H_2, depicted in Figure 4(b), is hierarchically convex. We note that this notion of hierarchically convex is merely an extension of convex as defined in [7].

Lemma 8. *If H is a subgraph of a hierarchical product $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ that is hierarchically convex, then for any two incident edges e and f of $E(H)$ that are in different layers, the unique square or semi-square of G that contains e and f is also in H.*

Proof. Let $G^{\square}$ be the Cartesian product associated with G. Suppose there exist incident edges e and f of H that are in different layers. As in Lemma 5, we shall assume e is in a G_j-layer and f is in a G_k-layer for some $1 \le j < k \le n$. Let $x_i \in U_i$ for $1 \le i \le k-1$ and $x_i \in V(G_i)$ for $k \le i \le n$. Write

$$e = uw = (x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n)(x_1, \ldots, y_j, \ldots, x_k, \ldots, x_n),$$

where $x_j y_j \in E(G_j)$. Similarly, write

$$f = wv = (x_1, \ldots, y_j, \ldots, x_k, \ldots, x_n)(x_1, \ldots, y_j, \ldots, y_k, \ldots, x_n),$$

where $x_k y_k \in E(G_k)$. Letting $z = (x_1, \ldots, x_j, \ldots, y_k, \ldots, x_n)$, observe that

$$P_1^{\square} = uwv \qquad \text{and} \qquad P_2^{\square} = uzv$$

are shortest $G^{\square}$-paths from u to v. If u, w, v, and z are contained in a square in G, then the edges of this square are also contained in H since H is hierarchically convex. If u, w, v and z are contained in a semi-square in G, then $x_j \notin U_j$ and so $uz \notin E(H)$. However, zv is contained in G, so it is also contained in H. Thus, the unique square or semi-square in G containing e and f is also contained in H. ■

Next we show that a subgraph H that is hierarchically convex in G is necessarily a subproduct. Note that if given two vertices (x_1, x_2) and (y_1, y_2) of a Cartesian product $G_1 \Box G_2$, the distance between (x_1, x_2) and (y_1, y_2) is given by $d((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$ (see [7, p. 51]).

Lemma 9. *If G is a hierarchical product and H is a hierarchically convex subgraph of G, then H is a subproduct.*

Proof. Let H be a subgraph which is hierarchically convex in G. Let a and b be two distinct vertices of $V(H)$. By associativity of the hierarchical product, it suffices to prove the lemma when $G = G_1(U_1) \sqcap G_2$. Write $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We wish to show that (a_1, b_2) and (b_1, a_2) are also vertices of $V(H)$. If $a_1 = b_1$, then any shortest $G^{\Box}$-path between a and b is of the form $(a_1, a_2)(a_1, x_2^1) \cdots (a_1, x_2^k)(a_1, b_2)$, where $x_2^i \in V(G_2)$ for $1 \leq i \leq k$. If $a_1 \notin U_1$, then no edge of this path is contained in G and so the result is trivially true. If $a_1 \in U_1$, then every edge of this path is contained in G and consequently contained in H as well as H is hierarchically convex. So we may assume that $a_1 \neq b_1$. If $a_2 = b_2$, then any shortest $G^{\Box}$-path between a and b is of the form $(a_1, a_2)(x_1^1, a_2) \cdots (x_1^k, a_2)(b_1, a_2)$, where $x_1^i \in V(G_1)$ for $1 \leq i \leq k$. Each edge of this path is contained in G and, therefore, H. So we may assume that $a_2 \neq b_2$.

Let $P^{\Box}$ be a shortest $G^{\Box}$-path between a and b. Every edge of $P^{\Box}$ is mapped into a single vertex by one of the projections p_1 or p_2 and into an edge by the other. It follows that $p_1(P^{\Box})$ is a shortest path in G_1 from a_1 to b_1 and $p_2(P^{\Box})$ is a shortest path in G_2 from a_2 to b_2. Write $p_1(P^{\Box}) = a_1 x_1^1 x_1^2 \cdots x_1^j b_1$, where $x_1^i \in V(G_1)$ for $1 \leq i \leq j$. Similarly, write $p_2(P^{\Box}) = a_2 x_2^1 x_2^2 \cdots x_2^k b_2$, where $x_2^i \in V(G_2)$ for $1 \leq i \leq k$. Define

$$\{a_1\} \times p_2(P^{\Box}) = (a_1, a_2)(a_1, x_2^1) \cdots (a_1, x_2^k)(a_1, b_2)$$

and

$$p_1(P^{\Box}) \times \{b_2\} = (a_1, b_2)(x_1^1, b_2) \cdots (x_1^j, b_2)(b_1, b_2).$$

