# ASYMPTOTIC SHARPNESS OF BOUNDS ON HYPERTREES 

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#### Abstract

The hypertree can be defined in many different ways. Katona and Szabó introduced a new, natural definition of hypertrees in uniform hypergraphs and investigated bounds on the number of edges of the hypertrees. They showed that a $k$-uniform hypertree on $n$ vertices has at most $\binom{n}{k-1}$ edges and they conjectured that the upper bound is asymptotically sharp. Recently, Szabó verified that the conjecture holds by recursively constructing an infinite sequence of $k$-uniform hypertrees and making complicated analyses for it. In this note we give a short proof of the conjecture by directly constructing a sequence of $k$-uniform $k$-hypertrees.


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## 1. Introduction

Paths, cycles and trees are among the most fundamental objects in graph theory. As we have known, trees have a number of interesting structural properties, and trees are the most common objects in all of graph theory. These concepts have been generalized to hypergraphs in a lot of different ways $[1,3,4]$.

Recently, Katona and Szabó [2] generalized the notion of trees to uniform hypergraphs and discussed lower and upper bounds on the number of edges of such hypertrees. They showed that a $k$-uniform hypertree on $n$ vertices has at most $\binom{n}{k-1}$ edges and they posed some conjectures for bounds on the number of edges in the hypertrees.

We now recall definitions of hypertrees for $k$-uniform hypergraphs given in [2]. Let $\mathcal{F}=(V, \mathcal{E})$ be a $k$-uniform hypergraph (with no multiple edges).

The hypergraph $\mathcal{F}$ is a chain if there exists a sequence $v_{1}, v_{2}, \ldots, v_{l}$ of its vertices such that every vertex appears at least once (possibly more times), $v_{1} \neq$ $v_{l}$ and $\mathcal{E}$ consists of $l-k+1$ distinct edges of the form $\left\{v_{i}, v_{i+1}, \ldots, v_{i+k-1}\right\}$, $1 \leq i \leq l-k+1$. The length of the chain is $l-k+1$, i.e., the number of its edges.

The hypergrah $\mathcal{F}$ is a semicycle if there exists a sequence $v_{1}, v_{2}, \ldots, v_{l}$ of its vertices such that every vertex appears at least once (possibly more times), $v_{1}=v_{l}$ and for all $1 \leq i \leq l-k+1,\left\{v_{i}, v_{i+1}, \ldots, v_{i+k-1}\right\}$ are distinct edges of $\mathcal{F}$. The length of the semicycle $\mathcal{F}$ is $l-k+1$, the number of its edges. From the definition it follows that every semicycle has at least 3 edges.

A $k$-uniform hypergraph $\mathcal{H}$ is chain-connected if every pair of its vertices is connected by a chain. A $k$-uniform hypergraph $\mathcal{H}$ is semicycle-free if it contains no semicycle as a subhypergraph. A hypertree is a $k$-uniform hypergraph $\mathcal{H}$ $(k \geq 2)$ such that $\mathcal{H}$ is chain-connected and semicycle-free. A hypertree is called an l-hypertree if every chain in it is of length at most $l$.

Katona and Szabó [2] investigated lower and upper bounds on the number of edges of hypertrees. They obtained the following results on the upper bounds.

Theorem 1 (Katona, Szabó [2]). If $\mathcal{H}$ is a semicycle-free $k$-uniform hypergraph on $n$ vertices, then $|\mathcal{E}(\mathcal{H})| \leq\binom{ n}{k-1}$, and this bound is asymptotically sharp for $k=3$.

Theorem 2 (Katona, Szabó [2]). Let $1 \leq l \leq k$ and $\mathcal{H}$ be a $k$-uniform l-hypertree on $n$ vertices. Then $|\mathcal{E}(\mathcal{H})| \leq \frac{1}{k-l+1}\binom{n}{k-1}$. This bound is asymptotically sharp in the case $l=2, k=3$.

Conjecture 3 (Katona, Szabó [2]). The upper bound in Theorem 1 can be reached by a sequence of $k$-hypertrees.

