Discussiones Mathematicae
Graph Theory 37 (2017) 789–795
doi:10.7151/dmgt.1947

ASYMPTOTIC SHARPNESS OF BOUNDS ON HYPERTREES

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Abstract

The hypertree can be defined in many different ways. Katona and Szabó introduced a new, natural definition of hypertrees in uniform hypergraphs and investigated bounds on the number of edges of the hypertrees. They showed that a k-uniform hypertree on n vertices has at most $\binom{n}{k-1}$ edges and they conjectured that the upper bound is asymptotically sharp. Recently, Szabó verified that the conjecture holds by recursively constructing an infinite sequence of k-uniform hypertrees and making complicated analyses for it. In this note we give a short proof of the conjecture by directly constructing a sequence of k-uniform k-hypertrees.

Keywords: hypertree, semicycle in hypergraph, chain in hypergraph.

2010 Mathematics Subject Classification: 05C65.

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1. Introduction

Paths, cycles and trees are among the most fundamental objects in graph theory. As we have known, trees have a number of interesting structural properties, and trees are the most common objects in all of graph theory. These concepts have been generalized to hypergraphs in a lot of different ways [1, 3, 4].

Recently, Katona and Szabó [2] generalized the notion of trees to uniform hypergraphs and discussed lower and upper bounds on the number of edges of such hypertrees. They showed that a k-uniform hypertree on n vertices has at most $\binom{n}{k-1}$ edges and they posed some conjectures for bounds on the number of edges in the hypertrees.

We now recall definitions of hypertrees for k-uniform hypergraphs given in [2]. Let $\mathcal{F} = (V, \mathcal{E})$ be a k-uniform hypergraph (with no multiple edges).

The hypergraph $\mathcal{F}$ is a *chain* if there exists a sequence $v_1, v_2, \ldots, v_l$ of its vertices such that every vertex appears at least once (possibly more times), $v_1 \neq v_l$ and $\mathcal{E}$ consists of $l-k+1$ distinct edges of the form $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$, $1 \leq i \leq l-k+1$. The length of the chain is $l-k+1$, i.e., the number of its edges.

The hypergrah $\mathcal{F}$ is a *semicycle* if there exists a sequence $v_1, v_2, \ldots, v_l$ of its vertices such that every vertex appears at least once (possibly more times), $v_1 = v_l$ and for all $1 \leq i \leq l-k+1$, $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ are distinct edges of $\mathcal{F}$. The length of the semicycle $\mathcal{F}$ is $l-k+1$, the number of its edges. From the definition it follows that every semicycle has at least 3 edges.

A k-uniform hypergraph $\mathcal{H}$ is *chain-connected* if every pair of its vertices is connected by a chain. A k-uniform hypergraph $\mathcal{H}$ is *semicycle-free* if it contains no semicycle as a subhypergraph. A *hypertree* is a k-uniform hypergraph $\mathcal{H}$ ($k \geq 2$) such that $\mathcal{H}$ is chain-connected and semicycle-free. A hypertree is called an *l-hypertree* if every chain in it is of length at most l.

Katona and Szabó [2] investigated lower and upper bounds on the number of edges of hypertrees. They obtained the following results on the upper bounds.

Theorem 1 (Katona, Szabó [2]). *If $\mathcal{H}$ is a semicycle-free k-uniform hypergraph on n vertices, then $|\mathcal{E}(\mathcal{H})| \leq \binom{n}{k-1}$, and this bound is asymptotically sharp for $k = 3$.*

Theorem 2 (Katona, Szabó [2]). *Let $1 \leq l \leq k$ and $\mathcal{H}$ be a k-uniform l-hypertree on n vertices. Then $|\mathcal{E}(\mathcal{H})| \leq \frac{1}{k-l+1}\binom{n}{k-1}$. This bound is asymptotically sharp in the case $l = 2, k = 3$.*

Conjecture 3 (Katona, Szabó [2]). *The upper bound in Theorem 1 can be reached by a sequence of k-hypertrees.*

Recently, Szabó [5] proved the above conjecture by recursively constructing a sequence of k-hypertrees. However, the construction is intricate and technical.

