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# ONE-THREE JOIN: A GRAPH OPERATION AND ITS CONSEQUENCES

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**Abstract**

In this paper, we introduce a graph operation, namely *one-three join*. We show that the graph $G$ admits a one-three join if and only if either $G$ is one of the basic graphs (bipartite, complement of bipartite, split graph) or $G$ admits a constrained homogeneous set or a bipartite-join or a join. Next, we define $\mathcal{M}_H$ as the class of all graphs generated from the induced subgraphs of an odd hole-free graph $H$ that contains an odd anti-hole as an induced subgraph by using one-three join and co-join recursively and show that the maximum independent set problem, the maximum clique problem, the minimum coloring problem, and the minimum clique cover problem can be solved efficiently for $\mathcal{M}_H$.

**Keywords:** one-three join, bipartite-join, homogeneous set, odd hole-free graphs.

**2010 Mathematics Subject Classification:** 05C75, 05C76.

## 1. Introduction

Graph operations are very useful in generating various graph classes and producing polynomial time algorithms to solve optimization problems on those classes of graphs [9, 15]. All $P_4$-free graphs (or co-graphs) can be generated from a single vertex by using the graph operations join and co-join recursively, and these operations are used to compute the clique number, independence number, and chromatic number of $P_4$-free graphs efficiently [15]. Chudnovsky and Seymour proved that every connected claw-free graph can be obtained from one of the basic claw-free graphs by simple expansion operations [9]. Chudnovsky *et al.* [11] introduced two graph operations, namely gluing operation and substitution operation and proved that the closure of a $\chi$-bounded class under these operations is $\chi$-bounded.

A *hole* is a chordless cycle of length at least five and an *anti-hole* is the complement of a hole. A hole (an anti-hole) is *odd* if it contains an odd number of vertices. The complexity status of the recognition problem, maximum independent set problem (MISP), and minimum coloring problem are still open for odd hole-free graphs [8, 13]. In fact, the complexity status of the MISP is unknown for hole-free graphs though the recognition problem for hole-free graphs can be solved in polynomial time [12]. Bienstock [3] proved that it is NP-complete to test whether a graph contains an odd hole passing through a specific vertex. Conforti *et al.* [14] proved that the recognition of odd hole-free graphs with cliques of bounded size can be done in polynomial time. The Table 1 summarizes the complexity results for the MISP in some subclasses of odd hole-free graphs.

| Graph Class | Complexity | Citation |
|---|---|---|
| (Hole, Co-chair)-free graphs | Polynomial time | [5] |
| (Hole, Dart)-free graphs | ” | [2] |
| (Hole, Diamond)-free graphs | ” | [6] |
| (Hole, Banner)-free graphs | ” | [7] |
| (Odd hole, Co-chair)-free graphs | ” | [5] |
| (Odd hole, Dart)-free graphs | ” | [8] |
| (Odd hole, Bull)-free graphs | ” | [8] |

Table 1. MISP for some subclasses of odd hole-free graphs.

In this paper, we introduce a graph operation one-three join. In Section 2, we show that the graph $G$ admits a one-three join if and only if either $G$ is one of the basic graphs (bipartite, complement of bipartite, split graph) or $G$ admits a constrained homogeneous set or a bipartite-join or a join, and it follows from a result of Feder *et al.* [16] that these graphs can be recognized in polynomial time. In Section 3, we define $\mathcal{M}_H$ as the class of all graphs generated from the induced subgraphs of an odd hole-free graph $H$ that contains an odd anti-hole as an induced subgraph by using one-three join and co-join recursively and show that the maximum independent set problem, the maximum clique problem, the minimum coloring problem, and the minimum clique cover problem can be solved efficiently for $\mathcal{M}_H$.

All graphs considered in this paper are finite, simple and undirected. For graph terminologies, we refer to [22]. For a graph $H$, we say that a graph $G$ is *$H$-free* if $G$ does not contain $H$ as an induced subgraph. Let $G[U]$ denote the subgraph induced by $U \subseteq V(G)$ in the graph $G$. The *complement* $G^c$ of a graph $G = (V, E)$ is the graph with vertex set $V$ and two vertices are adjacent in $G^c$ if and only if they are non-adjacent in $G$. A *clique* (*independent set*) is a subset of vertices of a graph $G$ which are pairwise adjacent (respectively, non-adjacent) in $G$. For a vertex $v$ in a graph $G$, $N(v)$ $(A(v))$ is the set of all vertices adjacent (respectively, non-adjacent) to $v$ in $G$. Let $U, W \subseteq V(G)$. Define $N(U) = \{x \in V(G) \setminus U : xu \in E(G)$ for some $u \in U\}$ and $N_W(U) = N(U) \cap W$. A graph $G = (V, E)$ is a *split graph* if there is a partition $V = S \cup K$ where $S$ is an independent set and $K$ is a clique. Let $A$ and $B$ be two disjoint subsets of $V(G)$. We define the set of edges $[A, B] = \{\{a, b\} : a \in A$ and $b \in B\}$. Often we denote an edge $\{a, b\}$ as $ab$ or $ba$ for convenience. The *join* $G_1 + G_2$ of vertex-disjoint graphs $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup [V(G_1), V(G_2)]$. The *co-join* $G_1 \cup G_2$ of vertex-disjoint graphs $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

For convenience, we use the following notation: Let $A$ and $B$ be two disjoint subsets of $V(G)$. We say $A \oplus B$ if $ab \in E(G)$ for all $a \in A$ and for all $b \in B$. In addition, $A \ominus B$ if $ab \notin E(G)$ for all $a \in A$ and for all $b \in B$. In particular, if $A = \{x\}$, then we simply denote $\{x\} \oplus B$ by $x \oplus B$. Similarly, $\{x\} \ominus B$ by $x \ominus B$.