Note that the concatenation of $\{a_1\} \times p_2(P^{\Box})$ and $p_1(P^{\Box}) \times \{b_2\}$ is a path of length $d_{G_1}(a_1, b_1) + d_{G_2}(a_2, b_2)$ making it a shortest $G^{\Box}$-path from (a_1, a_2) to (b_1, b_2). Furthermore, the path $p_1(P^{\Box}) \times \{b_2\}$ is contained in G, which implies this path is also contained in H since H is hierarchically convex. It follows that $(a_1, b_2) \in V(H)$. Similarly, we can define

$$\{b_1\} \times p_2(P^{\Box}) = (b_1, a_2)(b_1, x_2^1) \cdots (b_1, x_2^k)(b_1, b_2)$$

and

$$p_1(P^{\Box}) \times \{a_2\} = (a_1, a_2)(x_1^1, a_2) \cdots (x_1^j, a_2)(b_1, a_2),$$

where $p_1(P^{\Box}) \times \{a_2\}$ is contained in G. Thus, $(b_1, a_2) \in V(H)$, and we may conclude that H is a subproduct. ∎

Notice that every layer $G_j^a = \{a_1\} \sqcap \cdots \sqcap G_j \sqcap \cdots \sqcap \{a_n\}$, where $a_i \in U_i$ for $1 \leq i < j$, is a connected subproduct that is hierarchically convex in G.

Theorem 10. *Let ϕ be an isomorphism between the connected graphs G and H that are representable as hierarchical products*

$$G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n \quad \text{and} \quad H = H_1(W_1) \sqcap \cdots \sqcap H_{m-1}(W_{m-1}) \sqcap H_m$$

of prime graphs, where $U_i \subseteq V(G_i)$ for $i \in \{1, \ldots, n-1\}$ and $W_j \subseteq V(H_j)$ for $j \in \{1, \ldots, m-1\}$. Then for some $a \in V(G)$, there exists a permutation π of $\{1, \ldots, n\}$ such that $\phi(G_i^a) = H_{\pi(i)}^{\phi(a)}$ for $1 \leq i \leq n$ and $m = n$.

Proof. Fix $j \in \{1, \ldots, n\}$ and $a = (a_1, \ldots, a_n)$, where $a_i \in U_i$ for $1 \leq i < j$. Assume that $\phi(a) = b = (b_1, \ldots, b_m)$. As stated above, G_j^a is a connected subgraph of G that is hierarchically convex in G. Therefore, $\phi(G_j^a)$ is connected and is hierarchically convex in H. By Lemma 9, $\phi(G_j^a)$ is a subproduct; hence, we may write

$$(b_1, \ldots, b_m) \in \phi(G_j^a) = F_1(U_1) \sqcap \cdots \sqcap F_{m-1}(W_{m-1}) \sqcap F_m.$$

Since $G_j \cong G_j^a \cong \phi(G_j^a)$ is prime, $F_i = \{b_i\}$ for all except one i, $1 \leq i \leq j$, which we denote as $\pi(i)$. Moreover, $\phi(G_j^a)$ is connected, so $b_i \in W_i$ for each $i \in \{1, \ldots, \pi(i)-1\}$. Therefore, $\phi(G_j^a) \subseteq F_{\pi(i)}^{\phi(a)}$. This implies that $G_j^a \subseteq \phi^{-1}\left(F_{\pi(i)}^{\phi(a)}\right)$. Since $\phi^{-1}\left(F_{\pi(i)}^{\phi(a)}\right)$ is hierarchically convex in G, it is also a subproduct. Moreover, it is prime so $\phi^{-1}\left(F_{\pi(i)}^{\phi(a)}\right) \subseteq G_j^a$. Thus, $\phi(G_j^a) = F_{\pi(i)}^{\phi(a)}$. We claim that the map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is injective. If $\pi(i) = \pi(j)$, then

$$\phi(G_i^a) = F_{\pi(i)}^{\phi(a)} = \phi(G_j^a).$$

Since $F_{\pi(i)}^{\phi(a)}$ is nontrivial, it follows that G_i^a and G_j^a have a nontrivial intersection. This means $i = j$, so π is injective. Thus, $n \leq m$. Repeating this argument for ϕ^{-1} gives $m \leq n$, so $n = m$, and π is a permutation. ■