In this note we give a shorter proof of the conjecture by directly constructing a sequence of k-hypertrees.

We will prove the main result below in next section.

Theorem 4. *For $k \geq 3$, there exists an infinite sequence of k-hypertrees where the number of edges is asymptotically $\binom{n}{k-1}$.*

2. Proof of Theorem 4

Let $\mathcal{H} = (V, \mathcal{E})$ be an arbitrary k-uniform k-hypertree and let $V = \{v_1, v_2, \ldots, v_n\}$. Now let us define a new k-uniform hypergraph $\mathcal{H}' = (V \cup V', \mathcal{E} \cup \mathcal{E}')$, where $V' = \{1, 2, \ldots, k-1\}^n$, i.e., the set of n-dimensional vectors over $\{1, 2, \ldots, k-1\}$, and $\mathcal{E}' = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \mid v_i \in V$, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1} \in V'$, where the ith coordinate of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ is the smallest coordinate where all the coordinates are distinct$\}$.

By the definition of $\mathcal{E}'$, if $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$, then all of $1, 2, \ldots, k-1$ appear in the ith column of the $(k-1) \times n$ matrix

$$M = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix},$$

where every $\mathbf{u}_i$ is regarded as a row vector, but at least one of $1, 2, \ldots, k-1$ do not appear in the i'th column of the matrix M for each $i' < i$.

We first prove that $\mathcal{H}'$ is a k-uniform k-hypertree.

Lemma 5. *$\mathcal{H}'$ is a k-uniform k-hypertree.*

Proof. To prove that $\mathcal{H}'$ is a k-uniform k-hypertree, we need to verify that $\mathcal{H}'$ satisfies the following three properties.

(i) $\mathcal{H}'$ is chain-connected. Clearly, any two vertices of V are chain-connected, since $\mathcal{H}$ is a hypertree and all of its edges are edges of $\mathcal{H}'$. For any $\mathbf{u}_1, \mathbf{u}_2 \in V'$, let i denote the position of the first coordinate where they differ. Then we consider the vertices $\mathbf{u}_3, \ldots, \mathbf{u}_{k-1} \in V'$ each of which the first $i-1$ coordinates are the same as the first $i-1$ coordinates of $\mathbf{u}_1, \mathbf{u}_2$ but the ith coordinates of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ differ from each other. By the definition of $\mathcal{E}'$, we see that $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$. This implies that $\mathbf{u}_1, \mathbf{u}_2$ are connected by a chain of length one in $\mathcal{H}'$. For any $\mathbf{u}_1 \in V'$ and $v_i \in V$, let $\mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ be $k-2$ vertices in V' such that the first $i-1$ coordinates of each $\mathbf{u}_i$ ($2 \leq i \leq k-1$) are the same as the first $i-1$ coordinates of $\mathbf{u}_1$, but the ith coordinates of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ differ from each other. By the definition of $\mathcal{E}'$, $\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \in \mathcal{E}'$. So $\mathbf{u}_1$ and v_i are connected by a chain of length one.

(ii) $\mathcal{H}'$ is semicycle-free. Suppose, to the contrary, that $\mathcal{H}'$ contains a semicycle C. By the definition, we have $|e \cap e'| \leq 1$ for all $e \in \mathcal{E}$, $e' \in \mathcal{E}'$. This implies that all edges in C belong to either $\mathcal{E}$ or $\mathcal{E}'$ since $k \geq 3$. If all edges in C lie in $\mathcal{E}$, then C is also a semicycle of $\mathcal{H}$, which contradicts that $\mathcal{H}$ is semicycle-free. Therefore, all edges in C lie in $\mathcal{E}'$.

Without loss of generality, let $e_1 = \{v_1, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ be an edge in C. Then, by definition, the first coordinates of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ are the first coordinates that are different from each other. We may assume that i is the first coordinate of $\mathbf{u}_i$ for $1 \leq i \leq k-1$. Clearly, for any $1 < j \leq n$, $\{v_j, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ does not belong to $\mathcal{E}'$. Let e_1 and e_2 be two consecutive edges in C. Then, by the definition of the semicycle, $|e_1 \cap e_2| = k-1$. This implies that v_1 must be in e_2, and so each edge of C contains the vertex v_1.

