Discussiones Mathematicae
Graph Theory 37 (2017) 711–727
doi:10.7151/dmgt.1950

KALEIDOSCOPIC COLORINGS OF GRAPHS

Gary Chartrand, Sean English and Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

e-mail: gary.chartrand@wmich.edu

Abstract

For an r-regular graph G, let $c : E(G) \to [k] = \{1, 2, \ldots, k\}$, $k \geq 3$, be an edge coloring of G, where every vertex of G is incident with at least one edge of each color. For a vertex v of G, the multiset-color $c_m(v)$ of v is defined as the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ or $a_1a_2 \cdots a_k$, where a_i $(1 \leq i \leq k)$ is the number of edges in G colored i that are incident with v. The edge coloring c is called k-kaleidoscopic if $c_m(u) \neq c_m(v)$ for every two distinct vertices u and v of G. A regular graph G is called a k-kaleidoscope if G has a k-kaleidoscopic coloring. It is shown that for each integer $k \geq 3$, the complete graph K_{k+3} is a k-kaleidoscope and the complete graph K_n is a 3-kaleidoscope for each integer $n \geq 6$. The largest order of an r-regular 3-kaleidoscope is $\binom{r-1}{2}$. It is shown that for each integer $r \geq 5$ such that $r \not\equiv 3 \pmod 4$, there exists an r-regular 3-kaleidoscope of order $\binom{r-1}{2}$.

Keywords: edge coloring, vertex coloring, kaleidoscopic coloring, kaleidoscope.

2010 Mathematics Subject Classification: 05C15, 05C75.

1. Introduction

A well-known observation in graph theory concerning the degrees of the vertices of a graph is that every nontrivial graph contains at least two vertices having the same degree. Indeed, it is known that for every integer $n \geq 2$, there are exactly two graphs of order n having exactly two vertices of the same degree and these two graphs are complements of each other. Consequently, in any decomposition of the complete graph K_n of order n into two graphs, necessarily into a graph G and its complement $\overline{G}$, there are at least two vertices u and v such that $\deg_G u = \deg_G v$

(and so $\deg_{\overline{G}} u = \deg_{\overline{G}} v$ as well). In particular, for every decomposition of a complete graph K_n into two graphs G_1 and G_2 (where then $G_2 = \overline{G}_1$) such that each vertex of K_n is incident with at least one edge in each of G_1 and G_2, there is associated with each vertex v of K_n an ordered pair (a, b) of positive integers with $a = \deg_{G_1} v$ and $b = \deg_{G_2} v$. Consequently, for each such decomposition of K_n, there are at least two vertices with the same ordered pair. In fact, this is not only true of decompositions of the complete graph into two graphs but decompositions of every regular graph into two graphs. Indeed, for a given regular graph G, there is a question of whether there exists a decomposition of G into $k \geq 3$ graphs $G_1, G_2, \ldots, G_k$ such that (1) each vertex of G is incident with at least one edge of every graph G_i and (2) for every two vertices u and v of G, $\deg_{G_i} u \neq \deg_{G_i} v$ for some i. By assigning the color i $(1 \leq i \leq k)$ to each edge of G_i, we are led to the following graph coloring concept.

For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$ denote the set of positive integers that are at most k. For an r-regular graph G, let $c : E(G) \to [k]$, $k \geq 3$, be an edge coloring of G, where every vertex of G is incident with at least one edge of each color. Thus, $r \geq k$. For a vertex v of G, the *set-color* $c_s(v)$ of v is defined as the set of colors of the edges incident with v. Thus, $c_s(v) = [k]$ for every vertex v of G. That is, each such edge coloring of G induces a *set-regular* vertex coloring of G (see [5, 370–376]). The *multiset-color* $c_m(v)$ of v is defined as the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ or $a_1 a_2 \cdots a_k$, where a_i $(1 \leq i \leq k)$ is the number of edges in G colored i that are incident with v. Hence, each a_i is a positive integer and $\sum_{i=1}^{k} a_i = r$. Such an edge coloring c is called a *k-kaleidoscopic coloring* of G if $c_m(u) \neq c_m(v)$ for every two distinct vertices u and v of G. That is, each such edge coloring of G induces a *multiset-irregular* vertex coloring of G (see [5, 376–379]). An edge coloring of G is a *kaleidoscopic coloring* if it is a k-kaleidoscopic coloring for some integer $k \geq 3$. Thus, a kaleidoscopic coloring is both *set-regular* and *multiset-irregular*. A regular graph G is called a *k-kaleidoscope* if G has a k-kaleidoscopic coloring. Figure 1 shows a 6-regular 3-kaleidoscope G of order 8 together with a 3-kaleidoscopic coloring of G, where the multiset-color of a vertex v is indicated inside the vertex v.

In 1880 Tait [7] thought of an edge coloring approach to solve the Four Color Problem. He proved that the edges of a bridgeless 3-regular planar graph G can be colored with three colors so that every two adjacent edges are colored differently if and only if the regions of G can be colored with four colors so that every two adjacent regions are colored differently. Although Tait's approach never led to a solution of the Four Color Problem, he was able to prove that such a 3-coloring of the edges of G induces an appropriate 4-coloring of the regions of G. Tait's theorem can be considered as the beginning of a class of problems in which some type of coloring in a graph gives rise to another type of coloring in the graph possessing a property of interest. In recent years, a variety of edge colorings have been introduced which induce, in a number of ways, vertex colorings possessing desirable properties (see [1, 2, 3, 4, 6], for example). We refer to the book [5] for graph theory notation and terminology not described in this paper.

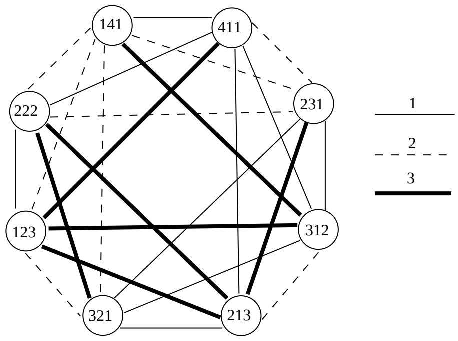

Figure 1. A 6-regular 3-kaleidoscope G of order 8.

Kaleidoscopic colorings can be used to model certain situations, one of which we now describe. Suppose that a hard-line network of n computers is to be constructed. Each of these computers requires k different types of connections. There are r locations on the back of each computer at which ports can be placed. Each computer needs to have at least one connection of each type and, for security reasons, no two computers can have more than one connection between them. In order to maximize the number of fail-safe connections, every port is to be used. Furthermore, it is advantageous for a computer technician to be able to distinguish the computers based only on the number of types of connections they have. For which values of n, k and r is such a situation possible?