Recently, Szabó [5] proved the above conjecture by recursively constructing a sequence of $k$-hypertrees. However, the construction is intricate and technical.

In this note we give a shorter proof of the conjecture by directly constructing a sequence of $k$-hypertrees.

We will prove the main result below in next section.
Theorem 4. For $k \geq 3$, there exists an infinite sequence of $k$-hypertrees where the number of edges is asymptotically $\binom{n}{k-1}$.

## 2. Proof of Theorem 4

Let $\mathcal{H}=(V, \mathcal{E})$ be an arbitrary $k$-uniform $k$-hypertree and let $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. Now let us define a new $k$-uniform hypergraph $\mathcal{H}^{\prime}=\left(V \cup V^{\prime}, \mathcal{E} \cup \mathcal{E}^{\prime}\right)$, where $V^{\prime}=\{1,2, \ldots, k-1\}^{n}$, i.e., the set of $n$-dimensional vectors over $\{1,2, \ldots, k-1\}$, and $\mathcal{E}^{\prime}=\left\{\left\{v_{i}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\} \mid v_{i} \in V, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1} \in V^{\prime}\right.$, where the $i$ th coordinate of the vectors $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ is the smallest coordinate where all the coordinates are distinct $\}$.

By the definition of $\mathcal{E}^{\prime}$, if $\left\{v_{i}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\} \in \mathcal{E}^{\prime}$, then all of $1,2, \ldots, k-1$ appear in the $i$ th column of the $(k-1) \times n$ matrix

$$
M=\left(\begin{array}{c}
\mathbf{u}_{1} \\
\mathbf{u}_{2} \\
\vdots \\
\mathbf{u}_{k-1}
\end{array}\right)
$$

where every $\mathbf{u}_{i}$ is regarded as a row vector, but at least one of $1,2, \ldots, k-1$ do not appear in the $i^{\prime}$ th column of the matrix $M$ for each $i^{\prime}<i$.

We first prove that $\mathcal{H}^{\prime}$ is a $k$-uniform $k$-hypertree.
Lemma 5. $\mathcal{H}^{\prime}$ is a $k$-uniform $k$-hypertree.
Proof. To prove that $\mathcal{H}^{\prime}$ is a $k$-uniform $k$-hypertree, we need to verify that $\mathcal{H}^{\prime}$ satisfies the following three properties.
(i) $\mathcal{H}^{\prime}$ is chain-connected. Clearly, any two vertices of $V$ are chain-connected, since $\mathcal{H}$ is a hypertree and all of its edges are edges of $\mathcal{H}^{\prime}$. For any $\mathbf{u}_{1}, \mathbf{u}_{2} \in V^{\prime}$, let $i$ denote the position of the first coordinate where they differ. Then we consider the vertices $\mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1} \in V^{\prime}$ each of which the first $i-1$ coordinates are the same as the first $i-1$ coordinates of $\mathbf{u}_{1}, \mathbf{u}_{2}$ but the $i$ th coordinates of $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ differ from each other. By the definition of $\mathcal{E}^{\prime}$, we see that $\left\{v_{i}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\} \in \mathcal{E}^{\prime}$. This implies that $\mathbf{u}_{1}, \mathbf{u}_{2}$ are connected by a chain of length one in $\mathcal{H}^{\prime}$. For any $\mathbf{u}_{1} \in V^{\prime}$ and $v_{i} \in V$, let $\mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ be $k-2$ vertices in $V^{\prime}$ such that the first $i-1$ coordinates of each $\mathbf{u}_{i}(2 \leq i \leq k-1)$ are the same as the first $i-1$ coordinates of $\mathbf{u}_{1}$, but the $i$ th coordinates of $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ differ from each other. By the
definition of $\mathcal{E}^{\prime},\left\{v_{i}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\} \in \mathcal{E}^{\prime}$. So $\mathbf{u}_{1}$ and $v_{i}$ are connected by a chain of length one.
(ii) $\mathcal{H}^{\prime}$ is semicycle-free. Suppose, to the contrary, that $\mathcal{H}^{\prime}$ contains a semicycle $C$. By the definition, we have $\left|e \cap e^{\prime}\right| \leq 1$ for all $e \in \mathcal{E}, e^{\prime} \in \mathcal{E}^{\prime}$. This implies that all edges in $C$ belong to either $\mathcal{E}$ or $\mathcal{E}^{\prime}$ since $k \geq 3$. If all edges in $C$ lie in $\mathcal{E}$, then $C$ is also a semicycle of $\mathcal{H}$, which contradicts that $\mathcal{H}$ is semicycle-free. Therefore, all edges in $C$ lie in $\mathcal{E}^{\prime}$.