3. Preliminary Domination Bounds

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* of G if $V(G) = N[D]$. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. Arguably the most well-known conjecture in domination theory is Vizing's Conjecture from 1968 that states $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$ (see [4]). As noted in the introduction and motivated by Vizing's Conjecture, we would like to provide bounds for $\gamma(G_1(U_1) \sqcap G_2)$ as a function of $\gamma(G_1)$ and $\gamma(G_2)$, which is now possible having shown prime factorization is unique in the class of connected graphs. Our first observation is that given two graphs G and H, the domination number of the hierarchical product of G and H is monotonic with respect to nested root sets.

Observation 11. *Given any graph G of order n and any graph H,*

$$\gamma(G(U_1) \sqcap H) \geq \gamma(G(U_2) \sqcap H) \geq \cdots \geq \gamma(G(U_n) \sqcap H) = \gamma(G \Box H),$$

where $U_1 \subset U_2 \subset \cdots \subset U_n$ and $|U_k| = k$ for $1 \leq k \leq n$.

Next, we determine bounds for the domination number of $G(U) \sqcap H$ when G is a path. Certainly if $U = \emptyset$, then $P_m(\emptyset) \sqcap H$ is simply the disjoint union of $|V(H)|$ paths, in which case $\gamma(P_m(\emptyset) \sqcap H) = |V(H)|\gamma(P_m)$. We now determine the value of $\gamma(P_m(U) \sqcap H)$ when U contains a single vertex. Throughout this section, we will use the following notation. Given a hierarchical product $G(U) \sqcap H$, we let $H_v = \{(v, y) \mid y \in V(H)\}$ and $G_y = \{(v, y) \mid v \in V(G)\}$.

Theorem 12. *Let $G = P_m$ and $U = \{x_j\}$, where $1 \leq j \leq m$. For any graph H of order n, $\gamma(P_m(U) \sqcap H) = n\gamma(P_m)$ if either $m \not\equiv 1 \pmod 3$ or $j \not\equiv 1 \pmod 3$. Otherwise, $\gamma(P_m(U) \sqcap H) = n(\gamma(P_m) - 1) + \gamma(H)$.*

Proof. We let $(a_1, a_2) = (m \pmod 3,\ j \pmod 3)$ represent the combination of congruence classes of m and j modulo 3. Since $G(\emptyset) \sqcap H$ is a subgraph of $P_m(U) \sqcap H$, we know that $\gamma(P_m(U) \sqcap H) \leq n\gamma(P_m)$, which shows the upper bound when $(a_1, a_2) \neq (1, 1)$. If $(a_1, a_2) = (1, 1)$, then given any minimum dominating set D_H of H, let

$$D_1 = \{(x_j, y) \mid y \in D_H\},$$

and

$$D_2 = \{(x_i, y) \mid i \equiv 0 \pmod 3 \text{ if } i > j,\ i \equiv 2 \pmod 3 \text{ if } i < j,\ y \in V(H)\}.$$

One can easily verify that $D_1 \cup D_2$ is a dominating set of $P_m(U) \sqcap H$ of order $\gamma(H) + n(\gamma(P_m) - 1)$.

Next, suppose that D is any dominating set of $P_m(U) \sqcap H$ and let $D_x = D \cap H_{x_j}$. We first consider the projection of D_x onto H. Let $D^* = \{y \in V(H) \mid (x_j, y) \in D\}$. Notice that for each $y \in D^*$, each vertex of $V(G_y) - \{(x_{j-1}, y),$ (x_j, y), $(x_{j+1}, y)\}$ is dominated from within its G-layer. It follows that

$$|D \cap G_y| \geq 1 + \gamma(P_{j-2}) + \gamma(P_{m-j-2}).$$

One can easily verify that $1 + \gamma(P_{j-2}) + \gamma(P_{m-j-2}) \geq \gamma(P_m)$ depending on (a_1, a_2).

