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STRONG EDGE-COLORING OF PLANAR GRAPHS

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Abstract

A *strong edge-coloring* of a graph is a proper edge-coloring where each color class induces a matching. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-colored with k colors. It is known that every planar graph G has a strong edge-coloring with at most $4\Delta(G) + 4$ colors [R.J. Faudree, A. Gyárfás, R.H. Schelp and Zs. Tuza, *The strong chromatic index of graphs*, Ars Combin. **29B** (1990) 205–211]. In this paper, we show that if G is a planar graph with $g \geq 5$, then $\chi'_s(G) \leq 4\Delta(G) - 2$ when $\Delta(G) \geq 6$ and $\chi'_s(G) \leq 19$ when $\Delta(G) = 5$, where g is the girth of G.

Keywords: strong edge-coloring, strong chromatic index, planar graph, discharging method.

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1. Introduction

All graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$ and $\Delta(G)$ to denote its vertex set, edge set and maximum degree, respectively. For a planar graph G, $F(G)$ denotes its face set, $d(v)$ denotes the *degree* of a vertex v in G. The *length* or *degree* of a face f, denoted by $d(f)$, is the length of a boundary walk around f in G. We call v a k*-vertex*, or a k^+*-vertex*, or a k^-*-vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call f a k*-face*, or a k^+*-face*, or a k^-*-face* if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [3].

A *proper* k*-edge-coloring* of a graph G is a mapping $f : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. A *strong edge-coloring* of a graph G is a proper edge-coloring where each color class induces a matching, i.e., every two edges at distance at most 2 receive distinct colors. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-colored with k colors.

Strong edge-coloring was introduced by Fouquet and Jolivet in 1983 [7, 8]. In 1985, Erdős and Nešetřil posed the following conjecture during a seminar in Prague (later published in [5]).

Conjecture 1 (Erdős and Nešetřil [5]). *For every graph G,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2, & \Delta(G) \text{ is even;} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta + 1), & \Delta(G) \text{ is odd.} \end{cases}$$

This conjecture was verified when $\Delta(G) \leq 3$ [1, 9]. When $\Delta(G)$ is large enough, Bruhn and Joos [4] showed currently the best known upper bound for the strong chromatic index of graphs by the following theorem

Theorem 2 (Bruhn and Joos [4]). *If G is a graph of sufficiently large maximum degree Δ, then $\chi'_s(G) \leq 1.93\Delta(G)^2$.*

In this paper, we mainly study the strong chromatic index of planar graphs with lower bounds on girth. The study on the strong chromatic index of planar graphs was started with the paper of Faudree *et al.* [6], who presented a construction of planar graphs of girth at least 4 which satisfies $\chi'_s(G) \leq 4\Delta(G) - 4$. Moreover, they proved the following theorem.

Theorem 3 (Faudree *et al.* [6]). *If G is a planar graph, then $\chi'_s(G) \leq 4\chi'(G)$.*

In the short and simple proof of Theorem 3, the authors used Vizing's Theorem and the Four Color Theorem. In particular, by Vizing's Theorem and Theorem 3, we can easily obtain that the strong chromatic index of every planar graph with $\Delta(G)$ at least 7 is at most $4\Delta(G)$.

Recently, Hudák *et al.* [10] considered planar graphs with girth at least 6 and obtained the following result.

Theorem 4 (Hudák *et al.* [10]). *If G is a planar graph with girth $g \geq 6$, then $\chi'_s(G) \leq 3\Delta(G) + 6$.*

Moreover, this result was improved by Bensmail *et al.* [2] to the following.

Theorem 5 (Bensmail *et al.* [2]). *If G is a planar graph with girth $g \geq 6$, then $\chi'_s(G) \leq 3\Delta(G) + 1$.*

For smaller values of the girth, they also obtained the following strengthening in the same paper.

Theorem 6 (Bensmail *et al.* [2])**.** *Let G be a planar graph with maximum degree $\Delta(G)$ and girth g. If G satisfies one of the following conditions, then $\chi'_s(G) \leq 4\Delta(G)$.*

(1) $\Delta(G) \geq 7$,

(2) $\Delta(G) \geq 5$ *and* $g \geq 4$,

(3) $g \geq 5$.

In this paper, we mainly improve the upper bound in (3) of Theorem 6 when $\Delta(G) \geq 5$; we show the following.

Theorem 7. *If G is a planar graph with $\Delta(G) \geq 6$ and girth $g \geq 5$, then $\chi'_s(G) \leq 4\Delta(G) - 2$.*

Theorem 8. *If G is a planar graph with $\Delta(G) = 5$ and girth $g \geq 5$, then $\chi'_s(G) \leq 19$.*

Before proving our results we introduce some definitions and notations.

Definition. Two edges are at *distance* 1 if they share one of their ends and they are at *distance* 2 if they are not at distance 1 and there exists an edge adjacent to both of them. We define $N_2(e)$ as the set of edges at distance at most 2 from the edge e. We denote by $SC(N_2(e))$ the set of colors used by edges in $N_2(e)$. We denote by $N(v)$ the neighborhood of the vertex v, i.e., the set of its adjacent vertices. We use $SC(v)$ to denote the set of colors used by edges which are incident to v. A k_l-vertex is a k-vertex adjacent to exactly l 2-vertices. A *bad* 5-*vertex* is a 5-vertex adjacent to three 2-vertices, otherwise it is a *good* 5-*vertex*. A *weak* 2-*path* is a path $v_1v_2v_3$ such that $d(v_1) = d(v_3) = 2$ and v_2 is a bad 5-vertex. A *bad* 5-*cycle* is a 5-cycle incident with a weak 2-path.

