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# ASYMPTOTIC SHARPNESS OF BOUNDS ON HYPERTREES

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### Abstract

The hypertree can be defined in many different ways. Katona and Szabó introduced a new, natural definition of hypertrees in uniform hypergraphs and investigated bounds on the number of edges of the hypertrees. They showed that a k-uniform hypertree on n vertices has at most  $\binom{n}{k-1}$  edges and they conjectured that the upper bound is asymptotically sharp. Recently, Szabó verified that the conjecture holds by recursively constructing an infinite sequence of k-uniform hypertrees and making complicated analyses for it. In this note we give a short proof of the conjecture by directly constructing a sequence of k-uniform k-hypertrees.

Keywords: hypertree, semicycle in hypergraph, chain in hypergraph.

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#### 1. INTRODUCTION

Paths, cycles and trees are among the most fundamental objects in graph theory. As we have known, trees have a number of interesting structural properties, and trees are the most common objects in all of graph theory. These concepts have been generalized to hypergraphs in a lot of different ways [1, 3, 4].

Recently, Katona and Szabó [2] generalized the notion of trees to uniform hypergraphs and discussed lower and upper bounds on the number of edges of such hypertrees. They showed that a k-uniform hypertree on n vertices has at most  $\binom{n}{k-1}$  edges and they posed some conjectures for bounds on the number of edges in the hypertrees.

We now recall definitions of hypertrees for k-uniform hypergraphs given in [2]. Let  $\mathcal{F} = (V, \mathcal{E})$  be a k-uniform hypergraph (with no multiple edges).

The hypergraph  $\mathcal{F}$  is a *chain* if there exists a sequence  $v_1, v_2, \ldots, v_l$  of its vertices such that every vertex appears at least once (possibly more times),  $v_1 \neq v_l$  and  $\mathcal{E}$  consists of l - k + 1 distinct edges of the form  $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}, 1 \leq i \leq l - k + 1$ . The length of the chain is l - k + 1, i.e., the number of its edges.

The hypergrah  $\mathcal{F}$  is a *semicycle* if there exists a sequence  $v_1, v_2, \ldots, v_l$  of its vertices such that every vertex appears at least once (possibly more times),  $v_1 = v_l$  and for all  $1 \le i \le l - k + 1$ ,  $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$  are distinct edges of  $\mathcal{F}$ . The length of the semicycle  $\mathcal{F}$  is l - k + 1, the number of its edges. From the definition it follows that every semicycle has at least 3 edges.

A k-uniform hypergraph  $\mathcal{H}$  is chain-connected if every pair of its vertices is connected by a chain. A k-uniform hypergraph  $\mathcal{H}$  is semicycle-free if it contains no semicycle as a subhypergraph. A hypertree is a k-uniform hypergraph  $\mathcal{H}$  $(k \geq 2)$  such that  $\mathcal{H}$  is chain-connected and semicycle-free. A hypertree is called an *l*-hypertree if every chain in it is of length at most *l*.

Katona and Szabó [2] investigated lower and upper bounds on the number of edges of hypertrees. They obtained the following results on the upper bounds.

**Theorem 1** (Katona, Szabó [2]). If  $\mathcal{H}$  is a semicycle-free k-uniform hypergraph on n vertices, then  $|\mathcal{E}(\mathcal{H})| \leq {n \choose k-1}$ , and this bound is asymptotically sharp for k = 3.

**Theorem 2** (Katona, Szabó [2]). Let  $1 \le l \le k$  and  $\mathcal{H}$  be a k-uniform l-hypertree on n vertices. Then  $|\mathcal{E}(\mathcal{H})| \le \frac{1}{k-l+1} \binom{n}{k-1}$ . This bound is asymptotically sharp in the case l = 2, k = 3.

**Conjecture 3** (Katona, Szabó [2]). The upper bound in Theorem 1 can be reached by a sequence of k-hypertrees.

Recently, Szabó [5] proved the above conjecture by recursively constructing a sequence of k-hypertrees. However, the construction is intricate and technical.