If $y \notin D^*$, then $|D \cap G_y| \geq \gamma(P_{j-1}) + \gamma(P_{m-j})$. We leave it to the reader to verify that

$$\gamma(P_{j-1}) + \gamma(P_{m-j}) = \begin{cases} \gamma(P_m) - 1 & \text{if } (a_1, a_2) = (1, 1), \\ \gamma(P_m) & \text{otherwise.} \end{cases}$$

Therefore, if $(a_1, a_2) \neq (1, 1)$, we have

$$\begin{aligned} |D| &= \sum_{y \in V(H)} |D \cap G_y| \\ &\geq |D^*|(1 + \gamma(P_{j-2}) + \gamma(P_{m-j-2})) + (n - |D^*|)(\gamma(P_{j-1}) + \gamma(P_{m-j})) \\ &\geq |D^*|\gamma(P_m) + (n - |D^*|)\gamma(P_m) = n\gamma(P_m). \end{aligned}$$

So assume that $(a_1, a_2) = (1, 1)$. Let $A = N_H(D^*) \setminus D^*$ and let $B = V(H) \setminus (A \cup D^*)$. Note that if $y \in B$, then $|D \cap G_y| \geq \gamma(P_m)$. Moreover, $D^* \cup B$ is a dominating set of H so $|D^*| + |B| \geq \gamma(H)$. This implies that $|A| \leq n - \gamma(H)$ and we have

$$\begin{aligned} |D| &= \sum_{y \in V(H)} |D \cap G_y| \geq |D^*|\gamma(P_m) + |B|\gamma(P_m) + |A|(\gamma(P_m) - 1) \\ &= n\gamma(P_m) - |A| \geq n\gamma(P_m) - (n - \gamma(H)) = n(\gamma(P_m) - 1) + \gamma(H). \end{aligned}$$ ■

We can use the above result to find a lower bound for the domination number of $P_m(U_i) \sqcap H$ for $U_1 \subset U_2 \subset \cdots \subset U_n$ and $|U_k| = k$ for $1 \leq k \leq n$ as a function of $\gamma(P_i)$ and $\gamma(P_{m-i})$. In what follows, we let P_0 represent the empty graph.

Theorem 13. *For any $m \in \mathbb{N}$, any $s \in \{1, \ldots, m\}$, and any graph H of order n,*

$$\gamma(P_m(U_s) \sqcap H) \geq \gamma(H)\gamma(P_s) + (n - \gamma(H))\gamma(P_{m-s}),$$

where $U_s = \{x_1, \ldots, x_s\}$.

Proof. Let H be any fixed graph with order n. We proceed by induction on m with base cases $m = 1, 2$, or 3.

Case 1. Suppose $m = 1$ and consider the graph $P_1(U_1) \sqcap H$. It follows that

$$\gamma(P_1(U_1) \sqcap H) = \gamma(P_1 \square H) \geq \gamma(H)\gamma(P_1).$$

Case 2. Suppose $m = 2$. Notice that when $s = 1$, Theorem 12 guarantees that

$$\gamma(P_2(U_1) \sqcap H) \geq n\gamma(P_2) = \gamma(H)\gamma(P_1) + (n - \gamma(H))\gamma(P_1).$$

On the other hand, when $s = 2$, we have

$$\gamma(P_2(U_2) \sqcap H) = \gamma(P_2 \Box H) \geq \gamma(H) = \gamma(H)\gamma(P_2) + (n - \gamma(H))\gamma(P_0).$$

Thus, the result holds when $m = 2$.

Case 3. Suppose $m = 3$. Notice that when $s = 1$, Theorem 12 guarantees that

$$\begin{aligned}\gamma(P_3(U_1) \sqcap H) \;&\geq\; n\gamma(P_3) = (n - \gamma(H))\gamma(P_3) + \gamma(H)\gamma(P_3)\\ &=\; \gamma(H)\gamma(P_1) + (n - \gamma(H))\gamma(P_2).\end{aligned}$$

Next, assume that $s = 2$ and consider the graph $P_3(U_2) \sqcap H$. Let D be any dominating set of $P_3(U_2) \sqcap H$ and set $D^* = \{y \in V(H) \mid (x_2, y) \in D\}$. Note that if $y \in D^*$, then $|D \cap G_y| \geq 1$ and if $y \notin D^*$, then $(x_3, y) \in D$. Therefore, in both cases we know $|D \cap G_y| \geq 1$ so that

$$\begin{aligned}|D| \;&=\; \sum_{y \in V(H)} |D \cap G_y| \geq n = (n - \gamma(H)) + \gamma(H)\\ &=\; (n - \gamma(H))\gamma(P_1) + \gamma(H)\gamma(P_2).\end{aligned}$$

Finally, assume that $s = 3$ and consider the graph $P_3(U_3) \sqcap H$. Then $\gamma(P_3(U_3) \sqcap H) = \gamma(P_3 \Box H) \geq \gamma(H) = \gamma(H)\cdot 1 + (n - \gamma(H))\cdot 0 = \gamma(H)\gamma(P_3) + (n - \gamma(H))\gamma(P_0)$.