Without loss of generality, let $e_{1}=\left\{v_{1}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\}$ be an edge in $C$. Then, by definition, the first coordinates of the vectors $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ are the first coordinates that are different from each other. We may assume that $i$ is the first coordinate of $\mathbf{u}_{i}$ for $1 \leq i \leq k-1$. Clearly, for any $1<j \leq n$, $\left\{v_{j}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\}$ does not belong to $\mathcal{E}^{\prime}$. Let $e_{1}$ and $e_{2}$ be two consecutive edges in $C$. Then, by the definition of the semicycle, $\left|e_{1} \cap e_{2}\right|=k-1$. This implies that $v_{1}$ must be in $e_{2}$, and so each edge of $C$ contains the vertex $v_{1}$.

If we write down the vertices of the semicycle in a sequence, denoting the vertices from $V$ by $v_{i}$ and those from $V^{\prime}$ by $\mathbf{u}_{j}$, there are $k$ possible sequences as follows:
(1) $v_{1}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}$ : only one edge, which obviously cannot be a semicycle.
(2) $\mathbf{u}_{1}, v_{1}, \mathbf{u}_{2}, u_{3}, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_{k}$ : only two edges. This sequence cannot be a semicycle because a semicycle must have at least three edges.
(3) $\mathbf{u}_{1}, \mathbf{u}_{2}, v_{1}, \mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_{k}, \mathbf{u}_{k+1}$ : there are three edges. By the definition of a semicycle, the first and the last vertices of the sequence must be the same. Because $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}, v_{1}, \mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1}\right\}$ is an edge of $\mathcal{E}^{\prime}$, the first coordinate of $\left\{\mathbf{u}_{1}\right.$, $\left.\mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\}$ differ from each other. We may assume $\{1,2, \ldots, k-1\}$ are respectively the first coordinate of $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\}$. Besides, $\mathbf{u}_{2}, v_{i}, \mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_{k}$ is also an edge of $\mathcal{E}^{\prime}$. The first coordinate of $\mathbf{u}_{2}, \mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_{k}$ differ from each other. So the first coordinate of $\mathbf{u}_{k}$ must be 1 , which is the same with the first coordinate of $\mathbf{u}_{1}$. Similarly, for the edge $v_{i}, \mathbf{u}_{3}, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_{k}, \mathbf{u}_{k+1}$, we may get that the first coordinate of $\mathbf{u}_{k+1}$ must be 2. Obviously, $\mathbf{u}_{1}$ and $\mathbf{u}_{k+1}$ differ in the first coordinate. As $\mathbf{u}_{1}$ and $\mathbf{u}_{k+1}$ are not the same vertices, this sequence cannot be a semicycle.
(k) $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}, v_{1}, \mathbf{u}_{k}, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{2 k-2}$. We assume that $\{1,2, \ldots, k-1\}$ are respectively the first coordinate of $\left\{\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\}$. According to the chainconnected properties, the $k$ edges in this sequence all contain the vertex $v_{1}$. So the first coordinates of the vertices in every edge except $v_{1}$ differ from each other. For $k \leq j \leq 2 k-2$, the first coordinate of $\mathbf{u}_{j}$ are the same as $\mathbf{u}_{j-k+1}$. So the first coordinate of $\mathbf{u}_{2 k-2}$ is $k-1$. As $\mathbf{u}_{1}$ and $\mathbf{u}_{2 k-2}$ are not the same vertices, this sequence cannot be a semicycle.