2. Complete Kaleidoscopes

We begin with some observations. Let G be an r-regular k-kaleidoscope of order n. Then $k \leq r < n$. First, it is impossible that $r = k$, for otherwise, any edge coloring c of G in which every vertex of G is incident with at least one edge of each color results in $c_m(v)$ being the k-tuple in which each term is 1. If $r = k+1$, then there are at most k distinct k-tuples, each of which has 2 as one term and 1 for all other terms. In this case, $n \leq k$, which is impossible. Therefore, $r \geq k+2$. Since the number of r-element multisets M whose elements belong to a k-element set S is $\binom{r-1}{r-k}$, we have the following bounds involving k, r and n.

Proposition 2.1. *If G is an r-regular k-kaleidoscope of order n, then*

$$k + 2 \leq r < n \leq \binom{r-1}{r-k} = \binom{r-1}{k-1}.$$

Proof. Let c be a k-kaleidoscopic coloring of G. Since we have already observed that $r \geq k+2$, it remains to show that $n \leq \binom{r-1}{r-k}$. The number of r-element multisets whose elements belong to the k-element set $[k]$ such that each multiset contains at least one element i for each i $(1 \leq i \leq k)$ is $\binom{(r-k)+k-1}{r-k} = \binom{r-1}{r-k} = \binom{r-1}{k-1}$. Hence, $n \leq \binom{r-1}{k-1}$. ■

As Proposition 2.1 indicates, for a given integer $k \geq 3$, the smallest possible value of r for an r-regular k-kaleidoscope is $r = k+2$ and the smallest possible order n of such a graph is $n = r+1$. Obviously, the graph in question is the complete graph K_{k+3}. We show for each $k \geq 3$ that K_{k+3} is, in fact, a k-kaleidoscope.

Theorem 2.2. *For each integer $k \geq 3$, the complete graph K_{k+3} is a k-kaleidoscope.*

Proof. We consider two cases, according to whether k is odd or k is even.

Case 1. *k is odd.* Then $k = 2\ell+1$ for some positive integer ℓ. Thus, $k+3 = 2\ell+4$. It is known that $K_{2\ell+4}$ can be decomposed into $\ell+1$ Hamiltonian cycles $H_1, H_2, \ldots, H_{\ell+1}$ and a 1-factor F. For each i with $1 \leq i \leq \ell$, let there be given a proper coloring H_i with the two colors $2i-1$ and $2i$. Furthermore, we assign the color $2\ell+1$ to each edge of F. Currently, each vertex of $K_{2\ell+4}$ is incident with exactly one edge of each of the colors $1, 2, \ldots, 2\ell+1 = k$ and incident with exactly two edges in $H_{\ell+1}$ that have not yet been assigned any color.

Let $H_{\ell+1} = C = (v_1, v_2, \ldots, v_{2\ell+4}, v_{2\ell+5} = v_1)$. For $1 \leq i \leq \ell+2$, assign the color i to the two edges of C incident with v_{2i}. This completes the edge coloring c of $K_{2\ell+4}$ and results in $c_s(v_i) = [k]$ for $1 \leq i \leq k+3$ and $c_m(v_i)$ equaling the multiset M_i, where

- M_1 contains two elements 1, two elements $\ell+2$ and one element of $[k] - \{1, \ell+2\}$;
- M_{2i+1} contains two elements i, two elements $i+1$ and exactly one element of $[k] - \{i, i+1\}$ for $1 \leq i \leq \ell+1$;
- M_{2i} contains three elements i and one element of $[k] - \{i\}$ for $1 \leq i \leq \ell+2$.

Thus, c is set-regular and multiset-irregular and so K_{k+3} is a k-kaleidoscope when k is odd.

Case 2. *k is even.* Then $k = 2\ell+2$ for some positive integer ℓ. Thus, $k+3 = 2\ell+5$. Let $G = K_{2\ell+5}$, let $v \in V(G)$ and let $G' = G - v = K_{2\ell+4}$. As in Case 1, the graph G' can be decomposed into $\ell+1$ Hamiltonian cycles $H_1, H_2, \ldots, H_{\ell+1}$ and a 1-factor F. Color the edges of $H_1, H_2, \ldots, H_\ell$ and F as in Case 1. At this point, each vertex of G' is incident with exactly one edge of each of the colors $1, 2, \ldots, 2\ell+1$ (and no edges colored $2\ell+2$) and incident with two edges in $H_{\ell+1}$ that have not yet been assigned any color.

Let $H_{\ell+1} = C = (v_1, v_2, \ldots, v_{2\ell+4}, v_1)$. For $1 \le i \le \ell + 2$, assign the color i to the edge $v_{2i-1}v_{2i}$ and the color $2\ell + 2$ to all other edges of C. This completes the edge coloring c' of $G' = K_{2\ell+4}$ and results in $c'_s(v_j) = [k]$ for $1 \le j \le 2\ell + 4$ and $c'_m(v_j) = M'_j$ for $1 \le j \le 2\ell + 4$ where $M'_{2i-1} = M'_{2i}$ contains two elements i and one element of $[k] - \{i\}$. We now consider the graph G. The edge coloring $c : E(G) \to [k]$ is defined by

$$c(e) = \begin{cases} c'(e) & \text{if } e \in E(G'), \\ i & \text{if } e = vv_{2i-1} \text{ for } 1 \le i \le \ell + 2, \\ \ell + 2 + i & \text{if } e = vv_{2i} \text{ for } 1 \le i \le \ell - 1, \\ 2\ell + 2 & \text{if } e = vv_{2i} \text{ for each } i \in \{\ell, \ell + 1, \ell + 2\}. \end{cases}$$

This completes the edge coloring c of G and results in $c_s(x) = [k]$ for all $x \in V(G)$, $c_m(v_i) = M_i$ and $c_m(v) = M$, where

- M_{2i-1} is the only multiset containing three elements i for $1 \le i \le \ell + 2$,
- M_{2i} is the only multiset containing exactly two elements i for $1 \le i \le \ell + 2$ and
- M is the only multiset containing exactly one element i for each i with $1 \le i \le \ell + 2$.

This is illustrated in Figure 2 for $k = 10$, where $c'_m(v_i) = M'_i$ is indicated inside the cycle C and $c_m(v_i) = M_i$ is indicated outside the cycle C for each i with $1 \le i \le 12$. Thus, c is set-regular and multiset-irregular and so $G = K_{k+3}$ is a k-kaleidoscope when k is even. ■

Not only is K_{k+3} a k-kaleidoscope but it is believed that every larger complete graph is also a k-kaleidoscope.

