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FORBIDDEN PAIRS AND (k, m)-PANCYCLICITY

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Abstract

A graph G on n vertices is said to be (k, m)-*pancyclic* if every set of k vertices in G is contained in a cycle of length r for each $r \in \{m, m+1, \ldots, n\}$. This property, which generalizes the notion of a vertex pancyclic graph, was defined by Faudree, Gould, Jacobson, and Lesniak in 2004. The notion of (k, m)-pancyclicity provides one way to measure the prevalence of cycles in a graph. We consider pairs of subgraphs that, when forbidden, guarantee hamiltonicity for 2-connected graphs on $n \geq 10$ vertices. There are exactly ten such pairs. For each integer $k \geq 1$ and each of eight such subgraph pairs $\{R, S\}$, we determine the smallest value m such that any 2-connected $\{R, S\}$-free graph on $n \geq 10$ vertices is guaranteed to be (k, m)-pancyclic. Examples are provided that show the given values are best possible. Each such example we provide represents an infinite family of graphs.

Keywords: hamiltonian, pancyclic, forbidden subgraph, cycle, claw-free.

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1. Introduction

Let $G = (V, E)$ denote a simple graph of order $n \geq 3$. We say G is *pancyclic* if G contains a cycle of each possible length, from 3 up to n. The notion of vertex pancyclicity was defined by Bondy in [2]. The graph G is *vertex pancyclic* if every vertex of G is contained in a cycle of each possible length. We consider the property (k, m)-pancyclicity, defined in 2004 by Faudree *et al.* [8], which is a generalization of vertex pancyclicity.

Definition (Faudree, Gould, Jacobson and Lesniak [8])**.** Given integers k and m with $0 \leq k \leq m \leq n$, a graph G of order n is said to be (k, m)-*pancyclic* if for any k-set $S \subseteq V$ and any integer r with $m \leq r \leq n$, there exists a cycle of length r in G that contains S.

Whenever $m > n$ or $k > n$, we define (k, m)-pancyclicity to be the same as hamiltonicity. Note that (k, n)-pancyclicity is hamiltonicity, $(0, 3)$-pancyclicity represents pancyclicity, and $(1, 3)$-pancyclicity is vertex pancyclicity. Note also that whenever a graph is (k, m)-pancyclic for some $k \geq 1$, then it must also be $(k-1, m)$-pancyclic and $(k, m+1)$-pancyclic.

Relationships between hamiltonian-type properties and bounds on the quantity $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}$ have been studied extensively. Ore proved in [9] that if $\sigma_2(G) \geq n$, then G is hamiltonian. In 1971, Bondy showed in [3] that the condition $\sigma_2(G) \geq n+1$ guarantees G is pancyclic. Then in 2004, Faudree *et al.* showed that this bound ensures much more than pancyclicity. Their result uses the notion of (k, m)-pancyclicity, and provides insight into the prevalence of cycles in such a graph.

Theorem A (Faudree, Gould, Jacobson and Lesniak [8])**.** *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n+1$, then G is $(k, 2k)$-pancyclic for each integer $k \geq 2$.*

Another technique that has been employed to ensure hamiltonian-type properties is the forbidding of a subgraph or subgraphs. Given a graph H, we say G is *H-free* if G does not contain H as an induced subgraph. In this context, H is called a *forbidden subgraph.* If $\mathcal{F}$ is a family of graphs, we say G is *$\mathcal{F}$-free* if G is F-free for each $F \in \mathcal{F}$.

In 2015, it was shown in [5] that if only claw-free graphs are considered, we may lower the $\sigma_2(G)$ bound to n in Theorem A and simultaneously guarantee $(k, k+3)$-pancyclicity as opposed to $(k, 2k)$-pancyclicity.

Theorem B [5]**.** *Let G be a claw-free graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is $(k, k+3)$-pancyclic for each integer $k \geq 1$.*

1.1. Pairs of forbidden subgraphs

A number of hamiltonian-type results have been obtained involving forbidden families of subgraphs. The claw is often a member of these forbidden families, as well as the graphs Z_1, Z_2, Z_3, B, N, and W, which are pictured in Figure 1.

A characterization of all pairs of subgraphs that, when forbidden, imply hamiltonicity in 2-connected graphs of order $n \geq 10$ was achieved in [7] by Faudree and Gould. Their result extended an earlier characterization by Bedrossian [1], which used graphs of small order to eliminate the pair $\{K_{1,3}, Z_3\}$.

Theorem C (Faudree and Gould [7])**.** *Let R and S be connected graphs ($R, S \neq P_3$) and let G be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is hamiltonian if, and only if, without loss of generality $R = K_{1,3}$ and S is one of the graphs $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$, or W.*

They went on to characterize the forbidden pairs that guarantee pancyclicity in 2-connected graphs of order $n \geq 10$.

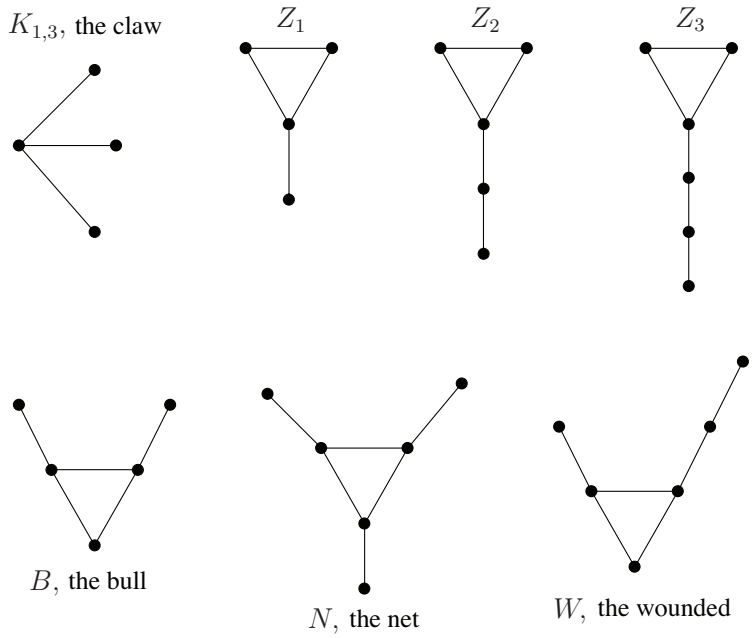

Figure 1. Some graphs that are commonly forbidden.