Without loss of generality, let $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{i-1}, v_{1}, \mathbf{u}_{i}, \ldots, \mathbf{u}_{t}$ be the sequence of vertices in $C$ such that $\left\{v_{1}, \mathbf{u}_{i}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}\right\}, i=1,2, \ldots, t-(k-2)$ are
all the edges of $C$. Note that every semicycle has at least 3 edges. Then $t \geq k+1$ and the first coordinates of $\mathbf{u}_{i}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}$ differ from each other. By the definition of the semicycle, it can be verified that $t \leq 2 k-2$ and $\mathbf{u}_{t}=\mathbf{u}_{1}$. So the length of $C$ is at most $k$. Hence, the first coordinate of $\mathbf{u}_{k}$ is the same as the first coordinate of $\mathbf{u}_{1}$, so the first coordinate of $\mathbf{u}_{k}$ is also 1 . In fact, it is easy to see that the first coordinate of $\mathbf{u}_{j}$ is the same as that of $\mathbf{u}_{j-k+1}$ for each $j, k \leq j \leq t \leq 2 k-2$. Thus the first coordinate of $\mathbf{u}_{t}$ is $t-k+1$. Obviously, $t-k+1 \neq 1$ as $t \leq 2 k-2$. This contradicts the fact that $\mathbf{u}_{1}=\mathbf{u}_{t}$.
(iii) $\mathcal{H}^{\prime}$ is a $k$-hypertree. For any $e \in \mathcal{E}, e^{\prime} \in \mathcal{E}^{\prime}$, since $\left|e \cap e^{\prime}\right| \leq 1$ and $k \geq 3$, all chains in $\mathcal{H}^{\prime}$ belong to either $\mathcal{E}$ or $\mathcal{E}^{\prime}$. Let $P$ be a chain in $\mathcal{H}^{\prime}$. If $P$ belongs to $\mathcal{E}, P$ is also a chain in $\mathcal{H}$. Since $\mathcal{H}$ is $k$-hypertree, every chain in it is of length at most $k$, so $P$ is of length at most $k$ in $\mathcal{H}^{\prime}$. If $P$ belongs to $\mathcal{E}^{\prime}$, as we noted in the proof in (ii), $P$ contains at most $2 k-1$ vertices. This implies that $P$ is of length at most $k$ in $\mathcal{H}^{\prime}$.

Return to the proof of Theorem 3.
By the construction of $\mathcal{H}^{\prime}$, we have $\left|V \cup V^{\prime}\right|=n+(k-1)^{n}$. Now we count the number of edges of $\mathcal{H}^{\prime}$. For each $v_{i} \in V$, let $\mathcal{E}_{i}^{\prime}=\left\{\left\{v_{i}, \mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1}\right\} \mid\right.$ $\left.\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{k-1} \in V^{\prime}\right\}$. Then $\mathcal{E}^{\prime}=\bigcup_{i=1}^{n} \mathcal{E}_{i}^{\prime}$. By the construction of $\mathcal{E}_{i}^{\prime}$, it is easy to see that

$$
\left|\mathcal{E}_{i}^{\prime}\right|=\left((k-1)^{k-1}-(k-1)!\right)^{i-1}\left((k-1)^{k-1}\right)^{n-i} .
$$

Hence,

$$
\left|\mathcal{E}^{\prime}\right|=x^{n-1}+y x^{n-2}+y^{2} x^{n-3}+\cdots+y^{n-1}
$$

where $x=(k-1)^{k-1}-(k-1)$ !, $y=(k-1)^{k-1}$. Therefore,

$$
\left|\mathcal{E}\left(\mathcal{H}^{\prime}\right)\right|=\left|\mathcal{E} \cup \mathcal{E}^{\prime}\right| \geq\left|\mathcal{E}^{\prime}\right|=\frac{y^{n}-x^{n}}{y-x}=\frac{\left[(k-1)^{k-1}\right]^{n}-\left[(k-1)^{k-1}-(k-1)!\right]^{n}}{(k-1)!}
$$