Conjecture 2.3. *For integers n and k with $n \ge k + 3 \ge 6$, the complete graph K_n is a k-kaleidoscope.*

In the case where $k = 3$, Conjecture 2.3 suggests that K_n is a 3-kaleidoscope when $n \ge 6$. We verify this special case of the conjecture. First, we make an observation. It is sometimes useful to look at kaleidoscopic colorings from another point of view. For a connected graph G of order $n \ge 3$ and a k-tuple factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of G, where each F_i has no isolated vertices for $1 \le i \le k$, we associate the ordered k-tuple $a_1 a_2 \cdots a_k$ with a vertex v of G where $\deg_{F_i} v = a_i$ for $1 \le i \le k$. Thus $\sum_{i=1}^{k} \deg_{F_i} v = \deg_G v$. If distinct vertices have distinct k-tuples, then we can assign the color i ($1 \le i \le k$) to each edge of F_i and obtain a k-kaleidoscopic coloring of G for which the multiset-color $c_m(v)$ of v is $a_1 a_2 \cdots a_k$. In this case, the factorization $\mathcal{F}$ is called *irregular*. Conversely, every k-kaleidoscopic coloring of G gives rise to an irregular k-tuple factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of G where the edges of F_i are those edges of G colored i

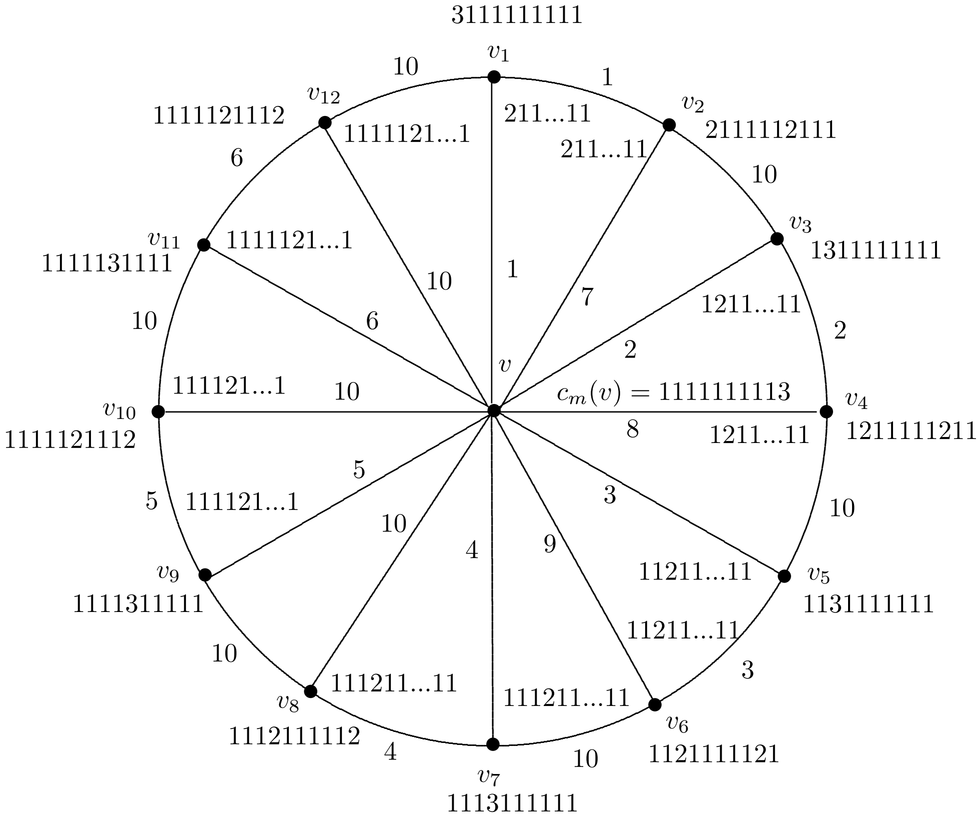

Figure 2. Illustrating a 10-kaleidoscopic coloring of K_{13} in the proof of Theorem 2.2.

and each F_i has no isolated vertices for $1 \leq i \leq k$. Hence, an edge coloring of a graph G is a kaleidoscopic coloring if and only if the corresponding factorization of G is irregular. Therefore, a graph G has a k-kaleidoscopic coloring if and only if G has an irregular k-tuple factorization.

Theorem 2.4. *For each integer $n \geq 6$, the complete graph K_n is a 3-kaleidoscope.*

Proof. By Theorem 2.2, K_6 is a 3-kaleidoscope. Figure 3 shows that K_7 and K_8 are also 3-kaleidoscopes. Hence, we may assume that $n \geq 9$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and let F be the unique connected graph of order n containing exactly two vertices with equal degree. Without loss of generality, we may assume that

$$\deg_F v_i = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ i-1 & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n. \end{cases}$$

Thus, $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lfloor \frac{n}{2} \rfloor + 1}$ are the only two vertices of F having the same degree $\lfloor \frac{n}{2} \rfloor$. To define an irregular factorization $\{F_1, F_2, F_3\}$ of K_n, we consider two cases, according to whether n is even or n is odd.

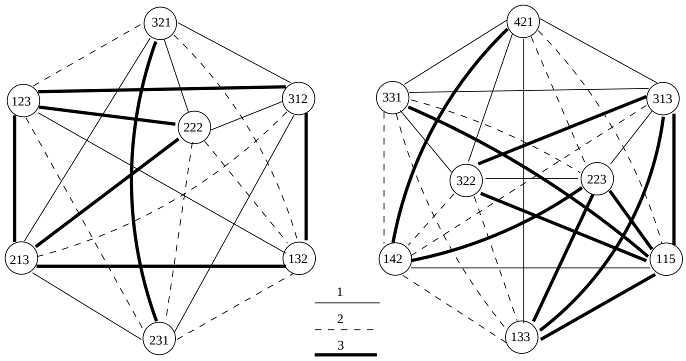

Figure 3. A 3-kaleidoscopic coloring for each of K_7 and K_8.

Case 1. *n is even.* Then $n = 2p$ for some integer $p \geq 5$. Let

$$M = \{v_2v_{n-1}, v_3v_{n-2}, \ldots, v_pv_{p+1}\} = \{v_iv_{n-i+1} : 2 \leq i \leq p\}$$

be a matching of size $p - 1$ in F. Let

$$\begin{aligned} F_1 &= F - \{v_3v_n, v_4v_n, v_pv_n\} - M, \\ F_2 &= \overline{F} + v_pv_n - \{v_1v_3, v_1v_4, v_1v_p, v_3v_4\}, \\ F_3 &= K_n - E(F_1) - E(F_2). \end{aligned}$$