A *homogeneous set* in a graph $G$ is a set $C$ of vertices of $G$ such that each vertex in $V(G) \setminus C$ is adjacent either to all or to none of the vertices of $C$ and $2 \leq |C| \leq |V(G)| - 1$. Next, we define the notion of constrained homogeneous set and bipartite-join. A graph $G$ admits a *constrained homogeneous set* if $G$ admits a vertex partition $V(G) = A \cup B \cup C$ where $A \neq \emptyset$ is a clique or an independent set, $B \neq \emptyset$ is an independent set, and $G[C]$ contains at least one edge such that $C \oplus A$ and $C \ominus B$ in $G$. A graph $G$ admits a *bipartite-join* if $G$ admits a vertex partition $V(G) = A_1 \cup A_2 \cup B_1 \cup B_2$ such that (a) each $A_i \neq \emptyset$ is an independent set in $G$, (b) $G[B_i]$ contains at least one edge in $G$ for all $i \in \{1, 2\}$, and (c) $A_i \oplus B_i$, $B_1 \oplus B_2$, and $A_i \ominus B_j$ in $G$, for $i \neq j$ and $i, j \in \{1, 2\}$. That is, removal of the bipartite graph $G[A_1 \cup A_2]$ results in the join of two graphs, namely $G[B_1]$ and $G[B_2]$. In Figure 1, the single line across the parts represent complete adjacency (all possible edges), the dotted line represent complete non-adjacency (no edges), and the wave implies that there are no restriction on the edges between the parts. The circle filled with dots represents an independent set.

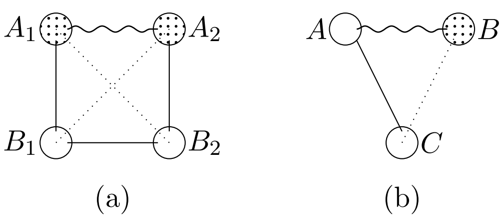


Figure 1. (a) Bipartite-join and (b) constrained homogeneous set.

By a result of Feder *et al.* [16], the graphs that admit a constrained homogeneous set (or a bipartite-join) can be recognized in polynomial time. It can be vertified that the time complexity to recognize (i) a constrained homogeneous set is $O(n^6)$ and (ii) a bipartite-join is $O(mn^8)$.

## 2. One-Three Join

In this section, we introduce a graph operation one-three join and characterize the class of graphs that admit a one-three join.

Let $H_1, H_2$ be two vertex-disjoint graphs. A *one-three join* of $H_1$ and $H_2$ is a graph $H$ with $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2) \cup F$ where $F \subseteq [V(H_1), V(H_2)]$ such that for every vertex $x \in V(H_i)$ and for every non-independent set (a set which induces at least one edge) $\{y_1, y_2, y_3\} \subseteq V(H_j)$, $H[\{x, y_1, y_2, y_3\}]$ contains a triangle for $i, j \in \{1, 2\}, i \neq j$ (see Figure 2). A graph $H$ *admits* a one-three join if the vertex set of $H$ can be partitioned as $V(H) = V_1 \cup V_2$ such that $H$ is a one-three join of $H[V_1]$ and $H[V_2]$, where $V_i \neq \emptyset$ for $i \in \{1, 2\}$. For convenience, let $\{[a], [b, c, d]\}$ denote the graph induced by $\{a, b, c, d\}$ in $H$ such that $a \in V(H_1)$ and $\{b, c, d\} \subseteq V(H_2)$. Also, let $\{[a, b, c], [d]\}$ denote the graph induced by $\{a, b, c, d\}$ in $H$ such that $\{a, b, c\} \subseteq V(H_1)$ and $d \in V(H_2)$.

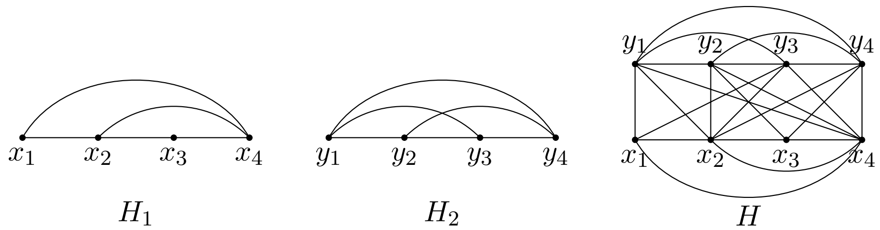


Figure 2. A one-three join $H$ of $H_1$ and $H_2$.