Suppose for some fixed k and all $m \in \{1, \ldots, k\}$ that the statement of the theorem is true. Consider the graph $P_m(U_s) \sqcap H$, where $m = k + 1$. We proceed by induction on s with the base case $s = 1$. From Theorem 12, we know that if $m \equiv 0, 2 \pmod 3$, then

$$\begin{aligned}\gamma(P_m(U_1) \sqcap H) \;&\geq\; n\gamma(P_m) = (n - \gamma(H))\gamma(P_m) + \gamma(H)\gamma(P_m)\\ &=\; (n - \gamma(H))\gamma(P_{m-1}) + \gamma(H)\gamma(P_m)\\ &\geq\; (n - \gamma(H))\gamma(P_{m-1}) + \gamma(H)\gamma(P_1).\end{aligned}$$

Similarly, if $m \equiv 1 \pmod 3$, then

$$\begin{aligned}\gamma(P_m(U_1) \sqcap H) \;&\geq\; n(\gamma(P_m) - 1) + \gamma(H) = n\gamma(P_{m-1}) + \gamma(H)\gamma(P_1)\\ &\geq\; (n - \gamma(H))\gamma(P_{m-1}) + \gamma(H)\gamma(P_1).\end{aligned}$$

So assume that for some fixed $j \in \{1, \ldots, m - 1\}$ that for all $1 \leq s \leq j$, we have

$$\gamma(P_m(U_s) \sqcap H) \geq \gamma(H)\gamma(P_s) + (n - \gamma(H))\gamma(P_{m-s}).$$

Consider $s = j + 1$ and let D be any dominating set of $P_m(U_s) \sqcap H$. Let $D^* = \{y \in V(H) \mid (x_s, y) \in D\}$, let $A = N_H(D^*) \setminus D^*$, and let $B = V(H) \setminus (A \cup D^*)$.

First, assume that $s = m$. Thus, $m > 3$ and we may define G' to be the subgraph of $P_m(U_s) \sqcap H$ induced by the set

$$\{\,(x_i, y) \mid 1 \le i \le s-3,\ y \in V(H)\,\}.$$

If $y \in D^*$, then $|D \cap G_y| \ge |D \cap G'_y| + |D \cap \{(x_{s-2}, y), (x_{s-1}, y)\}| + 1$. If $y \in A$, then $|D \cap G_y| \ge |D \cap G'_y| + |D \cap \{(x_{s-2}, y), (x_{s-1}, y)\}|$. If $y \in B$, then $|D \cap G_y| \ge |D \cap G'_y| + |D \cap \{(x_{s-2}, y)\}| + 1$. Thus,

$$\begin{aligned} |D| &= \sum_{y \in V(H)} |D \cap G_y| \ge |D \cap G'| + |D \cap H_{x_{s-2}}| + |D \cap H_{x_{s-1}}| + |D^*| \\ &\ge \gamma(P_{s-3} \square H) + |B| + |D^*|. \end{aligned}$$

Since $D^* \cup B$ is a dominating set of H, $|B| + |D^*| \ge \gamma(H)$ and

$$|D| \ge \gamma(P_{s-3} \square H) + \gamma(H).$$

Finally, note that $s - 3 = j - 2$ and by the inductive hypothesis, we know that for $m = j - 2$ and U_{j-2},

$$\begin{aligned} \gamma(P_{j-2}(U_{j-2}) \sqcap H) &= \gamma(P_{j-2} \square H) \\ &\ge \gamma(H)\gamma(P_{s-3}) + (n - \gamma(H))\gamma(P_0) = \gamma(H)\gamma(P_{s-3}). \end{aligned}$$