Hence, $E(F_3) = \{v_1v_3, v_1v_4, v_1v_p, v_3v_4\} \cup \{v_3v_n, v_4v_n\} \cup M$. In F_1, $\deg_{F_1} v_1 = \deg_{F_1} v_2 = \deg_{F_1} v_3 = 1$ and $\deg_{F_1} v_{n-2} = \deg_{F_1} v_n = n - 4$, while the remaining vertices have distinct degrees in F_1. Since

$$\begin{aligned} &\deg_{F_2} v_1 = n - 5,\ \deg_{F_2} v_2 = n - 3,\ \deg_{F_2} v_3 = n - 6, \\ &\deg_{F_3} v_{n-2} = 1 \text{ and } \deg_{F_3} v_n = 2, \end{aligned}$$

it follows that $\{F_1, F_2, F_3\}$ is an irregular factorization of K_n. By assigning color i to each edge in F_i for $i = 1, 2, 3$, we obtain a 3-kaleidoscopic coloring c for K_n. Figure 4 illustrates such a factorization for K_{10}, where the bold edges in F and in $\overline{F}$ play a special role in the creation of F_1, F_2, F_3. With this coloring c, the multiset-colors of the vertices of K_{10} are

$$\begin{aligned} &c_m(v_1) = 153,\ c_m(v_2) = 171,\ c_m(v_3) = 144,\ c_m(v_4) = 234,\ c_m(v_5) = 342, \\ &c_m(v_6) = 441,\ c_m(v_7) = 531,\ c_m(v_8) = 621,\ c_m(v_9) = 711,\ c_m(v_{10}) = 612. \end{aligned}$$

Case 2. *n is odd.* Then $n = 2p + 1$ for some integer $p \geq 4$. Let

$$M = \{v_2v_{n-1}, v_3v_{n-2}, \ldots, v_pv_{p+2}\} = \{v_iv_{n-i+1} : 2 \leq i \leq p\}$$

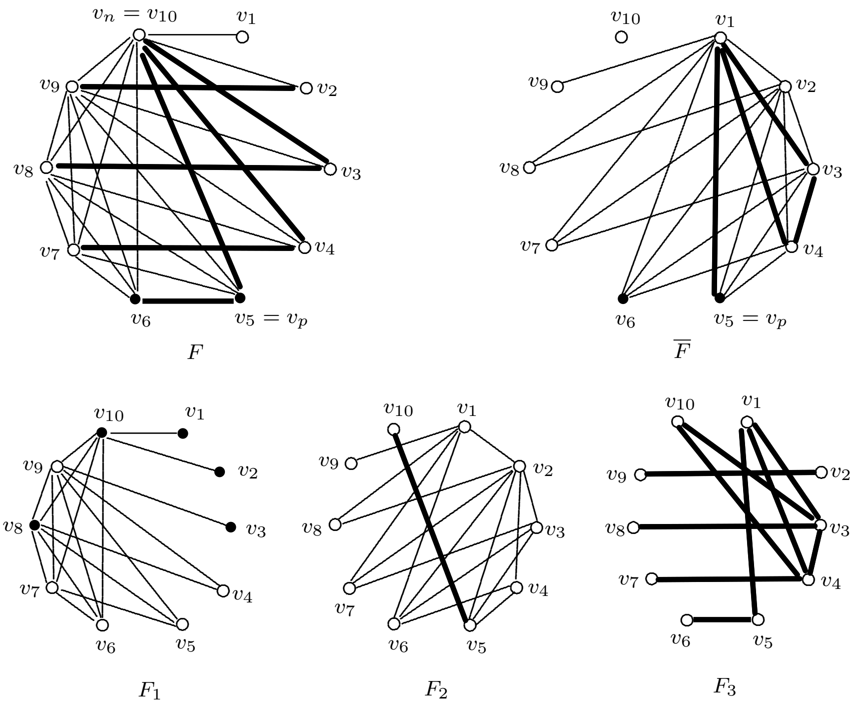

Figure 4. An irregular factorization $\mathcal{F} = \{F_1, F_2, F_3\}$ of K_{10}.

be a matching of size $p-1$ in F. Let

$$F_1 = F - \{v_3v_n, v_4v_n, v_{p+1}v_n\} - M,$$
$$F_2 = \overline{F} + v_{p+1}v_n - \{v_1v_3, v_1v_4, v_1v_{p+1}, v_2v_{p+1}, v_3v_4\},$$
$$F_3 = K_n - E(F_1) - E(F_2).$$

Hence, $E(F_3) = \{v_1v_3, v_1v_4, v_1v_{p+1}, v_2v_{p+1}, v_3v_4\} \cup \{v_3v_n, v_4v_n\} \cup M$. Observe that

(1) $\deg_{F_1} v_1 = \deg_{F_1} v_2 = \deg_{F_1} v_3 = 1$,
(2) if $n = 9$ (or $p = 4$), then $\deg_{F_1} v_4 = 2$ and $\deg_{F_1} v_5 = 3$; while if $n \geq 11$ (or $p \geq 5$), then $\deg_{F_1} v_p = \deg_{F_1} v_{p+1} = p - 1$,
(3) $\deg_{F_1} v_{n-2} = \deg_{F_1} v_n = n - 4$ and
(4) the remaining vertices have distinct degrees in F_1.

Since

$$\deg_{F_2} v_1 = n-5,\ \deg_{F_2} v_2 = n-4,\ \deg_{F_2} v_3 = n-6,$$
$$\deg_{F_2} v_p = p,\ \deg_{F_2} v_{p+1} = p-1,\ \deg_{F_3} v_{n-2} = 1 \text{ and } \deg_{F_3} v_n = 2,$$

it follows that $\{F_1, F_2, F_3\}$ is an irregular factorization of K_n. By assigning color i to each edge in F_i for $i = 1, 2, 3$, we obtain a 3-kaleidoscopic coloring c for K_n.

Figure 5 illustrates such a factorization for K_{11}, where the bold edges in F and in $\overline{F}$ play a special role in the creation of F_1, F_2, F_3. With this coloring c, the multiset-colors of the vertices of K_{11} are

$$c_m(v_1) = 163,\ c_m(v_2) = 172,\ c_m(v_3) = 154,\ c_m(v_4) = 244,$$
$$c_m(v_5) = 451\ c_m(v_6) = 442,\ c_m(v_7) = 541,\ c_m(v_8) = 631,$$
$$c_m(v_9) = 721,\ c_m(v_{10}) = 811,\ c_m(v_{11}) = 712.$$

