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DISTANCE 2-DOMINATION IN PRISMS OF GRAPHS

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Dedication

Ferran Hurtado passed away a few months after Eduardo Rivera-Campo and Rita Zuazua visited him and Mercè Mora in Barcelona, where most of this research was made. We all dedicate this final version to Ferran's memory.

Abstract

A set of vertices D of a graph G is a *distance* 2-*dominating* set of G if the distance between each vertex $u \in (V(G) - D)$ and D is at most two. Let $\gamma_2(G)$ denote the size of a smallest distance 2-dominating set of G. For any permutation π of the vertex set of G, the *prism of* G *with respect to* π is the graph πG obtained from G and a copy G' of G by joining $u \in V(G)$ with $v' \in V(G')$ if and only if $v' = \pi(u)$. If $\gamma_2(\pi G) = \gamma_2(G)$ for any permutation

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π of $V(G)$, then G is called a universal γ_2-fixer. In this work we characterize the cycles and paths that are universal γ_2-fixers.

Keywords: distance 2-dominating set, prisms of graphs, universal fixer.

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1. Introduction

Let $G = (V(G), E(G))$ be an undirected graph. A set $D \subseteq V(G)$ is a *dominating set* of G if each vertex of G not in D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G.

For any permutation π of the vertex set of G, the *prism of G with respect to* π is the graph πG obtained from G and a copy G' of G with vertex set $V(G')$ $= \{w' : w \in V(G)\}$, by joining $u \in V(G)$ to $v' \in V(G')$ if and only if $v = \pi(u)$.

A graph G is called a *universal γ-fixer* if $\gamma(\pi G) = \gamma(G)$ for all permutations π of $V(G)$. Domination in prisms were studied by Mynhardt and Xu [1] for several classes of graphs and it was conjectured that the edgeless graphs $\overline{K_n}$ are the only universal γ-fixers. Wash [5] proved this conjecture.

This concept was generalized for other types of domination. Mynhardt and Schurch [4] introduced the concept of paired domination in prisms. Lemanska and Zuazua [2] studied the concept of convex domination in prisms.

The *distance* $d_G(u, v)$ between two vertices u and v in a graph G is the length of a shortest uv-path in G. If there is no uv-path in G, then $d_G(u, v) = \infty$. The concept of distance k-dominating sets, for $k \geq 1$, was introduced by Meir and Moon [3], under the name k-covering. In particular, a set of vertices $D \subseteq V(G)$ is said to be a *distance* 2*-dominating* set of G if the distance between each vertex $u \in (V(G) - D)$ and D is at most two. The minimum cardinality of a distance 2-dominating set in G is the *distance* 2*-domination number of* G and is denoted by $\gamma_2(G)$. A 2-dominating set in G with cardinality $\gamma_2(G)$ is called a γ_2*-set of* G.

In this paper we study distance 2-domination in prisms. It is well known that $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$ for every graph G. However, while the second inequality still holds for distance 2-domination, the first one does not. In Section 2, we give some examples of families of graphs satisfying $\gamma_2(\pi G) < \gamma_2(G)$ for some permutations.

A graph G is called a *universal γ_2-fixer* if $\gamma_2(\pi G) = \gamma_2(G)$ for every permutation π of $V(G)$. As our main result, in Section 3 we characterize all paths and cycles that are universal γ_2-fixers.

2. Miscellaneous Results

In this section we show the existence of graphs G such that the prism πG has

distance 2-domination number less than or equal to the distance 2-domination number of G for some permutations. More precisely, we will see that a graph whose components are universal γ_2-fixers is not necessarily a universal γ_2-fixer (Corollary 3). On the other hand, we show that there are graphs with distance 2-domination number as large as desired, whereas this number is constant for at least one prism (Theorem 4). For any vertex u of a graph G let $N_G(u) = \{v : uv \in E(G)\}$ and $N_G[u] = \{u\} \cup N_G(u)$. For any subset S of vertices of G, let $N_G(S) = \bigcup_{u \in S} N_G(u)$.

Proposition 1. *For all positive integers r and s, and each permutation π, $\gamma_2(\pi(rK_s)) \leq \gamma_2(rK_s)$.*

Proof. First observe that $\gamma_2(rK_s) = r$. Now, let $\pi : V(rK_s) \to V(rK'_s)$ be a bijection and for $i = 1, 2, \ldots, r$ let $\{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ be the set of vertices of the ith copy of K_s.

Let F_r be the bipartite graph with vertex set $V(F_r) = \{w_1, w_2, \ldots, w_r\} \cup \{w'_1, w'_2, \ldots, w'_r\}$, where $w_i w'_j$ is an edge of F_r if and only if $\pi(v_{i,k}) = v'_{j,l}$ for some k and l.