If we write down the vertices of the semicycle in a sequence, denoting the vertices from V by v_i and those from V' by $\mathbf{u}_j$, there are k possible sequences as follows:

(1) v_1,$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$: only one edge, which obviously cannot be a semicycle.

(2) $\mathbf{u}_1, v_1, \mathbf{u}_2, u_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$: only two edges. This sequence cannot be a semicycle because a semicycle must have at least three edges.

(3) $\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$: there are three edges. By the definition of a semicycle, the first and the last vertices of the sequence must be the same. Because $\{\mathbf{u}_1, \mathbf{u}_2, v_1, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}\}$ is an edge of $\mathcal{E}'$, the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$ differ from each other. We may assume $\{1, 2, \ldots, k-1\}$ are respectively the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$. Besides, $\mathbf{u}_2, v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}$, $\mathbf{u}_k$ is also an edge of $\mathcal{E}'$. The first coordinate of $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k$ differ from each other. So the first coordinate of $\mathbf{u}_k$ must be 1, which is the same with the first coordinate of $\mathbf{u}_1$. Similarly, for the edge $v_i, \mathbf{u}_3, \ldots, \mathbf{u}_{k-1}, \mathbf{u}_k, \mathbf{u}_{k+1}$, we may get that the first coordinate of $\mathbf{u}_{k+1}$ must be 2. Obviously, $\mathbf{u}_1$ and $\mathbf{u}_{k+1}$ differ in the first coordinate. As $\mathbf{u}_1$ and $\mathbf{u}_{k+1}$ are not the same vertices, this sequence cannot be a semicycle.

$\vdots$

(k) $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, v_1, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_{2k-2}$. We assume that $\{1, 2, \ldots, k-1\}$ are respectively the first coordinate of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}$. According to the chain-connected properties, the k edges in this sequence all contain the vertex v_1. So the first coordinates of the vertices in every edge except v_1 differ from each other. For $k \leq j \leq 2k-2$, the first coordinate of $\mathbf{u}_j$ are the same as $\mathbf{u}_{j-k+1}$. So the first coordinate of $\mathbf{u}_{2k-2}$ is $k-1$. As $\mathbf{u}_1$ and $\mathbf{u}_{2k-2}$ are not the same vertices, this sequence cannot be a semicycle.

Without loss of generality, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{i-1}, v_1, \mathbf{u}_i, \ldots, \mathbf{u}_t$ be the sequence of vertices in C such that $\{v_1, \mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}\}$, $i = 1, 2, \ldots, t-(k-2)$ are all the edges of C. Note that every semicycle has at least 3 edges. Then $t \geq k+1$ and the first coordinates of $\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+(k-2)}$ differ from each other. By the definition of the semicycle, it can be verified that $t \leq 2k-2$ and $\mathbf{u}_t = \mathbf{u}_1$. So the length of C is at most k. Hence, the first coordinate of $\mathbf{u}_k$ is the same as the first coordinate of $\mathbf{u}_1$, so the first coordinate of $\mathbf{u}_k$ is also 1. In fact, it is easy to see that the first coordinate of $\mathbf{u}_j$ is the same as that of $\mathbf{u}_{j-k+1}$ for each j, $k \leq j \leq t \leq 2k-2$. Thus the first coordinate of $\mathbf{u}_t$ is $t-k+1$. Obviously, $t-k+1 \neq 1$ as $t \leq 2k-2$. This contradicts the fact that $\mathbf{u}_1 = \mathbf{u}_t$.

(iii) $\mathcal{H}'$ is a k-hypertree. For any $e \in \mathcal{E}$, $e' \in \mathcal{E}'$, since $|e \cap e'| \leq 1$ and $k \geq 3$, all chains in $\mathcal{H}'$ belong to either $\mathcal{E}$ or $\mathcal{E}'$. Let P be a chain in $\mathcal{H}'$. If P belongs to $\mathcal{E}$, P is also a chain in $\mathcal{H}$. Since $\mathcal{H}$ is k-hypertree, every chain in it is of length at most k, so P is of length at most k in $\mathcal{H}'$. If P belongs to $\mathcal{E}'$, as we noted in the proof in (ii), P contains at most $2k-1$ vertices. This implies that P is of length at most k in $\mathcal{H}'$. ■

Return to the proof of Theorem 3.