Theorem D (Faudree and Gould [7]). *Let R and S be connected graphs ($R, S \neq P_3$) and let G ($G \neq C_n$) be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is pancyclic if, and only if, without loss of generality $R = K_{1,3}$ and S is one of the graphs P_4, P_5, P_6, Z_1, or Z_2.*

In this paper, we investigate pairs of forbidden subgraphs that guarantee (k,m)-pancyclicity for integers $k \leq m$ in 2-connected graphs. Since a (k,m)-pancyclic graph must be hamiltonian, we need only consider the forbidden pairs from Theorem C. For each integer $k \geq 1$ and each pair $\{K_{1,3}, S\}$ where $S \in \{C_3, P_4, Z_1, Z_2, Z_3, B, N, W\}$, we determine the smallest integer m such that any 2-connected $\{K_{1,3}, S\}$-free graph is guaranteed to be (k,m)-pancyclic. A summary of results is provided at the end of the paper (see Theorem 7). We also give examples that show the provided values are best possible. Each such example represents an infinite family of graphs.

1.2. Notation

For terms and notation not defined here, we refer the reader to [4]. For a vertex $v \in V$, we denote by $d(v)$ the degree of v, and by $N(v)$ the neighborhood of v. Given a subgraph H of G and a vertex $v \in V(G)$, we let $N_H(v) = N(v) \cap V(H)$, and $d_H(v) = |N_H(v)|$. For $S \subseteq V$, let $N(S) = \{v \in V - S : vh \in E(G)$ for some

$h \in S\}$. The independence number of G will be denoted by $\alpha(G)$. For an integer $i \geq 1$ we let P_i denote a path on i vertices, and for an integer $j \geq 3$ we let C_j denote a j-cycle. Given distinct vertices $u, v \in V(G)$, a (u, v)-path is any path from u to v in G. Given a vertex u and a subgraph H such that $u \notin V(H)$, a (u, H)-path is any path in G from u to a vertex $v \in V(H)$.

Given a path P, we denote by (P) the set of all internal vertices of P, that is $V(P)$ minus the end vertices of P. Given a cycle C and a vertex $v \in V(C)$, we impose an orientation on C and let v^- (v^+) denote the vertex that appears directly before (after) v on C. We let xCy denote the path from x to y along C in the direction of the imposed orientation, while xC^-y will denote the path from x to y in the opposite direction along C. For a set of vertices $S \subseteq V$, we let $\langle S \rangle_G$ denote the subgraph induced by S in G. If the context is clear, we denote this induced subgraph by $\langle S \rangle$.

2. Forbidden Pairs That Guarantee Cycle Extendability

A non-hamiltonian cycle C is *extendable* if there exists a cycle C' in G such that $V(C') = V(C) \cup \{v\}$ for some $v \in V(G) - V(C)$. We say G is *cycle extendable* if every non-hamiltonian cycle in G is extendable. We will first examine the pairs of forbidden subgraphs that guarantee cycle extendability in 2-connected graphs. These pairs were completely characterized in [7].

Theorem E (Faudree and Gould [7])**.** *Let R and S be connected graphs ($R, S \neq P_3$) and let G be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is cycle extendable if, and only if, without loss of generality $R = K_{1,3}$ and S is one of the graphs C_3, P_4, Z_1, or Z_2.*

If $S = C_3$ then we must have $G = C_n$, and so G is (k, n)-pancyclic for all k in this case. If $S = Z_1$, it is easy to show that either $G = C_n$ or G is a complete graph minus at most a matching (see [7]). Thus we make the following observation.

Theorem 1. *Let $G \neq C_n$ be a 2-connected $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 5$. Then each of the following hold:*

(i) *G is $(1, 3)$-pancyclic;*
(ii) *G is $(k, 4)$-pancyclic for $k \in \{2, 3\}$;*
(iii) *G is (k, k)-pancyclic for each $k \geq 4$.*

These results are best possible.

Proof. Since G is isomorphic to K_n minus at most a matching, it is easy to see that for all $k \geq 4$, each k-set lies on a k-cycle. Since G is cycle extendable, part

(iii) follows. Parts (i) and (ii) follow similarly. The graph $H = K_n - uv$ shows that part (ii) is best possible, since any 2-set or 3-set of vertices containing $\{u, v\}$ in H is not contained in a triangle. ■

2.1. The pair $\{K_{1,3}, P_4\}$

Concerning the pair $\{K_{1,3}, P_4\}$, we show the following.

Theorem 2. *Let G be a 2-connected $\{K_{1,3}, P_4\}$-free graph with $n \geq 10$ vertices. Then the following hold:*

(i) *G is $(1, 4)$-pancyclic;*
(ii) *G is $(k, k+2)$-pancyclic for each integer $k \geq 2$.*

These results are best possible.

We will use two lemmas from [6] in the proof.