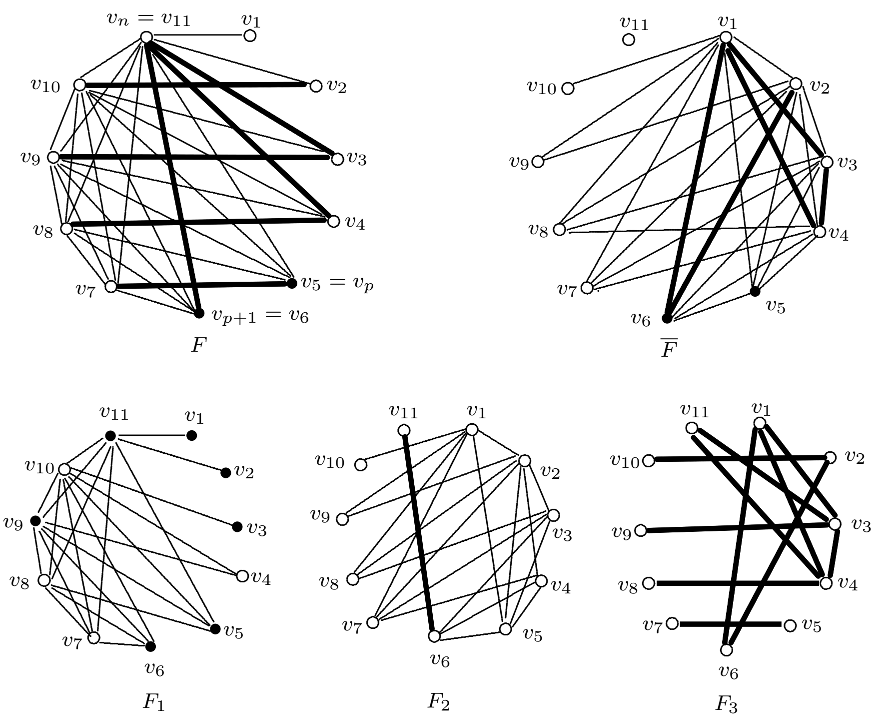

Figure 5. An irregular factorization $\{F_1, F_2, F_3\}$ of K_{11}.

Therefore, the complete graph K_n is a 3-kaleidoscope for each integer $n \geq 6$. ■

3. 3-Kaleidoscopes of Maximum Order

According to Proposition 2.1, the largest possible order of an r-regular 3-kaleidoscope is $\binom{r-1}{2}$. If $r \geq 7$ is an odd integer such that $r \equiv 3 \pmod 4$, then $\binom{r-1}{2}$ is odd and so no r-regular graphs of order $\binom{r-1}{2}$ exist for such odd integers r. On the other hand, there exists an r-regular 3-kaleidoscope of order $\binom{r-1}{2}$ for every integer $r \geq 5$ when $r \not\equiv 3 \pmod 4$.

Theorem 3.1. *For each integer $r \geq 5$ such that $r \not\equiv 3 \pmod 4$, there exists an r-regular 3-kaleidoscope of order $\binom{r-1}{2}$.*

Proof. Let $r \geq 5$ be an integer with $r \not\equiv 3 \pmod 4$; so either r is even or $r \equiv 1 \pmod 4$. We consider these two cases.

Case 1. *$r \geq 6$ is even.* We begin by constructing an r-regular graph G_r of order $\binom{r-1}{2}$. For $1 \leq i \leq r-2$, let V_i be a set of $r-1-i$ vertices and let $V(G_r) = \bigcup_{i=1}^{r-2} V_i$. Thus, the order of G_r is

$$|V(G_r)| = \sum_{i=1}^{r-2} |V_i| = \sum_{i=1}^{r-2} (r-i-1) = \sum_{i=1}^{r-2} i = \binom{r-1}{2}.$$

For $1 \leq i \leq r-2$, let $V_i = \{v_{i,j} : 1 \leq j \leq r-1-i\}$. The vertices of G_r are placed in a triangular array such that for each i with $1 \leq i \leq r-2$, the vertices of each set V_i are placed in a row where consecutive vertices are equally spaced two units apart, such that $V_1, V_2, \ldots, V_{r-2}$ are placed from top to bottom with each successive row of vertices one unit below the preceding. For $r = 6$, the vertices of $G_r = G_6$ are thus drawn as indicated in Figure 6.

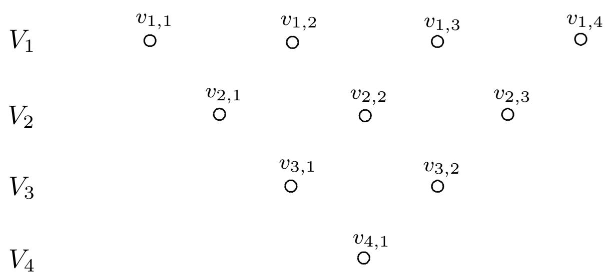

Figure 6. The location of the vertices of the graph G_6.

For $1 \leq i \leq r-2$, we now construct a subgraph H_i of G_r with vertex set V_i. The graph H_{r-2} is the trivial graph; while for $1 \leq i \leq r-3$, the graph H_i is the unique connected graph of order $r-1-i$ containing exactly two vertices of the same degree such that

$$\deg_{H_i} v_{i,1} \leq \deg_{H_i} v_{i,2} \leq \cdots \leq \deg_{H_i} v_{i,r-1-i}.$$

For $1 \leq i \leq r-2$, let

$$\begin{aligned} U_i &= \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lceil (r-2-i)/2 \rceil}\}, \\ W_i &= \{v_{i,\lceil (r-2-i)/2 \rceil}, v_{i,\lceil (r-2-i)/2 \rceil + 1}, \ldots, v_{i,r-1-i}\}. \end{aligned}$$

Then $H_i[U_i]$ is empty and $H_i[W_i]$ is complete. We now add additional edges to obtain a subgraph of G_r, which we denote by F_2. Since $r \geq 6$ is even, it follows that $r = 2p + 2$ for some integer $p \geq 2$. Let A be an independent set of "slanted edges", defined by

$$A = \begin{cases} \{v_{1,p+2i-1}\ v_{2,p+2i-2} : 1 \leq i \leq (p+1)/2\} & \text{if } p \text{ is odd,} \\ \{v_{1,p+2i}\ v_{2,p+2i-1} : 1 \leq i \leq p/2\} & \text{if } p \text{ is even.} \end{cases}$$

For each integer p and $1 \leq j \leq \lfloor p/2 \rfloor$, let E_j be an independent set of "vertical edges", defined by

$$E_j = \{v_{i,2p+2-i-2j}\ v_{i+2,2p+1-i-2j} : 1 \leq i \leq 2p+2-4j\}.$$

We now add the edges in A and the edges in E_j $(1 \leq j \leq \lfloor p/2 \rfloor)$ to those in the subgraphs H_i $(1 \leq i \leq r-2)$ where all edges are straight line segments. All edges in each subgraph H_i are therefore "horizontal edges" for $1 \leq i \leq r-3$. This completes the construction of F_2. The graph F_2 is illustrated for both $r = 6$ and $r = 8$ in Figure 7, where each slanted edge is indicated by a bold line and each vertical edge is indicated by a dashed line. Observe that $\deg_{F_2} v_{i,j} = j$ for all i and j.