**Lemma 1.** *A graph $H$ admits a one-three join if $H$ is one of the following:*

(i) *a bipartite graph,*
(ii) *a complement of a bipartite graph,*
(iii) *a split graph,*
(iv) *a join of two graphs,*
(v) *$H$ admits a constrained homogeneous set or a bipartite-join.*

***Proof.*** If $H$ is a bipartite graph with vertex partition $V_1$ and $V_2$ ($V_1$ and $V_2$ are independent sets), then $H$ is a one-three join of $H[V_1]$ and $H[V_2]$. Similarly, if $H$ is either complement of a bipartite graph or a split graph or a join of two graphs, then $H$ is a one-three join of the graphs induced by the corresponding partitions. If $H$ admits a constrained homogeneous set with vertex partition $V(H) = A \cup B \cup C$ satisfying the conditions in the definition of constrained homogeneous set, then we prove that $H$ is a one-three join of $H[A]$ and $H[B \cup C]$. Since $A$ is either a clique or an independent set in $H$, every non-independent set of three vertices in $A$ induces a triangle in $H$. Hence, for all $u \in B \cup C$ and a non-independent set $\{w_1, w_2, w_3\} \subseteq A$, $\{[w_1, w_2, w_3], [u]\}$ contains a triangle in $H$.

**Claim 1.** *Every edge in $H[B \cup C]$ has both end vertices in $C$.*

***Proof.*** Since $B \ominus C$ in $H$ and $B$ is an independent set, every edge in $H[B \cup C]$ has both end vertices in $C$. $\square$

Consider a vertex $x \in A$ and a non-independent set $\{u_1, u_2, u_3\}$ in $H[B \cup C]$ with an edge $u_1u_2$. By Claim 1, $u_1, u_2 \in C$. Since $A \oplus C$ in $H$, $\{x, u_1, u_2\}$ is a clique in $H$ and $\{[x], [u_1, u_2, u_3]\}$ contains a triangle in $H$. Hence $H$ is a one-three join of $H[A]$ and $H[B \cup C]$.

If $H$ admits a bipartite-join with vertex partition $V(H) = A_1 \cup A_2 \cup B_1 \cup B_2$ satisfying the conditions in the definition of bipartite-join, then we prove that $H$ is a one-three join of $H[A_1 \cup B_2]$ and $H[A_2 \cup B_1]$.

**Claim 2.** *Every edge in $H[A_i \cup B_j]$ has both end vertices in $B_j$ for $1 \leq i$, $j \leq 2$, $i \neq j$.*

***Proof.*** Since $A_i \ominus B_j$ in $H$ and $A_i$ is an independent set, every edge in $H[A_i \cup B_j]$ has both end vertices in $B_j$ for $1 \leq i, j \leq 2, i \neq j$. $\square$

Consider a vertex $v \in A_i \cup B_j$ and a non-independent set $\{w_1, w_2, w_3\}$ in $H[A_j \cup B_i]$ with an edge $w_1w_2$ for $1 \leq i, j \leq 2, i \neq j$. By Claim 2, $w_1, w_2 \in B_i$. Since $(A_i \cup B_j) \oplus B_i$ in $H$, $\{v, w_1, w_2\}$ is a clique in $H$. So $H[\{v, w_1, w_2, w_3\}]$ contains a triangle in $H$ for $1 \leq i, j \leq 2, i \neq j$. Hence $H$ is a one-three join of $H[A_1 \cup B_2]$ and $H[A_2 \cup B_1]$. ■

### 2.1. Characterization of graphs that admit one-three join

In this section, we prove that a graph $H$ admits a one-three join if and only if either $H$ is one of the basic graphs (bipartite, complement of bipartite, split graph) or $H$ admits a constrained homogeneous set or a bipartite-join or a join. Next, we discuss some observations on $H$.

**Observation 1.** *If a graph $H$ is a one-three join of $H_1$ and $H_2$ such that no edge in $H$ has one end vertex in $H_1$ and other in $H_2$ (i.e., $E(H) \cap [V(H_1), V(H_2)] = \emptyset$), then $V(H_i)$ is either an independent set or a clique for all $i \in \{1, 2\}$.*

[Hint: If not, any three vertices $\{u_1, u_2, u_3\}$ in $H_i$ induces $P_3$ or $K_2 \cup K_1$. For any $v \in V(H_j)$ $(i \neq j)$, applying one-three join on $\{[u_1, u_2, u_3], [v]\}$ leads to a contradiction.]

**Observation 2.** *If a graph $H$ is a one-three join of $H_1$ and $H_2$ such that $H_i$ is a disconnected graph that contains only non-trivial components for some $i \in \{1, 2\}$, then $H$ is a join of $H_1$ and $H_2$.*

***Proof.*** Follows using one-three join and since for every non-trivial component $M_r$ of $H_i$, $y \oplus V(M_r)$, for every $y \in V(H_j)$. ■

Let $H$ be a one-three join (not a join) of two vertex-disjoint graphs $H_1$ and $H_2$. Since $H$ is not a join of $H_1$ and $H_2$, there exists a vertex $v_1 \in V(H_1)$ such that $A_{H_2}(v_1) \neq \emptyset$ where $A_{H_2}(v_1) = \{u \in V(H_2) : uv_1 \notin E(H)\}$. Let $v_2 \in A_{H_2}(v_1)$, and let $N_i$ be the neighbours of $v_i$ in $H_i$ for $i \in \{1,2\}$. Let $I_i$ and $C_i$ be the set of all vertices which belong to the trivial and non-trivial components of $H_i \setminus (N_i \cup \{v_i\})$ for $i \in \{1,2\}$, respectively. So $V(H_i) = N_i \cup \{v_i\} \cup I_i \cup C_i$, $i \in \{1,2\}$ (see Figure 3). We adhere to the above notations whenever $H$ is a one-three join (not a join) of $H_1$ and $H_2$. Note that the graph $H$ in Figure 2(c) is not a join of $H_1$ and $H_2$. With respect to the non-adjacent vertices $x_3 \in V(H_1)$ and $y_1 \in V(H_2)$ in $H$, $N_1 = \{x_2, x_4\}, I_1 = \{x_1\}, C_1 = \emptyset$, $N_2 = \{y_2, y_3, y_4\}, I_2 = \emptyset$, and $C_2 = \emptyset$.