Lemma 3 (Egawa, Fujisawa, Fujita and Ota [6])**.** *Suppose G is a connected non-complete P_4-free graph, and let S be a minimum cutset of G. Then for every two vertices u and v with $u \in S$ and $v \in V - S$, we have $uv \in E$.*

The following lemma follows immediately from Lemma 3.

Lemma 4 (Egawa, Fujisawa, Fujita and Ota [6])**.** *Let $r \geq 2$, and suppose G is a connected P_4-free graph. Then G is $K_{1,r}$-free if and only if $\alpha(G) \leq r - 1$.*

We now prove Theorem 2.

Proof of Theorem 2. Let $a \in V$, and pick a shortest cycle C that contains a. Such a cycle must exist since G is hamiltonian. Now G is P_4-free, so the length of C must be 3 or 4. Since G is cycle extendable, we have shown that G is $(1, 4)$-pancyclic.

To prove part (ii) we first show that, given two vertices $u, v \in V$, there exists a 4-cycle in G that contains u and v. Since G is 2-connected, we may pick two internally vertex-disjoint (u, v)-paths P and Q so that (a) $|(P)|$ is as small as possible, and (b) subject to condition (a), $|(Q)|$ is as small as possible. Then $|(P)| \leq |(Q)| \leq 2$ because G is P_4-free. For the same reason, if $|(Q)| = 2$, then $|(P)| = 0$ and we have a 4-cycle containing u and v. Otherwise if $|(P)| \leq |(Q)| \leq 1$, then there exists a 3-cycle or 4-cycle that contains u and v, and as G is cycle extendable by Theorem E, there must be a 4-cycle containing u and v.

Now let $S \subset V$ and $|S| = k \geq 2$. We have shown there exists a 4-cycle that contains at least two vertices from S. We wish to obtain a cycle of length $t \geq 4$ that contains exactly two vertices which are not in S. Now G is cycle extendable, so if the 4-cycle contains fewer than two vertices that are not in S,

we may perform cycle extensions until we obtain a cycle $C = v_1v_2\cdots v_tv_1$ with $4 \le t \le k+2$ that contains exactly two vertices which are not in S.

If $t = k+2$ we are done, since G is cycle extendable. So assume $t < k+2$, and let $w \in S - V(C)$. We show we may obtain a cycle $\hat{C}$ of length $t+1$ that contains w and $S \cap V(C)$. Note that such a cycle can contain at most two vertices which are not in S. Note also that, since G is $K_{1,3}$-free, we have $\alpha(G) \le 2$ by Lemma 4.

First, suppose $d_C(w) \ge 2$. In the following, indices that are not in the set $\{1, 2, \ldots, t\}$ are taken modulo t. Let $v_i, v_j \in N_C(w)$ be such that $i < j$ and $N(w) \cap \{v_{i+1}, v_{i+2}, \ldots, v_{j-1}\} = \emptyset$. Now $wv_{j+1} \notin E$, or else we have the desired cycle $\hat{C}$. Hence $v_{i+1}v_{j+1} \in E$, or else $\{w, v_{i+1}, v_{j+1}\}$ is an independent set, contradicting the fact that $\alpha(G) \le 2$. But now $v_iC^-v_{j+1}v_{i+1}Cv_jwv_i$ is the desired cycle $\hat{C}$. Thus we assume $d_C(w) \le 1$.

Suppose $d_C(w) = 1$. Without loss of generality, assume $wv_1 \in E$. Now $wv_j \notin E$ for all $2 \le j \le t$, so $\{v_2, v_3, \ldots, v_t\}$ induces a clique since $\alpha(G) \le 2$. Since G is 2-connected, by Lemma 3 there exists a vertex $z \ne v_1$ such that $zw \in E$ and $zv_j \in E$ for all $j \in \{2, 3, \ldots, t\}$. There also exists a vertex $v_m \in V(C) - \{v_1\}$ that is not in S. If $m \ne t$, then $v_1wzv_{m+1}Cv_tv_{m-1}C^-v_1$ is the desired cycle $\hat{C}$. If $m = t$, then $v_1wzv_{t-1}C^-v_1$ is the cycle $\hat{C}$.

Lastly, suppose $d_C(w) = 0$. Since $\alpha(G) \le 2$, $V(C)$ induces a clique. As G is 2-connected, by Lemma 3 there exist distinct vertices z_1 and z_2 such that $wz_1, wz_2 \in E$ and $z_1x, z_2x \in E$ for all $x \in V(C)$. Let $V(C) - S = \{a, b\}$. As $\langle V(C)\rangle$ is a clique, we may modify the indices of C if necessary so that $a = v_1$ and $b = v_2$. Now $v_3Cv_tz_1wz_2v_3$ is the desired cycle $\hat{C}$.

We may repeat this argument as necessary to obtain a cycle containing S whose length is k, $k+1$, or $k+2$. Since G is cycle extendable, this completes the proof of part (ii).

To see that part (ii) is best possible, let $n \ge k+2$ and construct the graph H_1 in Figure 2 by removing $k-1$ edges incident to some vertex v in a copy of K_n. Now H_1 is 2-connected and $\{K_{1,3}, P_4\}$-free, but the k-set $S = V(H_1) - N(v)$ is not contained in a cycle of length $k+1$.

To see that part (i) is best possible, construct a graph G as follows. Let $H = K_{n-3}$, and add vertices x, y, and a such that $N_G(x) = N_G(y) = V(H) \cup \{a\}$ and $N_G(a) = \{x, y\}$. Then G is 2-connected and $\{K_{1,3}, P_4\}$-free, but the vertex a is not contained in a triangle. ■

2.2. The pair $\{K_{1,3}, Z_2\}$

We now consider $\{K_{1,3}, Z_2\}$, the only remaining pair from Theorem E that guarantees cycle extendability in 2-connected graphs. We prove the following.