In this note we give a shorter proof of the conjecture by directly constructing a sequence of k-hypertrees.

We will prove the main result below in next section.

**Theorem 4.** For  $k \ge 3$ , there exists an infinite sequence of k-hypertrees where the number of edges is asymptotically  $\binom{n}{k-1}$ .

### 2. Proof of Theorem 4

Let  $\mathcal{H} = (V, \mathcal{E})$  be an arbitrary k-uniform k-hypertree and let  $V = \{v_1, v_2, \ldots, v_n\}$ . Now let us define a new k-uniform hypergraph  $\mathcal{H}' = (V \cup V', \mathcal{E} \cup \mathcal{E}')$ , where  $V' = \{1, 2, \ldots, k-1\}^n$ , i.e., the set of n-dimensional vectors over  $\{1, 2, \ldots, k-1\}$ , and  $\mathcal{E}' = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} | v_i \in V, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1} \in V'$ , where the *i*th coordinate of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$  is the smallest coordinate where all the coordinates are distinct}.

By the definition of  $\mathcal{E}'$ , if  $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$ , then all of  $1, 2, \dots, k-1$  appear in the *i*th column of the  $(k-1) \times n$  matrix

$$M = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix},$$

where every  $\mathbf{u}_i$  is regarded as a row vector, but at least one of  $1, 2, \ldots, k-1$  do not appear in the *i*'th column of the matrix M for each i' < i.

We first prove that  $\mathcal{H}'$  is a k-uniform k-hypertree.

#### **Lemma 5.** $\mathcal{H}'$ is a k-uniform k-hypertree.

**Proof.** To prove that  $\mathcal{H}'$  is a k-uniform k-hypertree, we need to verify that  $\mathcal{H}'$  satisfies the following three properties.

(i)  $\mathcal{H}'$  is chain-connected. Clearly, any two vertices of V are chain-connected, since  $\mathcal{H}$  is a hypertree and all of its edges are edges of  $\mathcal{H}'$ . For any  $\mathbf{u}_1, \mathbf{u}_2 \in V'$ , let idenote the position of the first coordinate where they differ. Then we consider the vertices  $\mathbf{u}_3, \ldots, \mathbf{u}_{k-1} \in V'$  each of which the first i-1 coordinates are the same as the first i-1 coordinates of  $\mathbf{u}_1, \mathbf{u}_2$  but the *i*th coordinates of  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$  differ from each other. By the definition of  $\mathcal{E}'$ , we see that  $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$ . This implies that  $\mathbf{u}_1, \mathbf{u}_2$  are connected by a chain of length one in  $\mathcal{H}'$ . For any  $\mathbf{u}_1 \in V'$  and  $v_i \in V$ , let  $\mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$  be k-2 vertices in V' such that the first i-1coordinates of each  $\mathbf{u}_i$  ( $2 \leq i \leq k-1$ ) are the same as the first i-1 coordinates of  $\mathbf{u}_1$ , but the *i*th coordinates of  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$  differ from each other. By the definition of  $\mathcal{E}'$ ,  $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$ . So  $\mathbf{u}_1$  and  $v_i$  are connected by a chain of length one.

(ii)  $\mathcal{H}'$  is semicycle-free. Suppose, to the contrary, that  $\mathcal{H}'$  contains a semicycle C. By the definition, we have  $|e \cap e'| \leq 1$  for all  $e \in \mathcal{E}$ ,  $e' \in \mathcal{E}'$ . This implies that all edges in C belong to either  $\mathcal{E}$  or  $\mathcal{E}'$  since  $k \geq 3$ . If all edges in C lie in  $\mathcal{E}$ , then C is also a semicycle of  $\mathcal{H}$ , which contradicts that  $\mathcal{H}$  is semicycle-free. Therefore, all edges in C lie in  $\mathcal{E}'$ .