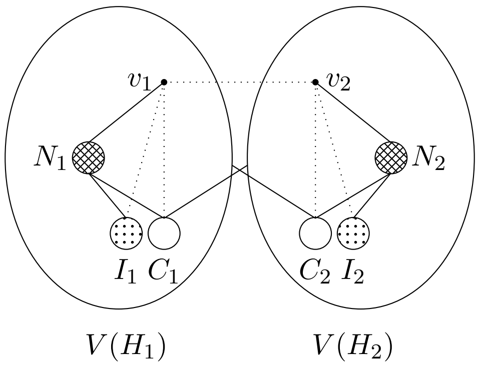


Figure 3. A schematic representation of the graph $H$ used in Lemma 2.

**Lemma 2.** *If $H$ is a one-three join (not a join) of $H_1$ and $H_2$, then*

(a) *$N_i$ is a clique in $H$ for $i \in \{1,2\}$.*

(b) *$N_i \oplus (V(H_i) \setminus N_i)$ in $H_i$ for $i \in \{1,2\}$. Moreover, $H$ is a join of $H[N_i]$ and $H \setminus N_i$ provided $N_i \neq \emptyset$ and $I_i \cup C_i \neq \emptyset$ for some $i \in \{1,2\}$.*

(c) *If $C_i \neq \emptyset$, then $C_i \oplus V(H_j)$ for $i, j \in \{1,2\}$.*

*In Figure 3, the circle filled with dots and cross lines represent independent set and clique in $H$, respectively.*

***Proof.*** (a) First we prove that $N_1$ is a clique in $H$. On the contrary, suppose that there exist $x, y \in N_1$ such that $xy \notin E(H)$. By one-three join of $H_1$ and $H_2$, $\{[x, v_1, y], [v_2]\}$ contains a triangle in $H$, a contradiction to the fact that $v_1v_2 \notin E(H)$. Hence $N_1$ is a clique in $H$. Similarly, $N_2$ is a clique in $H$.

(b) On the contrary, suppose that there exist $x \in N_1$ and $y \in V(H_1) \setminus N_1$ such that $xy \notin E(H)$. Clearly, $y \neq v_1$ and $H[v_1, x, y]$ contains exactly one edge $v_1x$. Since $H$ is a one-three join of $H_1$ and $H_2$, $\{[v_1, x, y], [v_2]\}$ contains a triangle in $H$, a contradiction to the fact that $v_1v_2 \notin E(H)$. Hence $N_1 \oplus (V(H_1) \setminus N_1)$ in $H_1$. Similarly, $N_2 \oplus (V(H_2) \setminus N_2)$ in $H_2$. Suppose $N_1 \neq \emptyset$ and $y \in I_1 \cup C_1 \neq \emptyset$; we prove that $N_1 \oplus V(H_2)$ in $H$. Take any $x \in N_1$ and $w \in V(H_2)$. Clearly, $H[v_1, x, y]$ induces a $P_3$ with middle vertex $x$. Since $H$ is a one-three join of $H_1$ and $H_2$, $\{[v_1, x, y], [w]\}$ contains a triangle in $H$ and $wx \in E(H)$. So, $N_1 \oplus V(H_2)$ in $H$. In addition, $N_1 \oplus (V(H_1) \setminus N_1)$ in $H$. Hence, $N_1 \oplus (V(H) \setminus N_1)$ in $H$ and $H$ is a join of $H[N_1]$ and $H \setminus N_1$.

(c) Assume $i = 1$ (a similar argument holds for $i = 2$). Consider an edge $x_1x_2 \in E(H[C_1])$ and a vertex $y \in V(H_2)$. Clearly, $H[v_1, x_1, x_2]$ contains exactly one edge $x_1x_2$. Since $H$ is a one-three join of $H_1$ and $H_2$, $\{[v_1, x_1, x_2], [y]\}$ contains a triangle in $H$ and $yx_1, yx_2 \in E(H)$. So, $y \in V(H_2)$ is adjacent to both end vertices of every edge in $H[C_1]$. Since $H[C_1]$ contains only non-trivial components, $C_1 \oplus V(H_2)$ in $H$. ■

**Lemma 3.** *A disconnected graph $H$ admits a one-three join if and only if $H$ is either bipartite, complement of bipartite, split graph or $H$ admits a constrained homogeneous set.*

***Proof.*** ($\Rightarrow$) Let $H$ be a one-three join of two vertex-disjoint graphs $H_1$ and $H_2$. If $E(H) \cap [V(H_1), V(H_2)] = \emptyset$, then $H$ is either a bipartite graph (union of independent sets), a split graph (union of an independent set and a clique) or a complement of bipartite graph (union of two cliques) by Observation 1. So $E(H) \cap [V(H_1), V(H_2)] \neq \emptyset$. Let $X_i$ and $Y_i$ be the set of all vertices that belong to the trivial and non-trivial components of $H_i$, respectively for $i \in \{1, 2\}$ (see Figure 4).