2. Proof of Theorem 7

We shall argue by contradiction to prove Theorem 7 and assume that G is a counterexample with $|E(G)|$ as small as possible. Let $G' = G - uv$ and $L = \{1, 2, \ldots, 4\Delta(G)-2\}$. By the minimality of G we can assume that it is connected and that it has $\chi'_s(G) \geq 4\Delta(G) - 1$. In the following two subsections, we first investigate the structure of the minimal counterexample G and then use discharging method to obtain a contradiction to complete the proof.

2.1. Structure of minimal counterexample

We first show the structure of minimal counterexample G by the following lemma. For each configuration of this lemma, we will show a contradiction by extending a strong $(4\Delta(G) - 2)$-edge-coloring ϕ of G' to a strong edge-coloring of G to complete the proof.

Lemma 9. *For a minimal counterexample G, each of the following holds:*

1. *G does not contain a 1-vertex adjacent to a 4^--vertex.*
2. *G does not contain a 2-vertex adjacent to a 3^--vertex.*
3. *G does not contain a 3-vertex adjacent to two 3^--vertices.*
4. *G does not contain a 4-vertex adjacent to a 2-vertex and a 3^--vertex.*
5. *G contains neither a 5-vertex adjacent to four 2^--vertices nor a bad 5-cycle.*
6. *G contains neither a bad 5-vertex adjacent to a 3^--vertex nor a bad 5-vertex with a 2-neighbor adjacent to a $(\Delta(G)-1)^-$-vertex.*
7. *G does not contain a 5_2-vertex adjacent to two other 3^--vertices.*
8. *If $k \geq 5$, then G does not contain a k-vertex adjacent to $k-3$ 1-vertices. Moreover, if the k-vertex is adjacent to $k-4$ 1-vertices, then it has no other 2^--neighbor. In particular, if a 5-vertex is adjacent to one 1-vertex, then it has no other 3^--neighbor.*
9. *If $k \geq 6$, then G does not contain a k-vertex adjacent to $k-1$ 2^--vertices; if the k-vertex is adjacent to $k-2$ 2^--vertices, then each 2^--vertex is not a 1-vertex. Moreover, if the k-vertex is adjacent to $k-2$ 2-vertices, then it has no other 3^--neighbor.*
10. *If $k \geq 6$ and $1 \leq \alpha \leq k-4$, then G does not contain a k-vertex adjacent to α 1-vertices and to $k-3-\alpha$ vertices of degree 2, such that this k-vertex is adjacent to another 3^--vertex.*

Proof. 1. Suppose G contains a 1-vertex u adjacent to a 4^--vertex v. We can extend ϕ to G by coloring uv with a color in $L\backslash SC_\phi(N_2(uv))$ because $|L\backslash SC_\phi(N_2(uv))| \geq 4\Delta(G)-2-3\Delta(G) \geq 4$.

2. Suppose G contains a 2-vertex u adjacent to a 3^--vertex v. W.l.o.g. assume that $d(v)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-2 \geq 4$, we can extend ϕ to G by coloring uv.

3. Suppose G contains a 3-vertex u adjacent to two 3^--vertices v, w. W.l.o.g. assume that $d(v)=d(w)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-5 \geq 1$, we can extend ϕ to G by coloring uv.

4. Suppose G contains a 4-vertex u adjacent to a 2-vertex v and a 3^--vertex w. W.l.o.g. assume that $d(w)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-5 \geq 1$, we can extend ϕ to G by coloring uv.

5. Suppose G contains a 5-vertex u adjacent to four 2-vertices v, w, x and y. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-8 \geq 4$, we can extend ϕ to G by coloring uv. Let a bad 5-cycle contains a bad 5-vertex u and two 2-vertices v, w. W.l.o.g. assume that $N(u)=\{v,w,x,y,z\}$, $N(v)=\{u,v_1\}$, $N(w)=\{u,w_1\}$, $N(y)=\{u,t\}$, $N(x)=\{u,x_1,x_2,\ldots,x_{\Delta-2},x_{\Delta-1}\}$, $N(t)=\{y,t_1,t_2,\ldots,t_{\Delta-2},t_{\Delta-1}\}$, $N(z)=\{u,z_1,z_2,\ldots,z_{\Delta-2},z_{\Delta-1}\}$, $N(v_1)=\{v,w_1,v_1^1,v_1^2,\ldots,v_1^{\Delta-3},v_1^{\Delta-2}\}$, $N(w_1)=\{v_1, w,w_1^1,w_1^2,\ldots,w_1^{\Delta-3},w_1^{\Delta-2}\}$ (see Figure 1). Clearly, we have $N_2(uv)=3\Delta(G)+4$. If $\Delta(G) \geq 7$, there must be a color $\alpha \in L\backslash SC_\phi(N_2(uv))$ since $(4\Delta(G)-2)-$

$(3\Delta(G)+4) \geq 1$. We color uv with it such that we get a strong $(4\Delta(G)-2)$-edge-coloring in G. Then we can extend the coloring ϕ to a strong $(4\Delta(G)-2)$-edge-coloring of G, a contradiction. Therefore, we assume that $\Delta(G)=6$ and $|L\backslash SC_\phi(N_2(uv))| = 0$. In this case, we may try to recolor uw. We color uv with $\phi(uw)$ and recolor uw with one color α of $\phi(v_1v_1^i)$, $i=1,2,3,4$. This is possible because $\{\phi(v_1v_1^i)\} \cap \{\phi(w_1w_1^i)\} = \emptyset$, $i=1,2,3,4$. Then we can extend ϕ to G.