Without loss of generality, let  $e_1 = \{v_1, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$  be an edge in C. Then, by definition, the first coordinates of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$  are the first coordinates that are different from each other. We may assume that i is the first coordinate of  $\mathbf{u}_i$  for  $1 \leq i \leq k-1$ . Clearly, for any  $1 < j \leq n$ ,  $\{v_j, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$  does not belong to  $\mathcal{E}'$ . Let  $e_1$  and  $e_2$  be two consecutive edges in C. Then, by the definition of the semicycle,  $|e_1 \cap e_2| = k-1$ . This implies that  $v_1$  must be in  $e_2$ , and so each edge of C contains the vertex  $v_1$ .

If we write down the vertices of the semicycle in a sequence, denoting the vertices from V by  $v_i$  and those from V' by  $\mathbf{u}_j$ , there are k possible sequences as follows:

(1)  $v_1, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ : only one edge, which obviously cannot be a semicycle.

(2)  $\mathbf{u}_1, v_1, \mathbf{u}_2, u_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ : only two edges. This sequence cannot be a semicycle because a semicycle must have at least three edges.

(3)  $\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$ : there are three edges. By the definition of a semicycle, the first and the last vertices of the sequence must be the same. Because  $\{\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}\}$  is an edge of  $\mathcal{E}'$ , the first coordinate of  $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$  differ from each other. We may assume  $\{1, 2, \ldots, k-1\}$  are respectively the first coordinate of  $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ . Besides,  $\mathbf{u}_2, v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ is also an edge of  $\mathcal{E}'$ . The first coordinate of  $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$  differ from each other. So the first coordinate of  $\mathbf{u}_k$  must be 1, which is the same with the first coordinate of  $\mathbf{u}_1$ . Similarly, for the edge  $v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$ , we may get that the first coordinate of  $\mathbf{u}_{k+1}$  must be 2. Obviously,  $\mathbf{u}_1$  and  $\mathbf{u}_{k+1}$  differ in the first coordinate. As  $\mathbf{u}_1$  and  $\mathbf{u}_{k+1}$  are not the same vertices, this sequence cannot be a semicycle.

(k)  $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, v_1, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{2k-2}$ . We assume that  $\{1, 2, \ldots, k-1\}$  are respectively the first coordinate of  $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ . According to the chainconnected properties, the k edges in this sequence all contain the vertex  $v_1$ . So the first coordinates of the vertices in every edge except  $v_1$  differ from each other. For  $k \leq j \leq 2k-2$ , the first coordinate of  $\mathbf{u}_j$  are the same as  $\mathbf{u}_{j-k+1}$ . So the first coordinate of  $\mathbf{u}_{2k-2}$  is k-1. As  $\mathbf{u}_1$  and  $\mathbf{u}_{2k-2}$  are not the same vertices, this sequence cannot be a semicycle.

Without loss of generality, let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{i-1}, v_1, \mathbf{u}_i, \dots, \mathbf{u}_t$  be the sequence of vertices in C such that  $\{v_1, \mathbf{u}_i, \mathbf{u}_{i+1}, \dots, \mathbf{u}_{i+(k-2)}\}, i = 1, 2, \dots, t - (k-2)$  are

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all the edges of C. Note that every semicycle has at least 3 edges. Then  $t \ge k+1$ and the first coordinates of  $\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}$  differ from each other. By the definition of the semicycle, it can be verified that  $t \le 2k - 2$  and  $\mathbf{u}_t = \mathbf{u}_1$ . So the length of C is at most k. Hence, the first coordinate of  $\mathbf{u}_k$  is the same as the first coordinate of  $\mathbf{u}_1$ , so the first coordinate of  $\mathbf{u}_k$  is also 1. In fact, it is easy to see that the first coordinate of  $\mathbf{u}_j$  is the same as that of  $\mathbf{u}_{j-k+1}$  for each  $j, k \le j \le t \le 2k - 2$ . Thus the first coordinate of  $\mathbf{u}_t$  is t - k + 1. Obviously,  $t - k + 1 \ne 1$  as  $t \le 2k - 2$ . This contradicts the fact that  $\mathbf{u}_1 = \mathbf{u}_t$ .