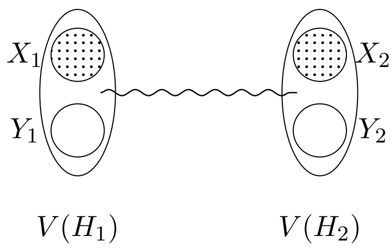


Figure 4. A disconnected graph $H$ in Lemma 3.

**Claim.** $X_1 \cup X_2 \neq \emptyset$.

***Proof.*** On the contrary, suppose $X_1 \cup X_2 = \emptyset$. There are two cases.

(a) $H_i$ is disconnected for some $i \in \{1, 2\}$. Then by Observation 2.1, $H$ is a join of $H_1$ and $H_2$, a contradiction to the fact that $H$ is disconnected.

(b) $H_i$ is connected for all $i \in \{1, 2\}$. Since $E(H) \cap [V(H_1), V(H_2)] \neq \emptyset$, $H$ is connected, a contradiction. So $X_1 \cup X_2 \neq \emptyset$. □

W.l.o.g. assume $X_1 \neq \emptyset$. Let $w \in X_1$. There are two cases.

*Case* 1. $Y_1 \neq \emptyset$. First we prove that $Y_1 \oplus V(H_2)$ in $H$. For an edge $uv \in E(H_1)$ and $y \in V(H_2)$, $\{[w,u,v],[y]\}$ contains a triangle in $H$ and $uy, vy \in E(H)$. So every vertex in $V(H_2)$ is adjacent to both end vertices of every edge in $H_1$. Since $Y_1$ contains only non-trivial components, $y \oplus Y_1$ in $H$ for all $y \in V(H_2)$. Hence $V(H_2) \oplus Y_1$ in $H$. Next, we prove that $V(H_2)$ is either an independent set or a clique. On the contrary, suppose that $V(H_2)$ is neither an independent set nor a clique in $H$. Then there exists a set of vertices $\{u_1,u_2,u_3\}$ in $H_2$ which induces either $P_3$ or $K_2 \cup K_1$. For any $v \in X_1$, by one-three join of $H_1$ and $H_2$, $\{[v],[u_1,u_2,u_3]\}$ contains a triangle in $H$ and $v$ has neighbours in $H_2$. So, every vertex $v \in X_1$ has neighbours in $H_2$. Indeed, $Y_1 \oplus V(H_2)$ in $H$. So $H$ is a connected graph, a contradiction. Hence, $V(H_2)$ is either an independent set or a clique. Clearly, $H$ admits a constrained homogeneous set with $V(H) = A \cup B \cup C$ where $A = V(H_2), B = X_1, C = Y_1$ (refer Figure 5(a)).

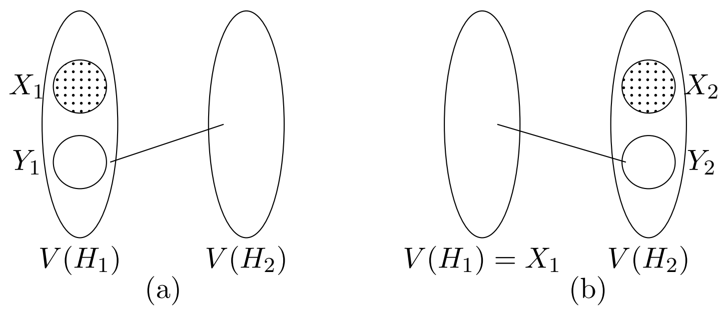


Figure 5. Constrained homogeneous set.

*Case* 2. $Y_1 = \emptyset$. There are three subcases.

*Case* 2.1. $Y_2 = \emptyset$. Then $H$ is a bipartite graph with partition $X_1 \cup X_2$.

*Case* 2.2. $Y_2 \neq \emptyset$ and $X_2 \neq \emptyset$. Then by Case 1, $H$ admits a constrained homogeneous set with $V(H) = A \cup B \cup C$ where $A = X_1$, $B = X_2$ and $C = Y_2$ (refer Figure 5(b)).

*Case* 2.3. $Y_2 \neq \emptyset$ and $X_2 = \emptyset$. Clearly, $H_2$ is connected. If not, by Observation 2, $H$ is a join of $H_1$ and $H_2$, a contradiction to the fact that $H$ is disconnected. Hence $H_2$ is connected. Next, we prove that $V(H_2)$ is a clique in $H$. On the contrary, suppose that there exists a pair of non-adjacent vertices $x, y$ in $H_2$. Since $H_2$ is connected, there exists a path $P(x,y) : (x =)x_1 - x_2 - x_3 - \cdots - x_k(= y)$ in $H_2$ where $k \geq 3$. Clearly, $\{x_1,x_2,x_3\}$ induces a $P_3$ in $H_2$. For any $v \in V(H_1)$, by one-three join of $H_1$ and $H_2$, $\{[v],[x_1,x_2,x_3]\}$ contains a triangle in $H$ and $v$ has neighbours in $H_2$. So, every vertex in $H_1$ has neighbours in $H_2$. Since $H_2$

is connected, $H$ is connected, a contradiction to the fact that $H$ is disconnected. Hence, $H$ is a split graph with partition $X_1 \cup Y_2$ (refer Figure 5(b)).

($\Leftarrow$) It follows from Lemma 1. ■

Next, we prove a characterization theorem for the graphs that admit one-three join.