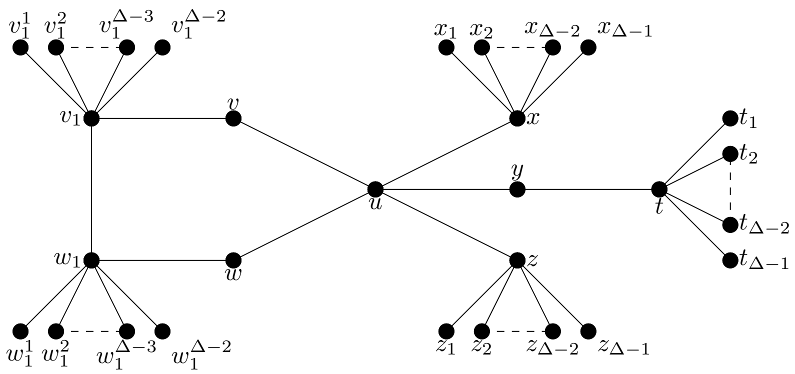

Figure 1. The configuration of Lemma 9.5.

6. Suppose a bad 5-vertex u is adjacent to three 2-vertices v, w, y and a 3^--vertex x. W.l.o.g. assume that $d(x)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-9 \geq 3$, we can extend ϕ to G by coloring uv. If a bad 5-vertex u is adjacent to three 2-vertices v, w, y and one 2-neighbor (say the vertex v) is adjacent to a $(\Delta(G)-1)^-$-vertex, then since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-5 \geq 1$, we can extend ϕ to G by coloring uv.

7. Suppose a 5-vertex u is adjacent to two 2-vertices v, w and two another 3^--vertices x, y. W.l.o.g. assume that $d(x)=d(y)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-10 \geq 2$, we can extend ϕ to G by coloring uv.

8. Let $k \geq 5$. Suppose G contains a k-vertex u with $N(u)=\{u_1,u_2,\ldots,u_k\}$ and $u_1=v$, such that each u_i with $i \in [\![k-3]\!]$ is a 1-vertex. Then, we can extend ϕ to G by coloring uv because $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-k+2 \geq 2$. Suppose now that the k-vertex is adjacent to $k-4$ 1-vertices and a 2^--vertex. W.l.o.g. assume that $d(u_{k-3})=2$. Since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-k+1 \geq 1$, we can extend ϕ to G by coloring uv. In particular, if a 5-vertex u is adjacent to a 1-vertex v and a 3^--vertex w, then since $|L\backslash SC_\phi(N_2(uv))| \geq \Delta(G)-5 \geq 1$, we can extend ϕ to G by coloring uv.

9. Let $k \geq 6$. Suppose G contains a k-vertex u with $N(u)=\{u_1,u_2,\ldots,u_k\}$ and $u_1=v$, such that each u_i with $i \in [\![k-1]\!]$ is a 2^--vertex. W.l.o.g. assume that $d(v)=d(u_2)=\cdots=d(u_{k-1})=2$. Then, we can extend ϕ to G by coloring uv because $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-2k+2 \geq 2$. Suppose that the k-vertex

is adjacent to $k-2$ 2^--vertices and u_1 is a 1-vertex. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-2k+4 \geq 4$, we can extend ϕ to G by coloring uv. Finally, assume that the k-vertex is adjacent to $k-2$ 2-vertices and a 3^--vertex u_{k-1}. W.l.o.g. assume that $d(u_{k-1})=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-2k+1 \geq 1$, we can extend ϕ to G by coloring uv.

10. Let $k \geq 6$ and $1 \leq \alpha \leq k-4$. Suppose G contains a k-vertex adjacent to α 1-vertices and to $k-3-\alpha$ vertices of degree 2, such that this k-vertex is adjacent to a 3^--vertex. W.l.o.g. assume that $d(w)=3$. Since $|L\backslash SC_\phi(N_2(uv))| \geq 2\Delta(G)-2k+\alpha+2 \geq 3$, we can extend ϕ to G by coloring uv. ■

2.2. Discharging method

In this section, we apply the discharging method to a planar graph G and complete the proof by a contradiction. Since G is a planar graph, we have

$$\sum_{v\in V(G)}\left(\frac{3}{2}d\,(v)-5\right)+\sum_{f\in F(G)}(d\,(f)-5)=-10.$$

We define the initial charge function $ch(x)$ of $x \in V\,(G) \cup F\,(G)$. Let $ch(v) = \frac{3}{2}d(v)-5$ if $v \in V\,(G)$ and $ch(f) = d(f)-5$ if $f \in F\,(G)$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V\,(G) \cup F\,(G)$ such that $ch'(x) \geq 0$ for all $x \in V\,(G) \cup F\,(G)$, then $0 \leq \sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -10$, a contradiction.

For $v \in V(G)$ and $f \in F(G)$, we define the discharging rules as follows.

R(1) Every face gives 2 to each incident 1-vertex.

R(2) Every k-face ($k \geq 6$) which is incident with a bad 5-vertex, gives $\frac{k-2\iota-5}{k-2\iota}$ to each incident 2^+-vertex, where ι denoted the number of 1-vertices incident with the k-face.

R(3) Every 4^+-vertex gives 1 to each adjacent 2-vertex. In particular, if a 6^+-face contains a weak 2-path, then the bad 5-vertex in the 6^+-face gives $\frac{5}{6}$ to each adjacent 2-vertex which is in the weak 2-path.