We count the limit of the ratio $\left|\mathcal{E}\left(\mathcal{H}^{\prime}\right)\right| /\binom{\left|V\left(\mathcal{H}^{\prime}\right)\right|}{k-1}$.

$$
\begin{aligned}
\frac{\left|\mathcal{E}\left(\mathcal{H} \mathcal{H}^{\prime}\right)\right|}{\binom{\left|V\left(\mathcal{H}^{\prime}\right)\right|}{k-1}} \geq & \frac{\frac{\left[(k-1)^{k-1}\right]^{n}-\left[(k-1)^{k-1}-(k-1)!\right]^{n}}{(k-1)!}}{\binom{n(k-1)^{n}}{k-1}} \\
= & \frac{\left[(k-1)^{k-1}\right]^{n}-\left[(k-1)^{k-1}-(k-1)!\right]^{n}}{(k-1)!} \\
& \cdot \frac{(k-1)!\left[n+(k-1)^{n}-(k-1)\right]!}{\left[n+(k-1)^{n}\right]!}
\end{aligned}
$$

$$
\begin{aligned}
& \geq \frac{\left[(k-1)^{k-1}\right]^{n}-\left[(k-1)^{k-1}-(k-1)!\right]^{n}}{\left[n+(k-1)^{n}\right]^{k-1}} \\
& =\frac{1-\left[\frac{(k-1)^{k-1}-(k-1)!}{(k-1)^{k-1}}\right]^{n}}{\left[\frac{n+(k-1)^{n}}{(k-1)^{n}}\right]^{k-1}}=\frac{1-\left[1-\frac{(k-1)!}{(k-1)^{k-1}}\right]^{n}}{\left[\frac{n}{(k-1)^{n}}+1\right]^{k-1}} \rightarrow 1(n \rightarrow \infty)
\end{aligned}
$$

On the other hand, by Theorem 2, we have

$$
\frac{\left|\mathcal{E}\left(\mathcal{H}^{\prime}\right)\right|}{\binom{\left|V\left(\mathcal{H}^{\prime}\right)\right|}{k-1}} \leq 1
$$

So, when $n \rightarrow \infty$, we obtain

$$
\frac{\left|\mathcal{E}\left(\mathcal{H}^{\prime}\right)\right|}{\binom{\left|V\left(\mathcal{H}^{\prime}\right)\right|}{k-1}} \rightarrow 1
$$

Thus, if $\left\{\mathcal{H}_{i}\right\}_{i=1}^{\infty}$ is a sequence of $k$-uniform $k$-hypertrees on $n(n \geq k)$ vertices such that $\lim _{n \rightarrow \infty}\left|V\left(\mathcal{H}_{i}\right)\right|=\infty$, then,

$$
\left|\mathcal{E}\left(\mathcal{H}_{i}^{\prime}\right)\right| \sim\binom{\left|V\left(\mathcal{H}_{i}^{\prime}\right)\right|}{k-1}
$$

Now let us review the construction given in [5]. In [5], the author constructed a $k$-hypertree $H_{i}^{k}=\left(V_{2^{i}, k}, E_{2^{i}, k}\right)$, where $\left|V_{2^{i}}\right|=2^{i}+F\left(2^{i}, k-1\right),\left|E_{2^{i}, k}\right|=$ $\binom{2^{2}}{k-1}+\left|D_{n, k}\right|$, and $D_{n, k}$ is the set of edges of a hypertree $F_{n, k}=\left(U_{n, k}, D_{n, k}\right)$. It is proved that $\left|E_{2^{i}, k}\right|$ is asymptotically $\binom{\left|V_{2^{i}}\right|}{k-1}$. The construction of $H_{i}^{k}$ and counting its number of edges are intricate and technical. This note provides an elegant construction of the desired $k$-hypertree by using vectors and matrices, and the proof is easy.

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