(iii)  $\mathcal{H}'$  is a k-hypertree. For any  $e \in \mathcal{E}$ ,  $e' \in \mathcal{E}'$ , since  $|e \cap e'| \leq 1$  and  $k \geq 3$ , all chains in  $\mathcal{H}'$  belong to either  $\mathcal{E}$  or  $\mathcal{E}'$ . Let P be a chain in  $\mathcal{H}'$ . If P belongs to  $\mathcal{E}$ , P is also a chain in  $\mathcal{H}$ . Since  $\mathcal{H}$  is k-hypertree, every chain in it is of length at most k, so P is of length at most k in  $\mathcal{H}'$ . If P belongs to  $\mathcal{E}'$ , as we noted in the proof in (ii), P contains at most 2k - 1 vertices. This implies that P is of length at most k in  $\mathcal{H}'$ .

Return to the proof of Theorem 3.

By the construction of  $\mathcal{H}'$ , we have  $|V \cup V'| = n + (k-1)^n$ . Now we count the number of edges of  $\mathcal{H}'$ . For each  $v_i \in V$ , let  $\mathcal{E}'_i = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\} \mid \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1} \in V'\}$ . Then  $\mathcal{E}' = \bigcup_{i=1}^n \mathcal{E}'_i$ . By the construction of  $\mathcal{E}'_i$ , it is easy to see that

$$|\mathcal{E}'_i| = ((k-1)^{k-1} - (k-1)!)^{i-1} ((k-1)^{k-1})^{n-i}.$$

Hence,

$$|\mathcal{E}'| = x^{n-1} + yx^{n-2} + y^2x^{n-3} + \dots + y^{n-1},$$

where  $x = (k-1)^{k-1} - (k-1)!$ ,  $y = (k-1)^{k-1}$ . Therefore,

$$|\mathcal{E}(\mathcal{H}')| = |\mathcal{E} \cup \mathcal{E}'| \ge |\mathcal{E}'| = \frac{y^n - x^n}{y - x} = \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}.$$

We count the limit of the ratio  $|\mathcal{E}(\mathcal{H}')| / {|V(\mathcal{H}')| \choose k-1}$ .

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \ge \frac{\frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}}{\binom{n+(k-1)^n}{k-1}}$$
$$= \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}$$
$$\cdot \frac{(k-1)![n+(k-1)^n - (k-1)]!}{[n+(k-1)^n]!}$$

$$\geq \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{[n+(k-1)^n]^{k-1}}$$
$$= \frac{1 - \left[\frac{(k-1)^{k-1} - (k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n+(k-1)^n}{(k-1)^n}\right]^{k-1}} = \frac{1 - \left[1 - \frac{(k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n}{(k-1)^n} + 1\right]^{k-1}} \to 1(n \to \infty).$$

On the other hand, by Theorem 2, we have

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \le 1.$$

So, when  $n \to \infty$ , we obtain

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \to 1.$$

Thus, if  $\{\mathcal{H}_i\}_{i=1}^{\infty}$  is a sequence of k-uniform k-hypertrees on  $n \ (n \ge k)$  vertices such that  $\lim_{n\to\infty} |V(\mathcal{H}_i)| = \infty$ , then,

$$|\mathcal{E}(\mathcal{H}'_i)| \sim igg( rac{|V(\mathcal{H}'_i)|}{k-1} igg).$$

Now let us review the construction given in [5]. In [5], the author constructed a k-hypertree  $H_i^k = (V_{2^i,k}, E_{2^i,k})$ , where  $|V_{2^i}| = 2^i + F(2^i, k - 1)$ ,  $|E_{2^i,k}| = {\binom{2^i}{k-1}} + |D_{n,k}|$ , and  $D_{n,k}$  is the set of edges of a hypertree  $F_{n,k} = (U_{n,k}, D_{n,k})$ . It is proved that  $|E_{2^i,k}|$  is asymptotically  $\binom{|V_{2^i}|}{k-1}$ . The construction of  $H_i^k$  and counting its number of edges are intricate and technical. This note provides an elegant construction of the desired k-hypertree by using vectors and matrices, and the proof is easy.

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