R(4) Every 4^+-vertex gives $\frac{1}{4}$ to each adjacent 3-vertex.

R(5) Every 5^+-vertex gives $\frac{3}{2}$ to each adjacent 1-vertex.

Let $f \in F(G)$ be a k-face. We have $k \geq 5$ by the condition on the girth. Note that if f has ι incident 1-vertices, then $k \geq 5+2\iota$. Since $ch(f) = d(f)-5$, if $k = 5+2\iota$, then $ch'(f) \geq k-5-2\iota = 0$. If $k \geq 6+2\iota$, then $ch'(f) \geq k-5-2\iota-\frac{k-5-2\iota}{k-2\iota}\times(k-2\iota) \geq 0$ by R(1) and R(2).

We next check the final charge of the vertex $v \in V(G)$.

Suppose v is a 1-vertex. By Lemma 9.1, v is adjacent to a 5^+-vertex. Thus v receives 2 from its incident face by R(1) and $\frac{3}{2}$ from its adjacent vertex by R(5). Hence, $ch'(v) \geq ch(v) + 2 + \frac{3}{2} = 0$.

Suppose v is a 2-vertex. Then v is adjacent to two 4^+-vertices by Lemma 9.2. Thus v receives 1 from each adjacent vertex by R(3). Hence, $ch'(v) \geq ch(v) + 1 \times 2 = 0$. In particular, if v is in a weak 2-path which is contained in a 6^+-face, then v must be adjacent to a $\Delta(G)$-vertex by Lemma 9.6. Thus, v receives $\frac{5}{6}$ from the bad 5-vertex, at least $\frac{1}{6}$ from the 6^+-face and 1 from the other neighbor of v by R(2) and R(3). Hence, $ch'(v) \geq ch(v) + \frac{5}{6} + \frac{1}{6} + 1 = 0$.

Suppose v is a 3-vertex. Then v is adjacent to at least two 4^+-vertices by Lemma 9.3. Thus v receives $\frac{1}{4}$ from each adjacent 4^+-vertex by R(4). Hence, $ch'(v) \geq ch(v) + \frac{1}{4} \times 2 = 0$.

Suppose v is a 4-vertex. Then v is adjacent to at most one 2-vertex by Lemma 9.4, and if v is adjacent to a 2-vertex, then v has no other 3^--neighbor. So v gives 1 to its adjacent 2-vertex by R(3). Hence, $ch'(v) \geq ch(v) - 1 = 0$. Otherwise, v has no 2-neighbor, and v gives $\frac{1}{4}$ to each adjacent 3-vertex by R(4). Hence, $ch'(v) \geq ch(v) - \frac{1}{4} \times 4 = 0$.

Suppose v is a 5-vertex. Then v is adjacent to at most one 1-vertex by Lemma 9.8. So we consider the following two cases.

(a) Assume v is adjacent to a 1-vertex. Then v has no other 3^--neighbor by Lemma 9.8. Thus v gives $\frac{3}{2}$ to its adjacent 1-vertex by R(5). Hence, $ch'(v) \geq ch(v) - \frac{3}{2} > 0$.

(b) Assume v is not adjacent to a 1-vertex. Then v is adjacent to at most three 2-vertices by Lemma 9.5. If the number of 2-neighbors of v is at most one, v gives 1 to each adjacent 2-vertex and $\frac{1}{4}$ to each adjacent 3-vertex by R(3) and R(4). Hence, $ch'(v) \geq ch(v) - \max\{1 + \frac{1}{4} \times 4, 5 \times \frac{1}{4}\} > 0$. If the number of 2-neighbors of v is two, then v is adjacent to at most one 3-neighbor by Lemma 9.7. Thus v gives 1 to each adjacent 2-vertex and $\frac{1}{4}$ to its adjacent 3-vertex by R(3) and R(4). Hence, $ch'(v) \geq ch(v) - 1 \times 2 - \frac{1}{4} > 0$. If v is a bad 5-vertex, then v is incident with at least one 6^+-face which contains a weak 2-path and not adjacent to any other 3^--neighbor by Lemma 9.5–9.7. So v gives $\frac{5}{6}$ to each adjacent 2-vertex which is in the weak 2-path and 1 to the third 2-neighbor and receives at least $\frac{1}{6}$ from incident 6^+-face by R(2) and R(3). Hence, $ch'(v) \geq ch(v) - \frac{5}{6} \times 2 - 1 + \frac{1}{6} = 0$.

Suppose v is a 6^+-vertex. We consider the following two cases.

(a) Assume v is not adjacent to a 1-vertex. Then v is adjacent to at most $k-2$ 2-vertices by Lemma 9.9. If the number of 2-neighbors of v is at most $k-3$, v gives 1 to each adjacent 2-vertex and $\frac{1}{4}$ to its adjacent 3-vertex by R(3) and R(4). Hence, $ch'(v) \geq ch(v) - 1 \times (k-3) - \frac{1}{4} \times 3 > 0$. If the number of 2-neighbors of v is exactly $k-2$, then v has no other 3-neighbor by Lemma 9.9. Thus v gives 1 to each adjacent 2-vertex by R(3). Hence, $ch'(v) \geq ch(v) - 1 \times (k-2) \geq 0$.