**Theorem 4.** *A graph $H$ admits a one-three join if and only if $H$ is one of the following:*

(i) *a bipartite graph,*
(ii) *a complement of a bipartite graph,*
(iii) *a split graph,*
(iv) *a join of two graphs,*
(v) *$H$ admits a constrained homogeneous set or a bipartite-join.*

***Proof.*** ($\Rightarrow$) Let $H$ be a one-three join of two vertex-disjoint graphs $H_1$ and $H_2$. If $H$ is disconnected, then the result follows from Lemma 3. If $H^c$ is not connected, then $H$ admits (iv). So $H$ and $H^c$ are connected. Since $H$ is not a join of $H_1$ and $H_2$, there exists a vertex $v_1 \in V(H_1)$ such that $A_{H_2}(v_1) \neq \emptyset$, say $v_2 \in A_{H_2}(v_1)$ (see Figure 3). Next, we consider three cases:

*Case* 1. $N_1 \cup N_2 = \emptyset$. There are three subcases.

*Case* 1.1. If $C_1 \cup C_2 = \emptyset$, then $H$ is a bipartite graph with vertex partition $\{v_1\} \cup I_1$ and $\{v_2\} \cup I_2$.

*Case* 1.2. If $C_1 = \emptyset$ and $C_2 \neq \emptyset$ (a similar argument follows for $C_1 \neq \emptyset$ and $C_2 = \emptyset$), then by Lemma 2(c), $C_2 \oplus V(H_1)$ in $H$. Hence, $H$ admits a constrained homogeneous set with vertex partition $A = \{v_1\} \cup I_1, B = \{v_2\} \cup I_2$, and $C = C_2$.

*Case* 1.3. If $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$, then by Lemma 2(c), $C_i \oplus V(H_j)$ in $H$ for $i, j \in \{1,2\}, i \neq j$. Hence, $H$ admits a bipartite-join with vertex partition $A_1 = \{v_1\} \cup I_1$, $A_2 = \{v_2\} \cup I_2$, $B_1 = C_2$ and $B_2 = C_1$.

*Case* 2. $N_1 \neq \emptyset$ and $N_2 = \emptyset$ (similar argument follows for $N_1 = \emptyset$ and $N_2 \neq \emptyset$). Clearly, $I_1 \cup C_1 = \emptyset$, else by Lemma 2(b), $H$ is a join of two graphs, a contradiction with the fact that $H^c$ is connected. There are two subcases.

*Case* 2.1. $C_2 = \emptyset$. Then by Lemma 2(a), $H$ is a split graph with partition $N_1 \cup \{v_1\}$ and $I_2 \cup \{v_2\}$.

*Case* 2.2. $C_2 \neq \emptyset$. Then by Lemma 2(c), $C_2 \oplus V(H_1)$ in $H$. Hence, $H$ admits a constrained homogeneous set with partition $A = \{v_1\} \cup N_1$, $B = \{v_2\} \cup I_2$ and $C = C_2$.

*Case* 3. $N_1 \neq \emptyset$ and $N_2 \neq \emptyset$. We prove that $I_i \cup C_i = \emptyset$ for every $i \in \{1,2\}$. If not, w.l.o.g. assume that $I_1 \cup C_1 \neq \emptyset$. Then, by Lemma 2(b), $H$ is join of

$H[N_1]$ and $H \setminus N_1$, a contradiction with the fact that $H^c$ is connected. Hence by Lemma 2(a), $H$ is a complement of a bipartite graph with partition $N_1 \cup \{v_1\}$ and $N_2 \cup \{v_2\}$.

($\Leftarrow$) It follows from Lemma 1. ■

As an implication of Theorem 4, Lemma 1, and by the known results of Feder *et al.* [16], observe that the graph admitting one-three join can be recognized in polynomial time ($O(mn^8)$).

## 3. Applications

In this section, we solve a few optimization problems on a subclass of odd hole-free graphs defined using one-three join and co-join. The following two observations follows from the definition of one-three join.

**Observation 3.** *If $H_1$ and $H_2$ are vertex-disjoint odd hole-free graphs, then a one-three join $H$ of $H_1$ and $H_2$ is also odd hole-free.*

[Hint: Suppose $M = \{v_1, v_2, \ldots, v_{2k+1}\}$ induces an odd hole in $H$. Then it is easy to verify that for every $v_r \in V(H_i) \cap M$, both $v_{r-1}, v_{r+1}$ belongs to $V(H_j)$, $1 \leq i$, $j \leq 2$, $i \neq j$ for $r$ mod $(2k+1)$.]

**Observation 4.** *If $H_1$ and $H_2$ are vertex-disjoint odd anti-hole-free graphs, then a one-three join $H$ of $H_1$ and $H_2$ is also odd anti-hole-free.*