Therefore,

$$|D| \ge \gamma(P_{s-3} \square H) + \gamma(H) \ge \gamma(H)\gamma(P_{s-3}) + \gamma(H) \ge \gamma(H)\gamma(P_m).$$

Next, assume that $s < m$ and let G'' be the graph induced by the set

$$\{\,(x_i, y) \mid 1 \le i \le s-2,\ y \in V(H)\,\}.$$

Notice that if $y \in D^*$, then

$$\begin{aligned} |D \cap G_y| &\ge |D \cap G''_y| + |D \cap \{(x_{s-1}, y)\}| + \gamma(P_{m-s-1}) + 1 \\ &\ge |D \cap G''_y| + |D \cap \{(x_{s-1}, y)\}| + \gamma(P_{m-s}). \end{aligned}$$

On the other hand, if $y \notin D^*$, then

$$|D \cap G_y| \ge |D \cap G''_y| + |D \cap \{(x_{s-1}, y)\}| + \gamma(P_{m-s}).$$

Summing over all $y \in V(H)$, we have

$$\begin{aligned} |D| &\ge \sum_{y \in V(H)} |D \cap G''_y| + |D \cap \{(x_{s-1}, y)\}| + \gamma(P_{m-s}) \\ &= |D \cap G''| + |D \cap H_{x_{s-1}}| + n\gamma(P_{m-s}) \ge \gamma(P_{s-2} \square H) + n\gamma(P_{m-s}). \end{aligned}$$

Notice that $s-2=j-1$ and by the inductive hypothesis, we know that for $m=j-1$ and U_{j-1},

$$\begin{aligned}\gamma(P_{j-1}(U_{j-1})\sqcap H) &= \gamma(P_{j-1}\Box H)\\ &\geq \gamma(H)\gamma(P_{s-2})+(n-\gamma(H))\gamma(P_0)=\gamma(H)\gamma(P_{s-2}).\end{aligned}$$

Therefore,

$$\begin{aligned}|D| &\geq \gamma(H)\gamma(P_{s-2})+n\gamma(P_{m-s})\\ &= \gamma(H)\gamma(P_{s-2})+(n-\gamma(H))\gamma(P_{m-s})+\gamma(H)\gamma(P_{m-s})\\ &\geq \gamma(H)\left(\gamma(P_s)-1\right)+(n-\gamma(H))\gamma(P_{m-s})+\gamma(H)\\ &= \gamma(H)\gamma(P_s)+(n-\gamma(H))\gamma(P_{m-s}).\end{aligned}$$

■

We would like to point out that although Jacobson and Kinch [8], and independently, Meir and Moon [12] proved $\gamma(P_m\Box H)\geq\gamma(P_m)\gamma(H)$ for any $m\in\mathbb{N}$, the above argument includes a unique proof for this same result. We also note that similar arguments may be used when considering $P_m(U)\sqcap H$ where U induces a disconnected graph.

Finally, we give a general lower bound for $\gamma(G(U)\sqcap H)$ when G and H are arbitrary graphs. The following lemma allows us to partition $V(G)$ prior to considering the hierarchical product. Recall that given a graph G, a *clique* is a subset of vertices in G such that its induced subgraph is complete.

Lemma 14. *Let G be any graph with $\gamma(G)=k$. There exists a partition $A_1,\ldots,A_k$ of $V(G)$ such that*

(i) *A_i is a clique for $1\leq i\leq k-1$,*

(ii) *for each $v\in A_k$, there exists $w\in A_i$ such that $vw\notin E(G)$ for $1\leq i\leq k-1$, and*

(iii) *for each $v\in A_i$ $1\leq i\leq k-1$, there exists $w\in A_k$ such that $vw\notin E(G)$.*

Proof. Let $D=\{x_1,\ldots,x_k\}$ be a γ-set of G. Choose a maximal clique from $N[x_1]\setminus(D-\{x_1\})$ that contains x_1 and call it A_1. Next, choose a maximal clique from $N[x_2]\setminus(D-\{x_2\}\cup A_1)$ that contains x_2 and call it A_2. Continue this process until $A_1,\ldots,A_{k-1}$ have been chosen and let $A_k=V(G)\setminus(\bigcup_{i=1}^{k-1}A_i)$. Note that for some $2\leq i\leq k-1$, A_i may only contain the vertex x_i.