(b) Assume v is adjacent to α 1-vertices with $\alpha \geq 1$. And we have $\alpha \leq k-4$

by Lemma 9.8 and v has no other 3^--neighbor while v is exactly adjacent to $k-4$ 1-vertices. Thus v gives $\frac{3}{2}$ to each adjacent 1-vertex by R(5). Hence, $ch'(v) \geq ch(v) - \frac{3}{2} \times (k-4) > 0$. So we suppose that $\alpha \leq k-5$. If the number of 2-neighbors of v is at most $k-4-\alpha$, then v gives $\frac{3}{2}$ to each adjacent 1-vertex, 1 to each adjacent 2-vertex and $\frac{1}{4}$ to its adjacent 3-vertex by R(3)–R(5). Thus $ch'(v) \geq ch(v) - \frac{3}{2} \times \alpha - 1 \times (k-4-\alpha) - \frac{1}{4} \times 4 = \frac{(k-\alpha)}{2} - 2 > 0$. So the the number of 2-neighbors of v is at least $k-3-\alpha$. And v has exactly $k-3-\alpha$ 2-neighbors because v has no $k-1-\alpha$ 2-neighbors by Lemma 9.9 and no $k-2-\alpha$ 2-neighbors also by Lemma 9.9 and $\alpha \geq 1$. Thus v gives $\frac{3}{2}$ to each adjacent 1-vertex and 1 to each adjacent 2-vertex by R(3), R(5) and Lemma 9.10. Hence, $ch'(v) \geq ch(v) - \frac{3}{2}\alpha - (k-3-\alpha) = \frac{(k-\alpha)}{2} - 2 > 0$.

Therefore, we have $0 \leq \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) < 0$. This contradiction completes the proof of Theorem 7.

3. Proof of Theorem 8

The proof of Theorem 8 in this section is just similar to the one in Section 2. We include it for completeness. Let G be a counterexample with $|E(G)|$ as small as possible and $L = \{1, 2, \ldots, 19\}$. By the minimality of G we can assume that it is connected and that it has $\chi'_s(G) \geq 20$.

3.1. Structure of minimal counterexample

The minimal counterexample G in this section also has the structure properties mentioned in Lemma 9.1–9.4 and Lemma 9.7–9.8. We omit the proofs here for simplicity. Now we only show the structure which is different from the ones in Section 2.1 in the following lemmas.

Lemma 10. *For a minimal counterexample G, each of the following holds:*

1. *G does not contain a 2-neighbor adjacent to a bad 5-vertex and a 4^--vertex.*
2. *G does not contain a 2-vertex adjacent to two bad 5-vertices.*
3. *G does not contain a 2-vertex adjacent to a bad 5-vertex and a 5_2-vertex which is adjacent to a 3-vertex.*
4. *G does not contain a bad 5-vertex adjacent to another 4^--vertex.*

Proof. 1. Suppose G contains a 2-neighbor u adjacent to a bad 5-vertex v and a 4^--vertex. Since $|L \backslash SC_\phi(N_2(uv))| \geq 19 - (5+5+2+2+4) \geq 1$, we can extend ϕ to G by coloring uv.

2. Suppose G contains a 2-vertex u adjacent to two bad 5-vertices v and w. W.l.o.g. assume that $N(u) = \{v, w\}$, $N(v) = \{u, v_1, v_2, v_3, v_4\}$, $N(w) = \{u, w_1, w_2, w_3, w_4\}$, $N(v_1) = \{v, v_1^1, v_1^2, v_1^3, v_1^4\}$, $N(v_2) = \{v, v_2^1, v_2^2, v_2^3, v_2^4\}$, $N(v_3) =$

$\{v, x\}$, $N(v_4) = \{v, y\}$, $N(x) = \{v_3, x_1, x_2, x_3, x_4\}$, $N(y) = \{v_4, y_1, y_2, y_3, y_4\}$, $N(w_3) = \{w, z\}$, $N(w_4) = \{w, t\}$, $N(z) = \{w_3, z_1, z_2, z_3, z_4\}$, $N(t) = \{w_4, t_1, t_2, t_3, t_4\}$ (see Figure 2). If there exists a color $\alpha \in L \backslash SC_\phi(N_2(uv))$, we color edge uv with it such that we get a strong edge-coloring in G. Then we can extend the coloring ϕ to a strong 19-edge-coloring of G, a contradiction. Otherwise, $|L \backslash SC_\phi(N_2(uv))| = 0$. W.l.o.g. we assume that $\phi(uw) = 1$, $\phi(ww_1) = 2$, $\phi(ww_2) = 5$, $\phi(ww_3) = 3$, $\phi(ww_4) = 4$, $\phi(vv_1) = 6$, $\phi(vv_2) = 9$, $\phi(vv_3) = 7$, $\phi(vv_4) = 8$, $\phi(v_3x) = 10$, $\phi(v_4y) = 11$, $\phi(v_1v_1^1) = 12$, $\phi(v_1v_1^2) = 13$, $\phi(v_1v_1^3) = 14$, $\phi(v_1v_1^4) = 15$, $\phi(v_2v_2^1) = 16$, $\phi(v_2v_2^2) = 17$, $\phi(v_2v_2^3) = 18$, $\phi(v_2v_2^4) = 19$. In this case we may try to recolor ww_3. If we cannot, then w.l.o.g. $\phi(w_3z) = 6$, $\phi(w_4t) = 7$, $\phi(zz_1) = 8$, $\phi(zz_2) = 9$, $\phi(zz_3) = 10$, $\phi(zz_4) = 11$, $\phi(w_1w_1^1) = 12$, $\phi(w_1w_1^2) = 13$, $\phi(w_1w_1^3) = 14$, $\phi(w_1w_1^4) = 15$, $\phi(w_2w_2^1) = 16$, $\phi(w_2w_2^2) = 17$, $\phi(w_2w_2^3) = 18$, $\phi(w_2w_2^4) = 19$, so we try to recolor ww_4. If we cannot, then w.l.o.g. $\phi(tt_1) = 8$, $\phi(tt_2) = 9$, $\phi(tt_3) = 10$, $\phi(tt_4) = 11$. We continue to try to recolor vv_3 and vv_4. If the recoloring is possible in one of the edges, then we will have a color free for uv. Otherwise, w.l.o.g. we obtain $\phi(xx_1) = \phi(yy_1) = 2$, $\phi(xx_2) = \phi(yy_2) = 3, \phi(xx_3) = \phi(yy_3) = 4, \phi(xx_4) = \phi(yy_4) = 5$. Now we recolor vv_4 and ww_3 with 1, uw with 3 and uv with 8. Thus we can extend the coloring ϕ to a strong 19-edge-coloring of G, a contradiction.