### The class $\mathcal{M}_H$

Let $H$ be an odd hole-free graph which contains an odd anti-hole as an induced subgraph and let $\mathcal{M}_H$ be the class of all graphs generated from the induced subgraphs of $H$ by using one-three join and co-join recursively. By Observation 3, $\mathcal{M}_H$ is a subclass of odd hole-free graphs. Note that every $P_4$-free graph can be generated by repeated application of join and co-join starting from a single vertex. Since every graph in the class $\mathcal{M}_H$ is generated by either co-join or one-three join (note that join is a special case of one-three join), it contains all $P_4$-free graphs. Note that $\mathcal{M}_H$ contains all complete graphs and its complements. Let $G_1$ and $G_2$ be two vertex-disjoint complete graphs (or a complete graph and an edgeless graph). Adding any edge between $V(G_1)$ and $V(G_2)$ preserves one-three join. Hence, $\mathcal{M}_H$ contains the complement of all bipartite graphs (respectively, all split graphs). Similarly, $\mathcal{M}_H$ contains all bipartite graphs (if $G_1$ and $G_2$ are edgeless graphs). The Strong Perfect Graph Theorem [10] states that a graph is perfect if and only if it is odd hole-free and odd anti-hole-free. So, $\mathcal{M}_H$ contains some imperfect graphs (by the definition of the class $\mathcal{M}_H$). An imperfect graph $G$ in Figure 6 is a member of $\mathcal{M}_{C_7{}^c}$ which is neither a join nor a co-join of two graphs. Note that $G$ admits a one-three join with partitions $V_1 = V(C_7^c) \cup \{v_9\}$ and $V_2 = \{v_8, v_{10}\}$. Hence $\mathcal{M}_H$ is odd hole-free, contains all $P_4$-free graphs, all bipartite graphs, complement of all bipartite graphs, all split graphs, and some imperfect graphs.

Next, we prove that $\mathcal{M}_H$ is induced hereditary.

**Theorem 5.** *If $K \in \mathcal{M}_H$, then every induced subgraph $U$ of $K$ belongs to $\mathcal{M}_H$.*

***Proof.*** Let us prove by induction on the number of vertices $n$ of the graph $K$. For $n = 1, 2, 3, 4$, the result is obvious. If $n = 5$, then every induced subgraph $U$ of $K$ contains at most 4 vertices and hence is a member of $\mathcal{M}_H$.

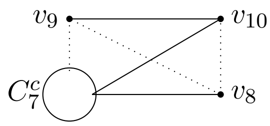


Figure 6. An imperfect graph $G$ in $\mathcal{M}_{C_7{}^c}$.

Induction hypothesis: Assume the result for $n = k$.
Induction step: We prove the result for $n = k + 1$. For a graph $K$ containing $k + 1$ vertices, let $U$ be an induced subgraph of $K$. If $K$ is an induced subgraph of $H$, then $U \in \mathcal{M}_H$. Else, there exists $G_1, G_2 \in \mathcal{M}_H$ such that $K$ is either a one-three join or a co-join of $G_1$ and $G_2$. Then there are two cases:

(i) $V(G_i) \cap V(U) \neq \emptyset$ for all $i \in \{1, 2\}$. Let $V(U_i) = V(G_i) \cap V(U)$. Clearly, $|V(G_i)| \leq k$ and hence by induction hypothesis, $U_i \in \mathcal{M}_H$ for all $i \in \{1, 2\}$. So $U$ is either a one-three join or a co-join of $U_1$ and $U_2$ and hence $U \in \mathcal{M}_H$.

(ii) $V(G_i) \cap V(U) \neq \emptyset$ for exactly one $i$ where $i \in \{1, 2\}$. W.l.o.g. assume $V(U) \subseteq V(G_1)$. Since $|V(G_1)| \leq k$, by induction hypothesis, $U \in \mathcal{M}_H$. ∎

Recall that a *clique* (*independent set*) is a subset of vertices of a graph $G$ which are pairwise adjacent (respectively, non-adjacent) in $G$. Next, we solve the maximum independent set problem, the maximum clique problem, the minimum coloring problem, and the minimum clique cover problem for the class $\mathcal{M}_H$. The MISP for a graph $G$ is to find an independent set with maximum cardinality in $G$ and the maximum weight independent set problem (MWISP) for a weighted graph $G$ is to find an independent set with maximum total weight in $G$. Let $\alpha_w(G)$ denotes the weighted independence number of $G$. The MWISP reduces to MISP if the weight of each vertex in the graph is equal to 1. The MISP is NP-hard in general, but solvable in polynomial time on various graph classes [2, 4–8, 20, 21]. The maximum clique problem for a graph $G$ is to find a clique with maximum cardinality in $G$. The minimum coloring problem for a graph $G$ is to determine the smallest number of colors in a vertex coloring of $G$. The minimum clique cover problem for a graph $G$ is to determine the smallest number of cliques of $G$ required to cover $V(G)$.

For a graph $H$ and $U \subseteq V(H)$, a graph $H^{'}$ is obtained by contracting $U$ in $H$ if $V(H^{'}) = V(H \setminus U) \cup \{u\}$ and $E(H^{'}) = E(H \setminus U) \cup E'$ where $E' = \{uw \;:\; w \in N(U)$ in $H\}$. Let $H$ be a graph that admits a constrained homogeneous set ($V(H) = A \cup B \cup C$). Let $H_c$ be the graph obtained from $H$ by contracting $C$ (contracted to $c$) where $V(H_c) = (V(H) \setminus C) \cup \{c\}$ (see Figure 7(a)). Similarly, for a graph $H$ that admits a bipartite-join ($V(H) = A_1 \cup A_2 \cup B_1 \cup B_2$), define $H_b$ as a graph obtained from $H$ by contracting $B_i$ (contracted to $b_i$) with $V(H_b) = (V(H) \setminus (B_1 \cup B_2)) \cup \{b_1, b_2\}$ for $i \in \{1, 2\}$ (see Figure 7(b)).

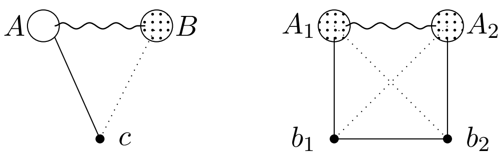


Figure 7. (a) $H_c$ and (b) $H_b$.