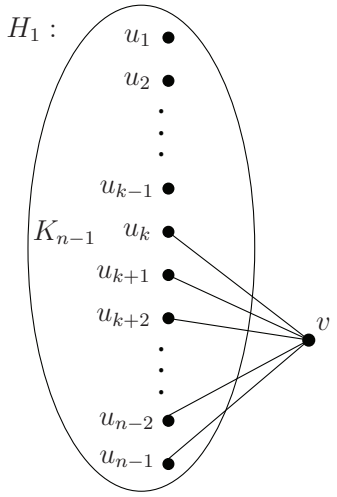

Figure 2. The k-set $V(H_1) - N(v)$ is not contained in a $(k+1)$-cycle.

Theorem 5. *Let $G \neq C_n$ be a 2-connected $\{K_{1,3}, Z_2\}$-free graph with $n \geq 10$ vertices. Then the following hold:*

(i) *G is $(1, 4)$-pancyclic;*

(ii) *G is $(k, 3k)$-pancyclic for each integer $k \geq 2$.*

These results are best possible.

Proof. We first prove part (i). Let $w \in V$. If w is contained in a triangle, then w is contained in a 4-cycle since G is cycle extendable. Thus we assume that w is not contained in a triangle. Since G is 2-connected and $G \neq C_n$, there exists a vertex of degree at least 3. This implies G contains a triangle, since G is claw-free.

As G is 2-connected, we let s be the smallest integer such that there exists a triangle H and a pair of (w, H)-paths (P, Q) which are vertex-disjoint (except for w) with P having length s. Now pick a triangle T so that there exists a pair of (w, T)-paths (P_T, Q_T) that are vertex-disjoint (except for w), P_T has length s, and Q_T has minimum possible length.

Let $T = axya$, and without loss of generality let $P_T = xx_1x_2 \cdots x_s$ and $Q_T = yy_1y_2 \cdots y_t$, where $x_s = w = y_t$. Note that P_T and Q_T are each induced paths and $s \leq t$.

Suppose $s \geq 2$. Now $yx_1 \notin E$, for otherwise w is connected to the triangle xyx_1x by Q_T and a path $x_1x_2 \cdots x_s$ that is shorter than P_T, violating the minimality of s. Similarly, $ax_1 \notin E$. Also $ax_2 \notin E$ by the minimality of s. Now we must have $yx_2 \in E$ or else $\{y, a, x, x_1, x_2\}$ induces a Z_2. But then $x_2 \neq w$ by the minimality of $t \geq 2$. Therefore $s \geq 3$, and avoiding a claw centered at x_2, we must have $yx_3 \in E$. This is a contradiction, since the triangle yx_2x_3y now violates the minimality of s.

Hence we must have $s = 1$. We may assume $t \geq 3$, for otherwise w is contained in a C_3 or C_4. First note that $xy_1 \notin E$, or else the minimality of t is violated via the triangle xyy_1x. Also $ay_i \notin E$ for all $2 \leq i \leq t$, by the minimality of t. Furthermore $ay_1 \notin E$, for otherwise $\{a, y, y_1, y_2, y_3\}$ induces a Z_2. But now $xy_2 \in E$ since otherwise $\{x, a, y, y_1, y_2\}$ induces a Z_2. This implies $t = 3$, for otherwise $\{x, w, y_2, y\}$ induces a claw centered at x. We now have a contradiction, since w is contained in a triangle.

Therefore w must be contained in a 4-cycle. Since G is cycle extendable, we have shown that G is $(1, 4)$-pancyclic. To see that part (i) is best possible, construct a graph G as follows. Remove an edge uv from a copy of K_{n-1}, and add a new vertex w such that $N_G(w) = \{u, v\}$. Now G fulfills the assumptions of Theorem 5, but G is not $(1, 3)$-pancyclic since w is not contained in a triangle.

Our proof of part (ii) will be by induction on k. We begin with a claim that will serve as the base case.

Claim. *Any set of two vertices is contained in a* 6*-cycle in* G.

Proof. Let $w, z \in V$. We consider two cases.

Case 1. Suppose z is contained in a triangle. If there is a triangle in G that contains both w and z, then the claim holds as G is cycle extendable. Thus suppose there is no triangle in G that contains both w and z.

As G is 2-connected, we let s be the smallest integer such that there exists a triangle H containing z and a pair of (w, H)-paths (P, Q) which are vertex-disjoint (except for w) with P having length s. Now pick a triangle T containing z so that there exists a pair of (w, T)-paths (P_T, Q_T) that are vertex-disjoint (except for w), P_T has length s, and Q_T has minimum possible length.

We may assume

$$|(P_T)| + |(Q_T)| \geq 3, \tag{1}$$

since otherwise we have the desired 6-cycle (using the fact that G is cycle extendable). Let $T = axya$, and let $P_T = xx_1x_2 \cdots x_s$ and $Q_T = yy_1y_2 \cdots y_t$, where $x_s = w = y_t$. Note that P_T and Q_T are each induced paths and $s \leq t$.

For each i with $2 \leq i \leq s$, we have $ax_i \notin E$, for otherwise the minimality of s is violated. Similarly $ay_i \notin E$ for all i with $2 \leq i \leq t$, by the minimality of t. By inequality (1), we know $t \geq 3$. Now $ay_1 \notin E$ or else $\{a, y, y_1, y_2, y_3\}$ induces a Z_2.