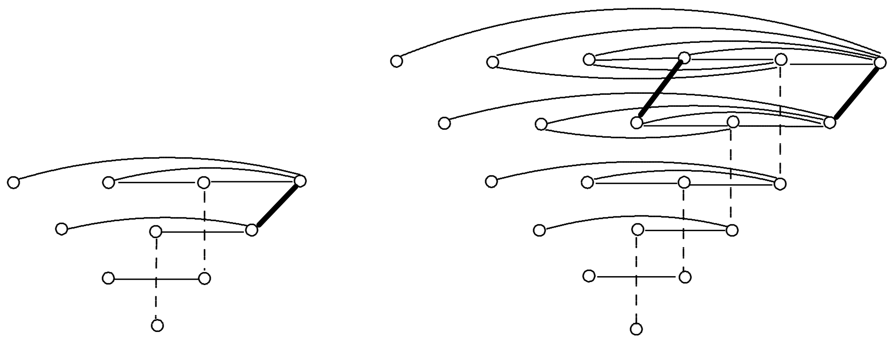

Figure 7. The subgraph F_2 for $r = 6$ and $r = 8$.

The subgraph F_1 is obtained by rotating the subgraph F_2 clockwise through an angle of $2\pi/3$ radians; while the subgraph F_3 is obtained by rotating the subgraph F_2 counter-clockwise through an angle of $2\pi/3$ radians (or by rotating the subgraph F_2 clockwise through an angle of $4\pi/3$ radians). This completes the construction of G_r. This is illustrated in Figure 8 for G_6, where each edge in F_1 is indicated by a thin solid line, each edge in F_2 is indicated by a dashed line and each edge in F_3 is indicated by a bold line. In Figure 8, the label ijk inside the vertex $v_{i,j}$ indicates that $\deg_{F_1} v_{i,j} = i$, $\deg_{F_2} v_{i,j} = j$ and $\deg_{F_3} v_{i,j} = k = 6 - (i + j)$.

We now verify that in the rotation of F_2 into F_1 and F_3, every pair of adjacent vertices in F_2 is rotated into a pair of nonadjacent vertices of F_2 and, consequently, G_r is decomposed into F_1, F_2 and F_3. Let each vertex $v_{i,j}$ of G_r be denoted by $v_{i,j,k}$ as well, where $k = r - (i + j)$. Then the vertex set of G_r can be decomposed

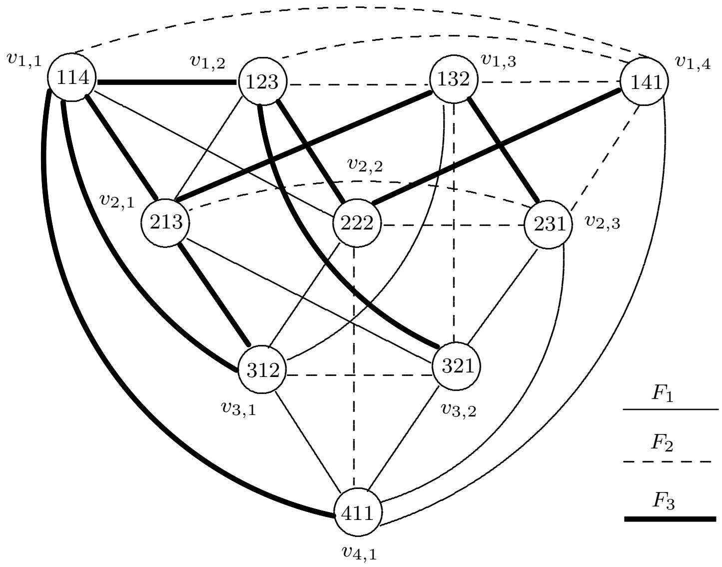

Figure 8. The subgraphs F_1, F_2 and F_3 in G_6.

into $\lfloor (r-1)/3 \rfloor$ triangular sets X_ℓ $(1 \le \ell \le \lfloor (r-1)/3 \rfloor)$ of vertices. For each such ℓ, the set X_ℓ consists of all those vertices $v_{i,j,k}$ such that $\ell = \min\{i,j,k\}$. In Figure 9, the sets X_1, X_2, X_3 are shown for G_{10}; while X_1 and X_2 are shown for G_8, where the vertices in X_1 are indicated by solid vertices, the vertices in X_2 are indicated by open vertices and the vertices in X_3 are indicated by double-circled vertices. The order of G_{10}, for example, is $\binom{9}{2} = 36 = \sum_{i=1}^{8} i = 1+2+3+4+5+6+7+8$. The number of vertices in X_1 is $8+7+6 = 21$, in X_2 is $5+4+3 = 12$ and in X_3 is $2+1+0 = 3$.

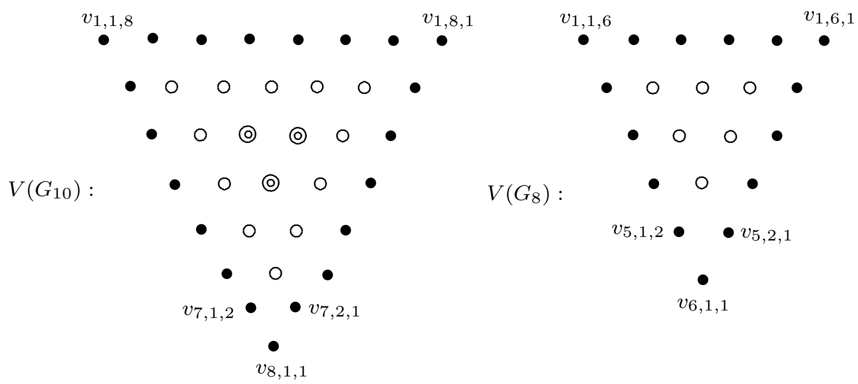

Figure 9. The triangular sets X_ℓ for G_{10} and G_8.

The edges of G_r that join two vertices of X_ℓ $(1 \le \ell \le \lfloor (r-1)/3 \rfloor)$ are those in H_ℓ (together with the edge $v_{1,r-2,1}v_{2,r-3,1}$ for $\ell = 1$), which is illustrated in Figure 10. Thus, during the rotations of F_2 into F_1 and F_3, every two adjacent vertices of F_2 are rotated into nonadjacent vertices in F_1 or F_3. All other edges of F_2 join two different triangular sets.

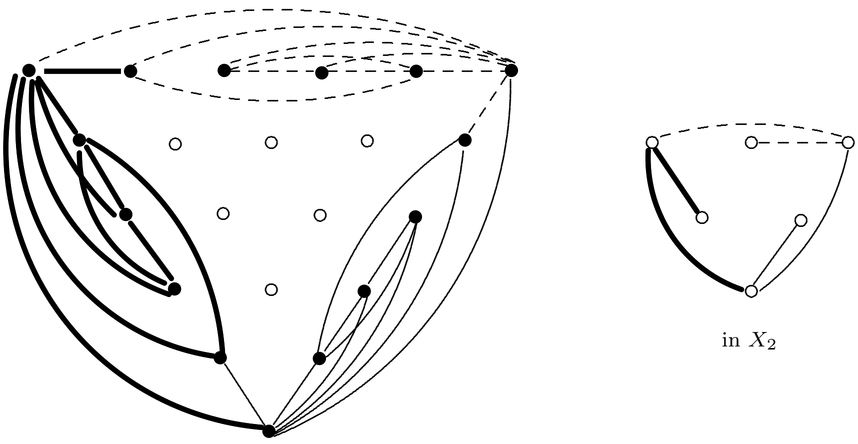

Figure 10. Edges in the two triangular sets X_1 and X_2 for G_8.