By the construction of $\mathcal{H}'$, we have $|V \cup V'| = n + (k-1)^n$. Now we count the number of edges of $\mathcal{H}'$. For each $v_i \in V$, let $\mathcal{E}'_i = \{\{v_i, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\} \mid \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1} \in V'\}$. Then $\mathcal{E}' = \bigcup_{i=1}^n \mathcal{E}'_i$. By the construction of $\mathcal{E}'_i$, it is easy to see that

$$|\mathcal{E}'_i| = ((k-1)^{k-1} - (k-1)!)^{i-1}((k-1)^{k-1})^{n-i}.$$

Hence,

$$|\mathcal{E}'| = x^{n-1} + yx^{n-2} + y^2x^{n-3} + \cdots + y^{n-1},$$

where $x = (k-1)^{k-1} - (k-1)!$, $y = (k-1)^{k-1}$. Therefore,

$$|\mathcal{E}(\mathcal{H}')| = |\mathcal{E} \cup \mathcal{E}'| \geq |\mathcal{E}'| = \frac{y^n - x^n}{y - x} = \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!}.$$

We count the limit of the ratio $|\mathcal{E}(\mathcal{H}')| \Big/ \binom{|V(\mathcal{H}')|}{k-1}$.

$$\begin{aligned}\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} &\geq \frac{\frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1}-(k-1)!]^n}{(k-1)!}}{\binom{n+(k-1)^n}{k-1}} \\ &= \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{(k-1)!} \\ &\quad \cdot \frac{(k-1)![n+(k-1)^n - (k-1)]!}{[n+(k-1)^n]!}\end{aligned}$$

$$\geq \frac{[(k-1)^{k-1}]^n - [(k-1)^{k-1} - (k-1)!]^n}{[n + (k-1)^n]^{k-1}}$$

$$= \frac{1 - \left[\frac{(k-1)^{k-1}-(k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n+(k-1)^n}{(k-1)^n}\right]^{k-1}} = \frac{1 - \left[1 - \frac{(k-1)!}{(k-1)^{k-1}}\right]^n}{\left[\frac{n}{(k-1)^n} + 1\right]^{k-1}} \to 1 (n \to \infty).$$

On the other hand, by Theorem 2, we have

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \leq 1.$$

So, when $n \to \infty$, we obtain

$$\frac{|\mathcal{E}(\mathcal{H}')|}{\binom{|V(\mathcal{H}')|}{k-1}} \to 1.$$

Thus, if $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a sequence of k-uniform k-hypertrees on n $(n \geq k)$ vertices such that $\lim_{n\to\infty} |V(\mathcal{H}_i)| = \infty$, then,

$$|\mathcal{E}(\mathcal{H}_i')| \sim \binom{|V(\mathcal{H}_i')|}{k-1}.$$

■

Now let us review the construction given in [5]. In [5], the author constructed a k-hypertree $H_i^k = (V_{2^i,k}, E_{2^i,k})$, where $|V_{2^i}| = 2^i + F(2^i, k-1)$, $|E_{2^i,k}| = \binom{2^i}{k-1} + |D_{n,k}|$, and $D_{n,k}$ is the set of edges of a hypertree $F_{n,k} = (U_{n,k}, D_{n,k})$. It is proved that $|E_{2^i,k}|$ is asymptotically $\binom{|V_{2^i}|}{k-1}$. The construction of H_i^k and counting its number of edges are intricate and technical. This note provides an elegant construction of the desired k-hypertree by using vectors and matrices, and the proof is easy.

Acknowledgments

This work was partially supported by the National Nature Science Foundation of China (grant no. 11571222, 11471210). We are very thankful to the referee for his/her careful reading of this paper and all helpful comments.

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Received 30 April 2016
Revised 14 July 2016
Accepted 14 July 2016