Suppose $yx_i \in E$ for some $i \in \{2, 3, \ldots, s\}$. Note that $i < s$ since $yw \notin E$. Then $yx_{i+1} \in E$, or else $\{a, x, y, x_i, x_{i+1}\}$ induces a Z_2. But by induction, this implies that $yx_{i+2}, yx_{i+3}, \ldots, yx_s \in E$, which is a contradiction since $yx_s \notin E$. Therefore $yx_i \notin E$ for each $i \in \{2, 3, \ldots, s\}$. We now consider four subcases.

Case 1.1. Let $|(P_T)| = 0$ and $z \in \{x, y\}$. By our choice of T and the minimality of t, we have $xy_1 \notin E$. Inequality (1) implies $t \geq 4$. Now $xy_2 \notin E$ or else $\{x, y, y_2, w\}$ induces a claw centered at x. But then $\{a, x, y, y_1, y_2\}$ induces a Z_2.

Case 1.2. Suppose $|(P_T)| = 0$ and $z \notin \{x, y\}$. We have $z = a$, and again $t \geq 4$. Now $aw \notin E$ or else z and w are contained in the triangle $axwa$. Note that $xy_{t-1} \in E$, for otherwise $\{a, y, x, w, y_{t-1}\}$ induces a Z_2. Also $xy_{t-2} \notin E$ since $\{x, w, y_{t-2}, y\}$ cannot induce a claw. Avoiding a Z_2 induced by the set $\{x, w, y_{t-1}, y_{t-2}, y_{t-3}\}$, we must have $xy_{t-3} \in E$. But now $\{x, a, y_{t-3}, w\}$ induces a claw centered at x.

Case 1.3. Let $|(P_T)| \geq 1$ and $z \in \{x, y\}$. First suppose $z = x$. Note that $ax_1 \notin E$, or else the minimality of s is violated by the triangle axx_1a. Similarly $yx_1 \notin E$. This is a contradiction, since $\{y, a, x, x_1, x_2\}$ now induces a Z_2.

Therefore we must have $z = y$. Now $yx_1 \notin E$, or else the minimality of s is violated. Thus $ax_1 \in E$ since otherwise $\{y, a, x, x_1, x_2\}$ induces a Z_2. Then $w = x_2$, since $\{a, x, x_1, x_2, x_3\}$ cannot induce a Z_2. Now $xy_{t-1} \notin E$, or else the paths $P = xy_{t-1}w$ and $Q = ax_1w$ violate the minimality of $t \geq 3$. Then $x_1y_{t-1} \in E$ or else $\{a, x, x_1, w, y_{t-1}\}$ induces a Z_2. But now $\{y_{t-1}, w, x_1, x, y\}$ induces a Z_2.

Case 1.4. Suppose $|(P_T)| \geq 1$ and $z \notin \{x, y\}$. We have $z = a$. Now $ax_1 \notin E$, since otherwise the minimality of s is violated via the triangle axx_1a. Hence $yx_1 \in E$, or else $\{y, a, x, x_1, x_2\}$ induces a Z_2. This implies $w = x_2$, since otherwise $\{y, x, x_1, x_2, x_3\}$ induces a Z_2. Now $xy_{t-1} \notin E$, or else the paths $P = xy_{t-1}w$ and $Q = yx_1w$ violate the minimality of $t \geq 3$. Thus $x_1y_{t-1} \in E$, since $\{y, x, x_1, w, y_{t-1}\}$ cannot induce a Z_2. This is a contradiction, since now $\{y_{t-1}, w, x_1, x, a\}$ induces a Z_2. Therefore the claim holds for Case 1.

Case 2. Suppose neither z nor w is contained in a triangle. By part (i), we know that z is contained in a 4-cycle. Also, since G is 2-connected and claw-free, we must have $d(w) = d(z) = 2$. Now let s be the smallest integer such that there exists a 4-cycle H containing z and a pair of (w, H)-paths (P, Q) which are vertex-disjoint (except for w) with P having length s. Now pick a 4-cycle C containing z so that there exists a pair of (w, C)-paths (P_C, Q_C) that are vertex-disjoint (except for w), P_C has length s, and Q_C has minimum possible length.

We may assume

$$|(P_C)| + |(Q_C)| \geq 2 \tag{2}$$

since otherwise we clearly have the desired 6-cycle. Since z is not contained in a triangle, C must be an induced 4-cycle. Let $P_C = xx_1x_2 \cdots x_s$ and $Q_C = yy_1y_2 \cdots y_t$, where $x_s = w = y_t$. Note that P_C and Q_C are induced paths. By inequality (2), we have $t \geq 2$. Since $d(z) = 2$, we know $z \notin \{x, y\}$. Let $V(C) - \{x, y, z\} = \{a\}$. For each i with $2 \leq i \leq s$, we have $ax_i \notin E$, for otherwise the minimality of s is violated. Similarly $ay_i \notin E$ for all $2 \leq i \leq t$, by the minimality of t. We now consider three subcases corresponding to the possible positions of x and y relative to z on C.

Case 2.1. Suppose $C = zaxyz$. Now $xy_1 \in E$ or else $\{y, z, x, y_1\}$ induces a claw. Note that $xy_i \notin E$ for all $2 \leq i \leq t$, since otherwise $\{x, a, y, y_i\}$ induces a claw. But now $w = y_2$, for otherwise $\{x, y, y_1, y_2, y_3\}$ induces a Z_2. By inequality (2) and the minimality of s, we must have $w = x_2$. Since w is not contained in a triangle, we know $x_1y_1 \notin E$. Also $ay_1 \notin E$, or $\{y_1, y, a, w\}$ induces a claw centered at y_1. Then $ax_1 \in E$, for otherwise $\{x, a, y_1, x_1\}$ induces a claw centered at x. But now ax_1wy_1yza is the desired 6-cycle.