The "vertical edges" in F_2 are rotated into edges of slopes -3 and 3; while the "slanted edges" are rotated into edges of slopes -1 and 0. This is illustrated in Figure 11. For example, the vertical edge $v_{1,5}v_{3,4}$ in F_2 is rotated into the edge $v_{4,1}v_{5,2}$ in F_1 (of slope -3) and the edge $v_{1,3}v_{2,1}$ in F_3 (of slope 3). Furthermore, the slanted edge $v_{1,6}v_{2,5}$ in F_2 is rotated into the edge $v_{5,1}v_{6,1}$ in F_1 (of slope -1) and the edge $v_{1,1}v_{1,2}$ in F_3 (of slope 0). Hence, each edge of G_r belongs to exactly one of F_1, F_2, F_3.

By the construction of F_1, F_2 and F_3, it follows that

$$\deg_{F_t} v_{i,j,k} = \begin{cases} i & \text{if } t = 1, \\ j & \text{if } t = 2, \\ k & \text{if } t = 3. \end{cases}$$

Furthermore, if $\deg_{F_2} v_{i,j,k} = \deg_{F_2} v_{a,b,c}$ (and so $j = b$), then $\deg_{F_1} v_{i,j,k} \neq \deg_{F_1} v_{a,b,c}$ (that is, $i \neq a$). Hence, $\{F_1, F_2, F_3\}$ is an irregular factorization of G_r. Assigning the color t $(1 \le t \le 3)$ to each edge of F_t, we obtain a 3-kaleidoscopic coloring of G_r for which the multiset-color $c_m(v_{i,j,k}) = ijk$ for all triples i, j, k. Therefore, G_r is an r-regular 3-kaleidoscope.

Case 2. $r \ge 5$ *and* $r \equiv 1 \pmod 4$. Since K_6 is a 5-regular 3-kaleidoscope by Theorem 2.4, we may assume that $r \ge 9$. We now construct an r-regular graph

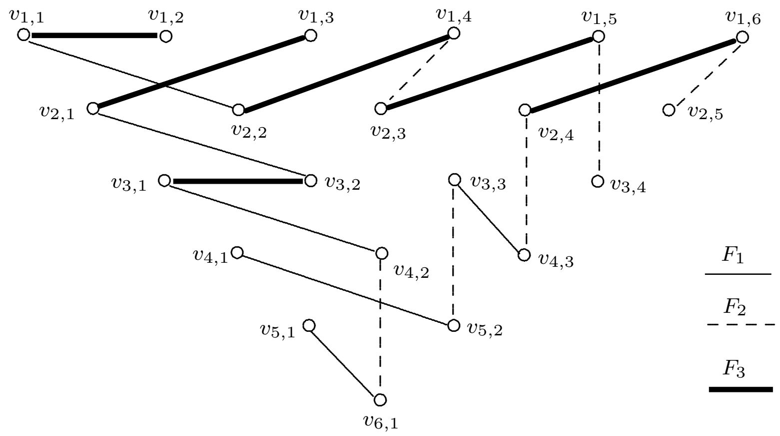

Figure 11. Vertical and slanted edges in G_8.

G_r of order $\binom{r-1}{2}$. Although the proof of this case is similar to the one employed in Case 1, we provide a discussion here for completion. For $1 \le i \le r-2$, let V_i be a set of $r-1-i$ vertices and let $V(G_r) = \bigcup_{i=1}^{r-2} V_i$. Thus, the order of G_r is $\binom{r-1}{2}$. The vertices of G_r are placed in a triangular array such that for each i with $1 \le i \le r-2$, the vertices of each set V_i are placed in a row where consecutive vertices are equally spaced two units apart, such that $V_1, V_2, \ldots, V_{r-2}$ are placed from top to bottom with each successive row of vertices one unit below the preceding.

As proceed in Case 1, for $1 \le i \le r-2$, we now construct a subgraph H_i of G_r with vertex set V_i. The graph H_{r-2} is the trivial graph; while for $1 \le i \le r-3$, the graph H_i is the unique connected graph of order $r-1-i$ containing exactly two vertices of the same degree such that

$$\deg_{H_i} v_{i,1} \le \deg_{H_i} v_{i,2} \le \cdots \le \deg_{H_i} v_{i,r-1-i}.$$

We now add additional edges to obtain a subgraph of G_r, which we denote by F_2. Since $r \ge 9$ and $r \equiv 1 \pmod 4$, it follows that $r = 4p+1$ for some integer $p \ge 2$. Let A be an independent set of "slanted edges", defined by

$$A = \{v_{1,2p+2i+1}\ v_{2,2p+2i} : 1 \le i \le p-1\}.$$

For each integer p and $1 \le j \le p$, let E_j be an independent set of "vertical edges", defined by

$$E_j = \{v_{i,4p+1-i-2j}\ v_{i+2,4p-i-2j} : 1 \le i \le 4p+1-4j\}.$$

We now add the edges in A and the edges in E_j $(1 \le j \le p)$ to those in the subgraphs H_i $(1 \le i \le r-2)$ where all edges are straight line segments. All edges in each subgraph H_i are therefore "horizontal edges" for $1 \le i \le r-3$.

This completes the construction of F_2. The graph F_2 is illustrated for $r = 9$ in Figure 12, where each slanted edge is indicated by a bold line and each vertical edge is indicated by a dashed line. Observe that $\deg_{F_2} v_{i,j} = j$ for all i and j.

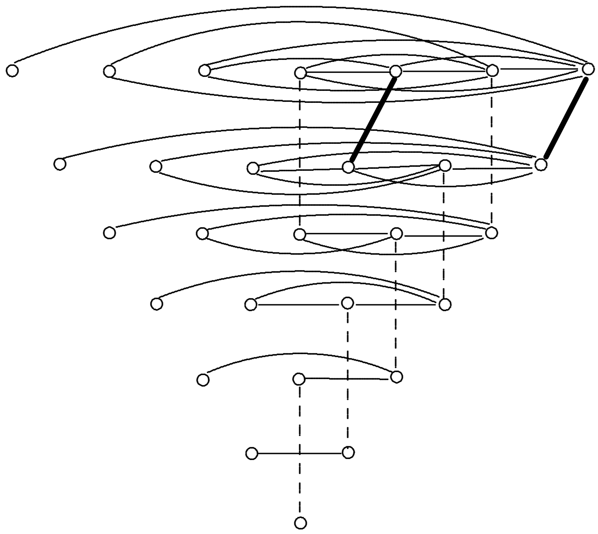

Figure 12. The subgraph F_2 for $r = 9$.