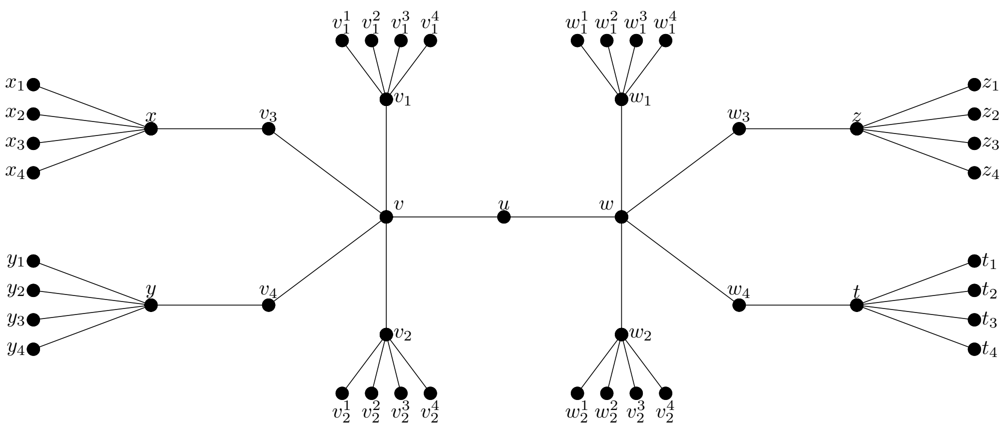

Figure 2. The configuration of Lemma 10.2.

3. Suppose G contains a 2-vertex u adjacent to a bad 5-vertex v and a 5_2-vertex w which is adjacent to a 3^--vertex. W.l.o.g. assume that $N(u) = \{v, w\}$, $N(v) = \{u, v_1, v_2, v_3, v_4\}$, $N(w) = \{u, w_1, w_2, w_3, w_4\}$, $N(v_1) = \{v, v_1^1, v_1^2, v_1^3, v_1^4\}$, $N(v_2) = \{v, v_2^1, v_2^2, v_2^3, v_2^4\}$, $N(v_3) = \{v, x\}$, $N(v_4) = \{v, y\}$, $N(x) = \{v_3, x_1, x_2, x_3, x_4\}$, $N(y) = \{v_4, y_1, y_2, y_3, y_4\}$, $N(w_3) = \{w, w_3^1, w_3^2\}$, $N(w_4) = \{w, t\}$, $N(t) = \{w_4, t_1, t_2, t_3, t_4\}$ (see Figure 3). If there exists a color $\alpha \in L \backslash SC_\phi(N_2(uv))$, we color edge uv with it such that we get a strong edge-coloring in G. Then we can

extend the coloring ϕ to a strong 19-edge-coloring of G, a contradiction. Otherwise, $|L\backslash SC_\phi(N_2(uv))| = 0$. W.l.o.g. we assume that $\phi(uw) = 1$, $\phi(ww_1) = 2$, $\phi(ww_2) = 5$, $\phi(ww_3) = 3$, $\phi(ww_4) = 4$, $\phi(vv_1) = 6$, $\phi(vv_2) = 9$, $\phi(vv_3) = 7$, $\phi(vv_4) = 8$, $\phi(v_3x) = 10$, $\phi(v_4y) = 11$, $\phi(v_1v_1^1) = 12$, $\phi(v_1v_1^2) = 13$, $\phi(v_1v_1^3) = 14$, $\phi(v_1v_1^4) = 15$, $\phi(v_2v_2^1) = 16$, $\phi(v_2v_2^2) = 17$, $\phi(v_2v_2^3) = 18$, $\phi(v_2v_2^4) = 19$. In this case we may try to recolor ww_4. If we cannot, then w.l.o.g. $\phi(w_3w_3^1) = 6$, $\phi(w_3w_2^2) = 7$, $\phi(w_4t) = 8$ $\phi(tt_1) = 9$, $\phi(tt_2) = 10$, $\phi(tt_3) = 11$, $\phi(tt_4) = 12$. $\phi(w_1w_1^1) = 13$, $\phi(w_1w_1^2) = 14$, $\phi(w_1w_1^3) = 15$, $\phi(w_1w_1^4) = 16$, $\phi(w_2w_2^1) = 17$, $\phi(w_2w_2^2) = 18$, $\phi(w_2w_2^3) = 19$. We continue to try to recolor vv_3 and vv_4. If the recoloring is possible in one of the edges, then we will have a color free for uv. Otherwise, w.l.o.g. we obtain $\phi(xx_1) = \phi(yy_1) = 2$, $\phi(xx_2) = \phi(yy_2) = 3$, $\phi(xx_3) = \phi(yy_3) = 4$, $\phi(xx_4) = \phi(yy_4) = 5$. Now we recolor vv_4 and ww_4 with 1, uw with 4 and uv with 8. This is possible since the color of $w_2w_2^4$ is not 1 or 4. Thus we can extend the coloring ϕ to a strong 19-edge-coloring of G, a contradiction.