Case 2.2. Suppose $C = zayxz$. We have $yx_1 \in E$, for otherwise $\{x, z, y, x_1\}$ induces a claw. Since $yw \notin E$, we know $s \geq 2$. If $yx_i \in E$ for some $i \geq 2$, then $\{y, a, x, x_i\}$ induces a claw. Hence $yx_i \notin E$ for all $2 \leq i \leq s$. But now $w = x_2$, for otherwise $\{y, x, x_1, x_2, x_3\}$ induces a Z_2.

Since w is not contained in a triangle, we have $x_1y_{t-1} \notin E$. Also $xy_{t-1} \notin E$, or $\{x, z, x_1, y_{t-1}\}$ induces a claw centered at x. But then we must have $w = y_2$, for otherwise $\{y, x, x_1, w, y_{t-1}\}$ induces a Z_2. Then $ay_1 \in E$, since otherwise $\{y, a, x, y_1\}$ induces a claw centered at y. But now ay_1wx_1xza is the desired 6-cycle.

Case 2.3. Suppose $C = zxayz$. We know $s \geq 2$ since $\{x, z, a, w\}$ cannot induce a claw. Now $ax_1 \in E$, or else $\{x, z, a, x_1\}$ induces a claw. But then $w = x_2$, for otherwise $\{a, x, x_1, x_2, x_3\}$ induces a Z_2. This implies $t \geq 3$, or else yy_1wx_1xzy is the desired cycle of length six.

Now $x_1y_{t-1} \notin E$ since w is not contained in a triangle. Also $xy_{t-1} \notin E$, since otherwise the pair of paths $Q = xy_{t-1}w$ and $P = ax_1w$ violate the minimality of t. But now $\{a, x, x_1, w, y_{t-1}\}$ induces a Z_2. This contradiction completes Case 2, and the proof of the claim. □

We now proceed with the inductive step of the proof for part (ii) of Theorem 5. Given an integer $k \geq 2$ with $3k \leq n$, suppose that any set of k vertices is contained in a cycle of length $3k$ in G. Let $S \subset V$ be a set of $k+1$ vertices with $3(k+1) \leq n$, and let C be a $3k$-cycle that contains at least k vertices of S. We wish to prove there exists a $(3k+3)$-cycle containing S. Now if $S \subset V(C)$, then since G is cycle extendable, there exists a cycle of length $3k+3$ that contains S. Therefore we assume $S - V(C) = \{w\}$.

Let $\mathcal{F}$ be the set of pairs of paths in G defined by: $\{P, Q\} \in \mathcal{F}$ if and only if P and Q are (w, C)-paths that are vertex-disjoint (except for w) and $|(P)| + |(Q)|$ is minimal among all such pairs of (w, C)-paths. Since G is 2-connected, $\mathcal{F}$ is nonempty. Now pick a pair of paths $\{P, Q\} \in \mathcal{F}$ so that P and Q have endpoints that are as close as possible on C. Let x and y denote the endpoints of P and Q, respectively, on C. Let $P = xx_1x_2 \cdots x_s$ and $Q = yy_1y_2 \cdots y_t$, where $x_s = w = y_t$. By their definition, P and Q are induced paths. We now consider two cases.

Case 1. Assume x and y occur consecutively on C. Let $y = x^+$. If $|(P)| + |(Q)| \leq 2$, then since G is cycle extendable, we may clearly extend C to obtain a

cycle of length $3(k+1)$ that contains S. Therefore suppose $|(P)|+|(Q)| \geq 3$.

Without loss of generality we may assume that $|(P)| \geq |(Q)|$, and hence $|(P)| \geq 2$. For each i with $2 \leq i \leq s$, we have $x^-x_i, y^+x_i \notin E$ by the minimality of $|(P)|+|(Q)|$. Now $x^-x_1 \notin E$, or else $\{x^-, x, x_1, x_2, x_3\}$ induces a Z_2.

Suppose $|(P)| \geq 3$. Further, suppose $yx_2 \in E$. Then $xy^+ \in E$, since otherwise $\{y, x, y^+, x_2\}$ induces a claw. But now $yx_3 \in E$, or else $\{x, y^+, y, x_2, x_3\}$ induces a Z_2. Also $yx_4 \in E$, for otherwise $\{x, y^+, y, x_3, x_4\}$ induces a Z_2. This yields a contradiction since now $\{y, x, x_2, x_4\}$ induces a claw centered at y. Thus $yx_2 \notin E$.

Now suppose $yx_1 \in E$. Then $yx_3 \in E$ or else the set $\{y, x, x_1, x_2, x_3\}$ induces a Z_2. But now $xy^+ \in E$, since otherwise $\{y, x, y^+, x_3\}$ induces a claw centered at y. This yields a contradiction, since now $\{x, y^+, y, x_3, x_2\}$ induces a Z_2. Hence $yx_1 \notin E$. Avoiding a claw centered at x, we then have $yx^- \in E$. But this is a contradiction, since $\{x^-, y, x, x_1, x_2\}$ now induces a Z_2.

Therefore $|(P)| = 2$. Also $|(Q)| \geq 1$ since $|(P)|+|(Q)| \geq 3$. Note that $w = x_3$. First suppose $yx_1 \in E$. Then $yx_2 \in E$, for otherwise $\{y, x, x_1, x_2, w\}$ induces a Z_2. In order to avoid a claw induced by $\{y, x, y^+, x_2\}$, we must have $xy^+ \in E$. This is a contradiction, since now $\{x, y^+, y, x_2, w\}$ induces a Z_2. Therefore $yx_1 \notin E$ must hold. Then $x^-y \in E$, since otherwise $\{x, x^-, y, x_1\}$ induces a claw. We must also have $yx_2 \in E$, for otherwise $\{x^-, y, x, x_1, x_2\}$ induces a Z_2. But now $\{x^-, x, y, x_2, w\}$ induces a Z_2. This contradiction completes Case 1.