**Observation 5.** (a) *If $H$ admits a constrained homogeneous set, then $H_c$ is either a bipartite graph or a split graph with vertex partition $A$ and $B \cup \{c\}$.*
(b) *If $H$ admits a bipartite-join, then $H_b$ is a bipartite graph with vertex partition $A_1 \cup \{b_2\}$ and $A_2 \cup \{b_1\}$.*

For a graph $H$ that admits a constrained homogeneous set, let $H_c^{\alpha}$ be a weighted graph obtained from $H_c$ with vertex weights $w(v) = 1$ if $v \neq c$ and $w(c) = \alpha(H[C])$. Clearly, $\alpha(H) = \alpha_w(H_c^{\alpha})$. Similarly, if $H$ admits a bipartite-join, then let $H_b^{\alpha}$ be a weighted graph obtained from $H_b$ with vertex weights $w(v) = 1$ if $v \neq b_i$ and $w(b_i) = \alpha(H[B_i])$ for every $i \in \{1, 2\}$. Also, $\alpha(H) = \alpha_w(H_b^{\alpha})$. Hence the maximum independent set problem for $H$ can be solved efficiently since MWISP can be efficiently solved for split and bipartite graphs [1, 5, 18, 19] where $H$ either admits a constrained homogeneous set or a bipartite-join, provided $\alpha(H[C])$ and $\alpha(H[B_i])$ is known for all $i \in \{1, 2\}$. In a similar manner, $H_c^{\omega}$ and $H_b^{\omega}$ are defined by replacing $\alpha$ by $\omega$ in the definition of the graphs $H_c^{\alpha}$ and $H_b^{\alpha}$, respectively.

Note that the MWISP for (a) bipartite graphs can be solved in $O(n^4)$ time [1] (b) for split graphs and co-bipartite graphs can be solved in linear time [5, 18].

**Theorem 6.** *The MISP for $\mathcal{M}_H$ can be solved efficiently.*

***Proof.*** Every graph $K$ in $\mathcal{M}_H$ is either an induced subgraph of $H$ or obtained by co-join or one-three join of two graphs in $\mathcal{M}_H$. If $K$ is an induced subgraph of $H$, then MISP can be solved in $O(1)$ time [4] (graphs with constant size). If $K$ is a co-join of two graphs say $U_1$ and $U_2$, then $\alpha(K) = \alpha(U_1) + \alpha(U_2)$. The MISP for $K$ can be solved efficiently provided $\alpha(U_1)$ and $\alpha(U_2)$ can be computed efficiently. If $K$ is a one-three join of two graphs say $U_1$ and $U_2$, then by Theorem 4, $K$ is one of the following.

*Case* (i) If $K$ is a bipartite graph or complement of a bipartite graph or a split graph, then the MISP for $K$ can be solved efficiently [1, 5, 18, 19] since $U_1$ and $U_2$ are either independents or cliques.

*Case* (ii) If $K$ is a join of $U_1$ and $U_2$, then $\alpha(K) = \max\{\alpha(U_1), \alpha(U_2)\}$. So, the MISP for $K$ can be solved efficiently if $\alpha(U_1)$ and $\alpha(U_2)$ can be computed efficiently.

*Case* (iii) If $K$ admits a constrained homogeneous set, then $K_c^\alpha$ is either a weighted bipartite graph or a weighted split graph. Hence, the MWISP for $K_c^\alpha$ can be solved efficiently provided the MISP for $K[C]$ can be solved efficiently.

*Case* (iv) If $K$ admits a bipartite-join, then $K_b^\alpha$ is a weighted bipartite graph. Hence, the MWISP for $K_b^\alpha$ can be solved efficiently provided MISP for $K[B_1]$ and $K[B_2]$ can be solved efficiently.

Note that the graphs $U_1, U_2, K[C], K[B_1], K[B_2] \in \mathcal{M}_H$ (by Theorem 5). Repeat the above procedure for these graphs, until we obtain an induced subgraph of $H$. Hence, the recursive decomposition leads to a binary tree with at most $n-1$ internal nodes and at most $n$ leaves. Note that the tree can be constructed in $O(mn^9)$ time as the recognition problem for one-three join can be solved in $O(mn^8)$ time. Using bottom-up approach, we solve the MISP or MWISP for each internal node of the tree. Since the graph induced by each node is either a join of two graphs or a (weighted) bipartite graph or (weighted) complement of a bipartite graph or a (weighted) split graph, the MISP or MWISP for the graph induced by each internal node can be solved in $O(n^4)$ time. So, the time complexity to solve MISP or MWISP for all the internal nodes is $O(n^5)$. Since the construction of the tree and solving the MISP for all internal nodes are executed in parallel, the MISP for $K$ can be solved in $O(mn^9)$ time. ■

By the same arguments as in Theorem 6, we obtain the following theorem.

**Theorem 7.** *The maximum independent set problem, maximum clique problem, minimum coloring problem and minimum clique cover problem for $\mathcal{M}_H$ can be solved efficiently.*

## 4. Conclusions

We studied some algorithmic graph problems such as maximum independent set problem, the maximum clique problem, the minimum coloring problem, and the minimum clique cover problem for the class $\mathcal{M}_H$ of graphs (a subclass of odd hole-free graphs) obtained by the graph operation *one-three join*. The main result of the paper is the characterization of graphs that admit one-three join, which is useful in the recognition problem.

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