The subgraph F_1 is obtained by rotating the subgraph F_2 clockwise through an angle of $2\pi/3$ radians; while the subgraph F_3 is obtained by rotating the subgraph F_2 counter-clockwise through an angle of $2\pi/3$ radians (or by rotating the subgraph F_2 clockwise through an angle of $4\pi/3$ radians). This completes the construction of G_r.

Next, we show that in the rotation of F_2 into F_1 and F_3, every pair of adjacent vertices in F_2 is rotated into a pair of nonadjacent vertices of F_2 and, consequently, G_r is decomposed into F_1, F_2 and F_3. Let each vertex $v_{i,j}$ of G_r be denoted by $v_{i,j,k}$ as well, where $k = r - (i + j)$. Then the vertex set of G_r can be decomposed into $\lfloor r/3 \rfloor$ triangular sets X_ℓ $(1 \le \ell \le \lfloor r/3 \rfloor)$ of vertices. For each such ℓ, the set X_ℓ consists of all those vertices $v_{i,j,k}$ such that $\ell = \min\{i, j, k\}$. The order of G_9, for example, is $\binom{8}{2} = 28 = \sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7$. The number of vertices in X_1 is $7 + 6 + 5 = 18$, in X_2 is $4 + 3 + 2 = 9$ and in X_3 is 1.

The edges of G_r that join two vertices of X_ℓ $(1 \le \ell \le \lfloor r/3 \rfloor)$ are those in H_ℓ (together with the edge $v_{1,r-2,1}v_{2,r-3,1}$ for $\ell = 1$), which is illustrated in Figure 13 for X_1. Thus, during the rotations of F_2 into F_1 and F_3, every two adjacent vertices of F_2 are rotated into nonadjacent vertices in F_1 or F_3. All other edges of F_2 join two different triangular sets.

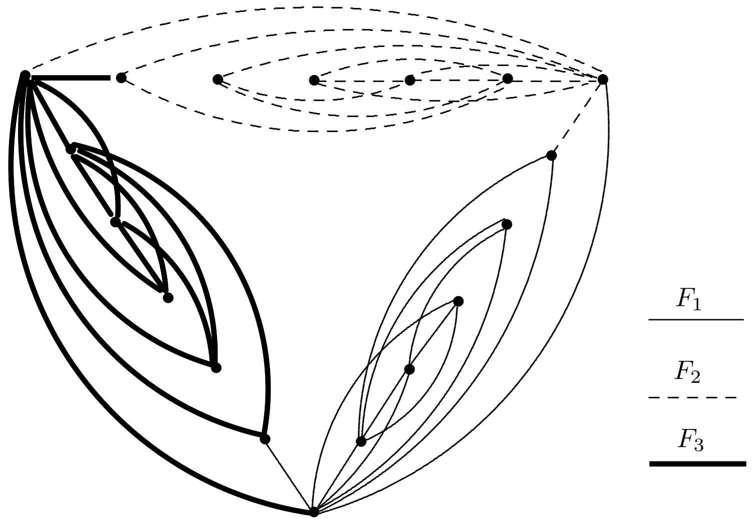

Figure 13. Edges in the triangular set X_1 for G_9.

The "vertical edges" in F_2 are rotated into edges of slopes -3 and 3; while the "slanted edges" are rotated into edges of slopes -1 and 0. This is illustrated in Figure 14. For example, the vertical edge $v_{1,6}v_{3,5}$ in F_2 is rotated into the edge $v_{5,1}v_{6,2}$ in F_1 (of slope -3) and the edge $v_{1,3}v_{2,1}$ in F_3 (of slope 3). Furthermore, the slanted edge $v_{1,7}v_{2,6}$ in F_2 is rotated into the edge $v_{6,1}v_{7,1}$ in F_1 (of slope -1) and the edge $v_{1,1}v_{1,2}$ in F_3 (of slope 0). Hence, each edge of G_r belongs to exactly one of F_1, F_2, F_3. An argument similar to the one in Case 1 shows that G_r is an r-regular 3-kaleidoscope. ■

Acknowledgment

We greatly appreciate the valuable suggestions made by the referees that resulted in an improved paper.

References

[1] M. Aigner, E. Triesch and Zs. Tuza, *Irregular assignments and vertex-distinguishing edge-colorings of graphs*, Combinatorics' 90 Elsevier Science Pub., New York (1992) 1–9.

[2] A C. Burris and R.H. Schelp, *Vertex-distinguishing proper edge colorings*, J. Graph Theory **26** (1997) 73–82.
doi:10.1002/(SICI)1097-0118(199710)26:2⟨73::AID-JGT2⟩3.0.CO;2-C

[3] J. Černý, M. Horňák and R. Soták, *Observability of a graph*, Math. Slovaca **46** (1996) 21–31.

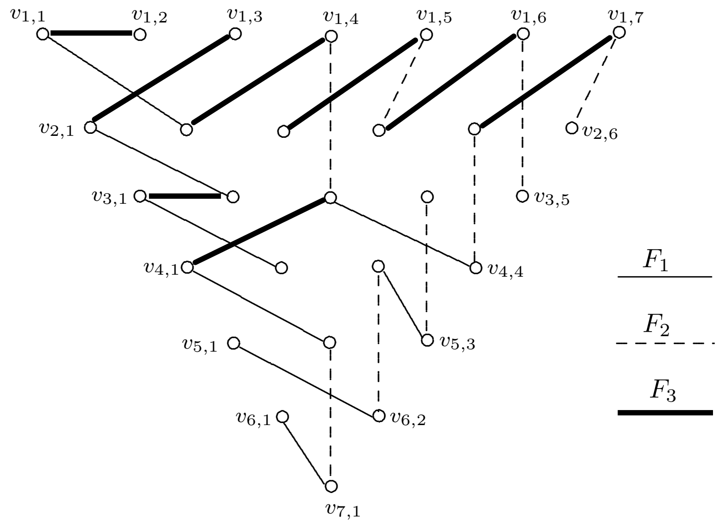

Figure 14. Vertical and slanted edges in G_9.

[4] G. Chartrand. S. English and P. Zhang, *Binomial colorings of graphs*, Bull. Inst. Combin. Appl. **76** (2016) 69–84.

[5] G. Chartrand and P. Zhang, Chromatic Graph Theory (Chapman & Hall/CRC Press, Boca Raton, FL, 2009).

[6] O. Favaron, H. Li and R. H. Schelp, *Strong edge colorings of graphs*, Discrete Math. **159** (1996) 103–109.
doi:10.1016/0012-365X(95)00102-3

[7] P.G. Tait, *Remarks on the colouring of maps*, Proc. Royal Soc. Edinburgh **10** (1880) 501–503, 729.
doi:10.1017/S0370164600044643

Received 11 September 2015
Revised 18 April 2016
Accepted 13 June 2016