Case 2. Assume x and y do not occur consecutively on C. Suppose the length of xCy is less than or equal to the length of yCx.

For each i with $2 \leq i \leq s$, we have $x^+x_i, x^-x_i \notin E$ or else the minimality of $|(P)|+|(Q)|$ is violated. We must have $x^+x_1 \notin E$, for otherwise the pair $\{x^+x_1Pw, Q\} \in \mathcal{F}$ has endpoints that are closer together on C than x and y.

Suppose $|(P)| \geq 2$. Now $x^-x_1 \notin E$ or else $\{x, x^-, x_1, x_2, x_3\}$ induces a Z_2. Since G is claw-free, we must have $x^-x^+ \in E$. But now $\{x^-, x^+, x, x_1, x_2\}$ induces a Z_2. So we must have $|(P)| \leq 1$. By symmetry we also have $|(Q)| \leq 1$.

Suppose $|(P)| = 1$. If $x^-x_1 \notin E$, then $x^-x^+ \in E$ in order to avoid a claw, and then $\{x^-, x^+, x, x_1, w\}$ induces a Z_2. Hence $x^-x_1 \in E$. If $|(Q)| = 1$, then by symmetry $y^+y_1 \in E$. Therefore in any case, it is clear we may extend C to obtain a cycle C' with vertex set $V(C) \cup (P) \cup (Q)$ that satisfies $|V(C') - V(C)| \leq 2$ and $|N(w) \cap V(C')| \geq 2$.

Relabel the vertices so that $C' = u_1u_2 \cdots u_mu_1$, where $u_1, u_i \in N(w)$ for some $i > 1$ with $N(w) \cap \{u_2, u_3, \ldots, u_{i-1}\} = \emptyset$ and i as small as possible. If u_1 and u_i occur consecutively on C', then $u_1wu_2C'u_mu_1$ is a cycle of length at most $3(k+1)$ that contains S. Thus we assume that u_1 and u_i do not occur consecutively on C'.

Since G is claw-free, we must have $u_mu_2, u_{i-1}u_{i+1} \in E$. If $i = 3$, then $u_1wu_3C'u_mu_2u_1$ is a cycle of length at most $3(k+1)$ that contains S. If $i = 4$,

then $u_1wu_4u_3u_5C'u_mu_2u_1$ is a cycle of length at most $3(k+1)$ that contains S. Hence we assume $i \geq 5$.

Now $u_iu_2 \notin E$, since otherwise $u_1wu_iu_2C'u_{i-1}u_{i+1}C'u_mu_1$ is the desired cycle. Also $u_iu_m \notin E$ by symmetry. Therefore $u_1u_i \in E$, since $\{u_m, u_2, u_1, w, u_i\}$ cannot induce a Z_2. Note that $u_iu_3 \notin E$, or else $u_1wu_iu_3C'u_{i-1}u_{i+1}C'u_mu_2u_1$ is the desired cycle. But now $u_1u_3 \in E$, or else $\{u_i, w, u_1, u_2, u_3\}$ induces a Z_2. We may assume $i > 5$, for otherwise $u_1wu_5u_4u_6C'u_mu_2u_3u_1$ is the desired cycle. Note that $u_iu_4 \notin E$, or else $u_1wu_iu_4C'u_{i-1}u_{i+1}C'u_mu_2u_3u_1$ is the desired cycle. Now we have $u_1u_4 \in E$, or else $\{u_i, w, u_1, u_3, u_4\}$ induces a Z_2. An easy inductive argument now verifies that $u_1u_j \in E$ for all $2 \leq j \leq i-2$. But then $u_1wu_iu_{i-1}u_{i+1}C'u_mu_2C'u_{i-2}u_1$ is the desired cycle.

We have shown that there must exist a cycle of length at most $3(k+1)$ which contains S. Since G is cycle extendable, this completes Case 2.

We have thus shown by induction that every set S of $k \geq 2$ vertices is contained in a $3k$-cycle whenever $3k \leq n$. Since G is cycle extendable, this completes the proof of part (ii).

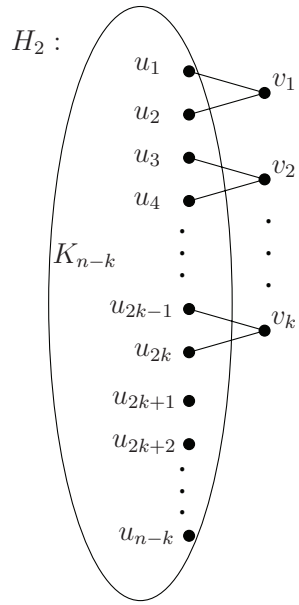

Figure 3. The set $\{v_1, v_2, \ldots, v_k\}$ is not contained in a $(3k-1)$-cycle.

To see that part (ii) is best possible, construct the graph H_2 in Figure 3 as follows. Given $n \geq 3k$, take a copy of K_{n-k} with vertex set $\{u_1, u_2, \ldots, u_{n-k}\}$. Let H_2 be the graph obtained by adding k distinct vertices $v_1, v_2, \ldots, v_k$ such that $N(v_i) = \{u_{2i-1}, u_{2i}\}$ for all i with $1 \leq i \leq k$. Then H_2 is 2-connected and $\{K_{1,3}, Z_2\}$-free, but the set $S = \{v_1, v_2, \ldots, v_k\}$ is not contained in a cycle of length $3k-1$. ■

3. Four Pairs for Which Hamiltonicity Is Best Possible

Given an integer $k \geq 1$, the forbidden pairs in the following theorem do not guarantee (k,m)-pancyclicity for any integer $m < n$ under the given conditions.