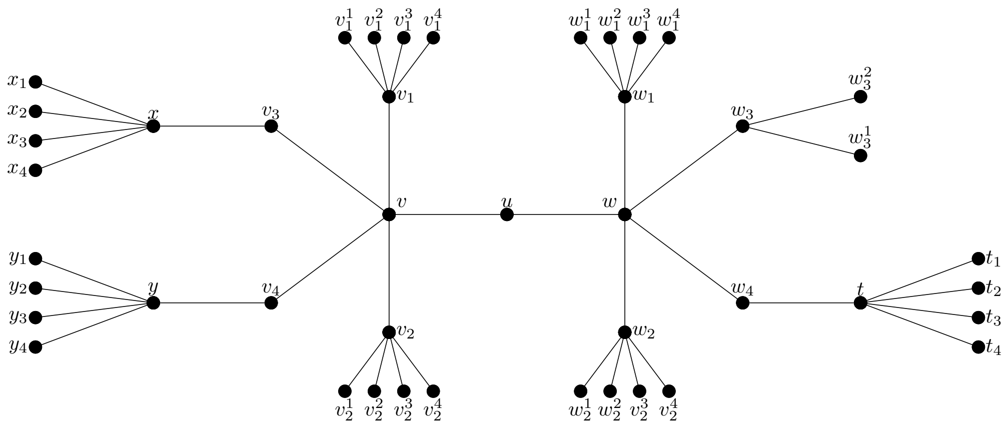

Figure 3. The configuration of Lemma 10.3.

4. Suppose G contains a bad 5-vertex u adjacent to a 4^--vertex w. W.l.o.g. assume $d(w) = 4$ and v is one 2-neighbor of the bad 5-vertex. Since $|L\backslash SC_\phi(N_2(uv))| \geq 19-(5+4+2+2+5) \geq 1$. We can extend ϕ to G by coloring uv. ■

3.2. Discharging method

Now we apply the discharging method to a planar graph G and complete the proof by a contradiction. Since G is a planar graph, we have

$$\sum_{v\in V(G)} \left(\frac{3}{2}d(v) - 5\right) + \sum_{f\in F(G)} (d(f) - 5) = -10.$$

We define the initial charge function $ch(x)$ of $x \in V(G) \cup F(G)$. Let $ch(v) = \frac{3}{2}d(v) - 5$ if $v \in V(G)$ and $ch(f) = d(f) - 5$ if $f \in F(G)$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to change the initial charge function $ch(x)$ to the final charge function $ch'(x)$ on $V(G) \cup F(G)$ such that $ch'(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then $0 \leq \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -10$, a contradiction.

For $v \in V(G)$ and $f \in F(G)$, we define the discharging rules as follows.

R(1) Every face gives 2 to each incident 1-vertex.

R(2) Every 4-vertex gives 1 to its adjacent 2-vertex.

R(3) Every 4-vertex gives $\frac{1}{4}$ to each adjacent 3-vertex.

R(4) Every good 5-vertex gives:

R(4.1) $\frac{3}{2}$ to each adjacent 1-vertex.

R(4.2) $\frac{7}{6}$ to each adjacent 2-vertex if this 2-vertex is adjacent to a bad 5-vertex, otherwise 1 to each adjacent 2-vertex.

R(4.3) $\frac{1}{4}$ to each adjacent 3-vertex.

R(5) Every bad 5-vertex gives $\frac{5}{6}$ to each adjacent 2-vertex.

Let $f \in F(G)$ be a k-face. We have $k \geq 5$ by the condition on the girth. Note that if f has ι incident 1-vertices, then $k \geq 5 + 2\iota$. Since $ch(f) = d(f) - 5$, $ch'(f) \geq k - 5 - 2\iota \geq 0$ by R(1).

We next check the final charge of the vertex $v \in V(G)$.

Suppose v is a 1-vertex. Then v is adjacent to a 5-vertex by Lemma 9.1 and Lemma 9.8. Thus v receives 2 from its incident face and $\frac{3}{2}$ from its adjacent 5-vertex by R(1) and R(4.1). Hence, $ch'(v) \geq ch(v) + 2 + \frac{3}{2} = 0$.

Suppose v is a 2-vertex. Then v is adjacent to two 4^+-vertices by Lemma 9.2. If v is adjacent to at most one 5-vertex, then v receives 1 from its each neighbor by R(2) and R(4.2). Hence, $ch'(v) \geq ch(v) + 1 \times 2 = 0$. Otherwise, v is adjacent to two 5-vertices. And v is not adjacent to two bad 5-vertices by Lemma 10.2. Then v receives $\frac{7}{6}$ from its good 5-neighbor and $\frac{5}{6}$ from its bad 5-neighbor by R(4.2) and R(5). Hence, $ch'(v) \geq ch(v) + \frac{5}{6} + \frac{7}{6} = 0$.

Suppose v is a 3-vertex. Then v is adjacent to at least two 4^+-vertices by Lemma 9.3. So v receives $\frac{1}{4}$ from each neighbor by R(3) and R(4.3) and Lemma 10.4. Hence, $ch'(v) \geq ch(v) + \frac{1}{4} \times 2 = 0$.