Let $w\in A_i$ for some $1\leq i\leq k-1$. If w is adjacent to every vertex of A_k, then $\bigcup_{j\notin\{i,k\}}\{x_j\}\cup\{w\}$ is a dominating set of G, contradicting $\gamma(G)=k$. So there exists some $v\in A_k$ such that $wv\notin E(G)$.

On the other hand, if there exists $u\in A_k$ that is adjacent to every vertex of A_i for some $1\leq i\leq k-1$, then A_i was not a maximal clique. Thus, there exists some $v\in A_i$ such that $uv\notin E(G)$. ■

Using the partition in Lemma 14, the following two results give lower bounds for $\gamma(G(U) \sqcap H)$ when $\gamma(G) = 3$ or $\gamma(G) = 4$.

Theorem 15. *Let G be any graph with $\gamma(G) = 3$, and let A_1, A_2, A_3 be any partition of $V(G)$ satisfying the conditions of Lemma 14. For any graph H of order n, $\gamma(G(U) \sqcap H) \geq 2n$, where $U \subseteq A_1$ or $U \subseteq A_2$.*

Proof. Assume that $U \subseteq A_1$. Let D be any minimum dominating set of $G(U) \sqcap H$. Let E represent the vertices of $\pi_H(D)$ that have exactly two pre-images in D. First note that if $v \in V(H) \setminus \pi_H(D)$, then each vertex of $\{(x, v) \mid x \in A_3\}$ is not dominated by D, which cannot happen. So $\pi_H(D) = V(H)$. Suppose there exists $v \in \pi_H(D)$ where v has only one pre-image in D, call it (x, v). If $x \in A_1$, then there exists some vertex $w \in A_2 \cup A_3$ that is not adjacent to x for otherwise $\gamma(G) = 1$. However, this implies D does not dominate (w, v) which is a contradiction. If $x \in A_3$, then by the choice of A_1, A_2, and A_3, there exists $w \in A_2$ that is not adjacent to x, another contradiction. If $x \in A_2$ and x is adjacent to each vertex of A_3, then $\{x, z\}$ dominates G for any $z \in A_1$. However, this contradicts $\gamma(G) = 3$ so there exists $w \in A_3$ that is not adjacent to x. It follows that for each $y \in V(H)$, $|D \cap G_y| \geq 2$. Therefore, $|D| \geq 2|E| + 3|V(H) \setminus E| \geq 2n$. A similar argument shows the same result holds when $U \subseteq A_2$. ■

Theorem 16. *Let G be any graph with $\gamma(G) = 4$, and let A_1, A_2, A_3, A_4 be any partition of $V(G)$ satisfying the conditions of Lemma 14. For any graph H of order n, $\gamma(G(U) \sqcap H) \geq 3n$, where $U \subseteq A_i$ for some $1 \leq i \leq 3$.*

Proof. Assume $U \subseteq A_1$ and let D be any minimum dominating set of $G(U) \sqcap H$. Let E represent the vertices of $\pi_H(D)$ that have exactly two pre-images in D. As in the proof of Theorem 15, $\pi_H(D) = V(H)$ and no vertex of $\pi_H(D)$ has exactly one pre-image in D. Suppose that $v \in E$ and let (x, v) and (y, v) represent the pre-images of v in D. Note that $\{x, y\}$ must dominate $G \setminus A_1$ since each vertex of the form (w, v), where $w \in A_2 \cup A_3 \cup A_4$, is dominated by either (x, v) or (y, v). However, this implies that $\{a, x, y\}$ dominates G for any $a \in A_1$ as A_1 is a clique. This contradiction shows that no such $v \in E$ exists so $E = \emptyset$. It follows that $|D| \geq 3n$. A similar argument shows the same result holds when $U \subseteq A_2$ or $U \subseteq A_3$. ■

We point out that the above argument can be generalized for any graph G with domination number $k \geq 5$. Yet there exist many open questions regarding domination in the hierarchical product. For instance, given two graph G and H, where $\gamma(G) = 3$, can we partition $V(G)$ into three sets A_1, A_2, A_3 in a slightly different way in order to obtain a lower bound for $G(U) \sqcap H$ when U is not restricted to precisely one of A_i? In general, can we find a lower bound for $G(U) \sqcap H$ regardless of the structure of G or choice of U? Is it possible to determine the gap between $\gamma(G \Box H)$ and $\gamma(G(U) \sqcap H)$ based on the choice of U?

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