Theorem 6. *Let G be a 2-connected $K_{1,3}$-free graph on n vertices.*

(i) *If $n \geq 5$ and G is M-free for some $M \in \{B, N, W\}$, then G is (k,n)-pancyclic for all $k \geq 1$.*

(ii) *If $n \geq 10$ and G is Z_3-free, then G is (k,n)-pancyclic for all $k \geq 1$.*

These results are best possible.

Proof. For part (i), we construct a graph H_3 (see Figure 4) with $|V(H_3)| = n \geq k \geq 1$ as follows. Let $r \geq 1$ be an integer. Pick integers $l_1, l_2, \ldots, l_r \geq 1$ such that $\Sigma_{i=1}^{r} l_i \leq k$, and let $A_i = K_{l_i}$ for each i with $1 \leq i \leq r$. Add $2r$ distinct vertices $x_1, y_1, x_2, y_2, \ldots, x_r, y_r$ such that for each $i \in \{1, 2, \ldots, r\}$, we have $x_i w, y_i w \in E$ for all $w \in V(A_i)$. For each i with $1 \leq i \leq r-1$, add a path Q_i with $|V(Q_i)| \geq 2$ such that $y_i Q_i x_{i+1}$ is an induced path. Lastly, add a path Q_r with $|V(Q_r)| \geq 2$ such that $y_r Q_r x_1$ is an induced path. Now H_3 is 2-connected and $\{K_{1,3}, M\}$-free for each $M \in \{B, N, W\}$, but any k-set $S \subseteq V(H_3)$ with $\bigcup_{i=1}^{r} V(A_i) \subseteq S$ is not contained in a cycle of length $n-1$.

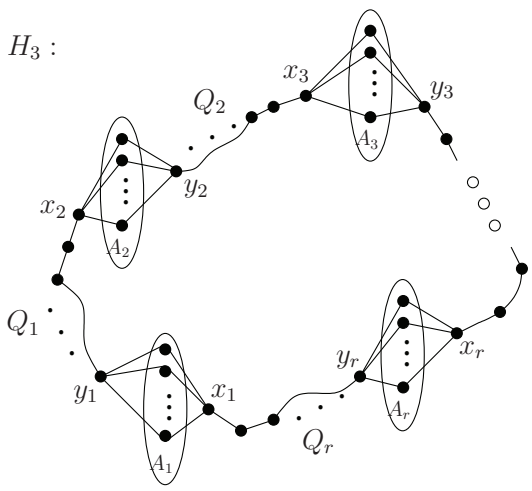

Figure 4. The set $\bigcup_{i=1}^{r} V(A_i)$ is not contained in an $(n-1)$-cycle.

Regarding part (ii), for some integer t with $0 \leq t \leq k-1$, construct the graph H_4 in Figure 5 as follows. Let $G_1 = K_{t+2r}$ for some integer r with $4r \geq k-t$, and let $V(G_1) = \{u_1, u_2, \ldots, u_{t+2r}\}$. Add a matching $M = \{x_1 y_2, x_3 y_4, \ldots, x_{2r-1} y_{2r}\}$ such that $x_i u_{t+i} \in E$ for all odd i with $1 \leq i \leq 2r-1$, and $y_i u_{t+i} \in E$ for all even i with $2 \leq i \leq 2r$. Let $S \subseteq V(H_4)$ be any k-set satisfying $\{u_1, u_2, \ldots, u_{t+1}\} \subseteq S$.

The graph H_4 is 2-connected and $\{K_{1,3}, Z_3\}$-free, but there is no cycle of length $n-1 = t+4r-1$ that contains S. ■

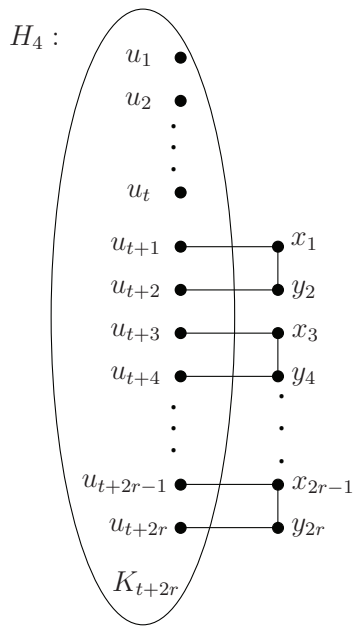

Figure 5. The set $\{u_1, u_2, \ldots, u_{t+1}\}$ is not contained in an $(n-1)$-cycle.

The following theorem summarizes results from this paper.

Theorem 7. *Let G be a 2-connected $K_{1,3}$-free graph of order $n \geq 10$, and let $k \geq 2$.*

(i) *If $G \neq C_n$ is Z_1-free, then G is (k,k)-pancyclic for all $k \geq 4$.*
(ii) *If G is P_4-free, then G is $(k, k+2)$-pancyclic.*
(iii) *If $G \neq C_n$ is Z_2-free, then G is $(k, 3k)$-pancyclic.*
(iv) *If G is C_3-free, then G is (k,n)-pancyclic.*
(v) *If G is Z_3-free, then G is (k,n)-pancyclic.*
(vi) *If G is B-free, then G is (k,n)-pancyclic.*
(vii) *If G is N-free, then G is (k,n)-pancyclic.*
(viii) *If G is W-free, then G is (k,n)-pancyclic.*

These results are best possible under the given conditions.

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