Suppose v is a 4-vertex. Then v is adjacent to at most one 2-vertex and v has no other 3^--neighbor while v is adjacent to a 2-vertex by Lemma 9.4. Thus v gives 1 to its adjacent 2-vertex by R(2). Hence, $ch'(v) \geq ch(v) - 1 = 0$. If v has no 2-neighbor, v gives $\frac{1}{4}$ to each adjacent 3-vertex by R(3). Hence, $ch'(v) \geq ch(v) - \frac{1}{4} \times 4 = 0$.

Suppose v is a 5-vertex. If v is a bad 5-vertex, then v has no other 4^--neighbor by Lemma 10.4. Thus v gives $\frac{5}{6}$ to each adjacent 2-vertex by R(5). Hence, $ch'(v) \geq ch(v) - \frac{5}{6} \times 3 = 0$. Otherwise, we may assume v is a good 5-vertex. So v is adjacent to at most two 2-vertices. If v is adjacent to exactly two 2-vertices, v is adjacent to at most one another 3^--vertex by Lemma 9.7. In particular, if one of the two 2-neighbors of v is adjacent to a bad 5-vertex, then v has no other adjacent 3^--vertex by Lemma 10.3. So v gives $\frac{7}{6}$ to each adjacent 2-vertex by R(4.2). Hence, $ch'(v) \geq ch(v) - \frac{7}{6} \times 2 > 0$. Otherwise, v gives 1 to each adjacent 2-vertex and $\frac{1}{4}$ to its adjacent 3-vertex by R(4.2) and R(4.3). Hence, $ch'(v) \geq ch(v) - 1 \times 2 - \frac{1}{4} > 0$. If the number of 2-neighbors of v is at most one, then $ch'(v) \geq ch(v) - \max\{\frac{7}{6} + \frac{1}{4} \times 4, \frac{1}{4} \times 5\} > 0$ by R(4).

Therefore, we have $0 \leq \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) < 0$. This contradiction completes the proof of Theorem 8.

4. Discussion

As mentioned in introduction, we summarize all the results from Theorem 3 to Theorem 7 and show in the following table (see Table 1).

Girth	$\Delta \geq 7$	$\Delta = 6$	$\Delta = 5$	$\Delta = 4$
No girth restriction	4Δ	$4\Delta + 4$	$4\Delta + 4$	$4\Delta + 4$
$g \geq 4$	4Δ	4Δ	4Δ	$4\Delta + 4$
$g \geq 5$	$4\Delta - 2$	$4\Delta - 2$	$4\Delta - 1$	4Δ
$g \geq 6$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$

Table 1. Known upper bounds on the strong chromatic index of planar graphs mentioned in this paper.

Faudree *et al.* [6] presented a construction of planar graphs of girth 4 which satisfies $\chi'_s(G) \leq 4\Delta(G) - 4$. Thus, the bounds we have shown in this paper are not tight. However, we could easily obtain the following corollary.

Corollary 11. *Let G be a planar graph with girth $g = 5$ and $\Delta \in \{5, 6\}$. Then $\chi'_s(G) \leq 3\Delta(G) + 4$.*

In [10], Hudák *et al.* proposed the following conjecture.

Conjecture 12. *There exists a constant C such that for every planar graph G of girth k (where $k \geq 5$) $\chi'_s(G) \leq \left\lceil \frac{2k(\Delta(G)-1)}{k-1} \right\rceil + C$.*

Combined with the above conjecture and Theorem 5, we propose the following natural questions which we could not find an answer.

Question 1. Let G be a planar graph with girth $g = 5$ and $\Delta \in \{5, 6\}$. Is it true that $\chi'_s(G) \leq 3\Delta(G) + 1$?

Question 2. Let G be a planar graph with girth $g = 5$ and $\Delta \geq 7$. Is it true that $\chi'_s(G) \leq 3\Delta(G) + 4$?

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References

[1] L.D. Andersen, *The strong chromatic index of a cubic graph is at most* 10, Discrete Math. **108** (1992) 231–252.
doi:10.1016/0012-365X(92)90678-9

[2] J. Bensmail, A. Harutyunyan, H. Hocquard and P. Valicov, *Strong edge-colouring of sparse planar graphs*, Discrete Appl. Math. **179** (2014) 229–234.
doi:10.1016/j.dam.2014.07.006

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York-Amsterdam-Oxford, 1982).

[4] H. Bruhn and F. Joos, *A stronger bound for the strong chromatic index*, Electron. Notes Discrete Math. **49** (2015) 277–284.
doi:10.1016/j.endm.2015.06.038

[5] P. Erdős, *Problems and results in combinatorial analysis and graph theory*, Discrete Math. **72** (1988) 81–92.
doi:10.1016/0012-365X(88)90196-3

[6] R.J. Faudree, A. Gyárfás, R.H. Schelp and Zs. Tuza, *The strong chromatic index of graphs*, Ars Combin. **29B** (1990) 205–211.

[7] J.L. Fouquet and J.L. Jolivet, *Strong edge-colorings of graphs and applications to multi-k-gons*, Ars Combin. **16A** (1983) 141–150.

[8] J.L. Fouquet and J.L. Jolivet, *Strong edge-coloring of cubic planar graphs*, Progress in Graph Theory **111** (1984) 247–264.

[9] P. Horák, H. Qing and W.T. Trotter, *Induced matchings in cubic graphs*, J. Graph Theory **17** (1993) 151–160.
doi:10.1002/jgt.3190170204

[10] H. Hudák, B. Lužar, R. Soták and R. Škrekovski, *Strong edge-coloring of planar graphs*, Discrete Math. **324** (2014) 41–49.
doi:10.1016/j.disc.2014.02.002

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