Since π is a bijection, F_r satisfies Hall's Condition, that is, for any subset $S \subseteq \{w_1, w_2, \ldots, w_r\}$ we have $|N_{F_r}(S)| \geq |S|$.

Therefore, the graph F_r has a perfect matching $w_1 w'_{i_1}, w_2 w'_{i_2}, \ldots, w_r w'_{i_r}$. By construction of F_r, this implies the existence of vertices $v_{1,k_1}, v_{2,k_2}, \ldots, v_{r,k_r}$ of the r different copies of K_s and $v'_{j_1,l_1}, v'_{j_2,l_2}, \ldots, v'_{j_r,l_r}$ each one of a different copy of K'_s such that $\pi(v_{i,k_i}) = v'_{j_i,l_i}$. Hence, $\{v_{1,k_1}, v_{2,k_2}, \ldots, v_{r,k_r}\}$ is a 2-dominating set of $\pi(rK_s)$. Thus, $\gamma_2(\pi\,(r\,K_s)) \leq r = \gamma_2(r\,K_s)$. ■

Theorem 2. *For each integer $s \geq 2$ there is a permutation π such that*

$$\gamma_2(\pi((3s-1)K_s)) < \gamma_2((3s-1)K_s).$$

Proof. Denote by G_s the graph $(3s-1)K_s$. For $i = 1, 2, \ldots, 3s-1$ let $\{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ be the set of vertices of the ith copy of K_s. Let $\pi : V(G_s) \to V(G'_s)$ be a bijection satisfying the following conditions:

$$\pi(v_{i,1}) = v'_{i,1} \text{ for } i = 1, 2, \ldots, s,$$
$$\pi\Big(\bigcup_{i=1}^{s} \{v_{i,2}, v_{i,3}, \ldots, v_{i,s}\}\Big) = \bigcup_{j=s+1}^{2s-1} \Big\{v'_{j,1}, v'_{j,2}, \ldots, v'_{j,s}\Big\},$$
$$\pi\Big(\bigcup_{i=s+1}^{2s-1} \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}\Big) = \bigcup_{j=2s}^{3s-1} \Big\{v'_{j,2}, v'_{j,3}, \ldots, v'_{j,s}\Big\},$$
$$\pi(v_{i,1}) = v'_{i,1} \text{ for } i = 2s, 2s+1, \ldots, 3s-1, \text{ and}$$
$$\pi\Big(\bigcup_{i=2s}^{3s-1} \{v_{i,2}, v_{i,3}, \ldots, v_{i,s}\}\Big) = \bigcup_{j=1}^{s} \Big\{v'_{j,2}, v'_{j,3}, \ldots, v'_{j,s}\Big\}.$$

It is easy to check that $\{v_{1,1}, v_{2,1}, \ldots, v_{s,1}\} \cup \{v'_{2s,1}, v'_{2s+1,1}, \ldots, v'_{3s-1,1}\}$ is a 2-dominating set for πG_s and therefore $\gamma_2(\pi G_s) \leq 2s < 3s - 1 = \gamma_2(G_s)$ (see Figure 1). ■

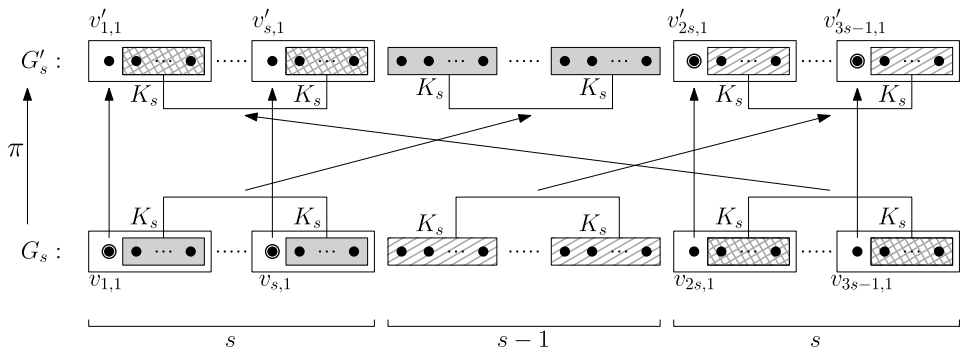

Figure 1. The graph $\pi((3s-1)K_s)$. The set of $s(s-1)$ vertices in gray (resp. lined, squared) rectangles below map to the set of $s(s-1)$ vertices in gray (resp. lined, squared) rectangles above. The set of encircled vertices is a distance 2-dominating set.

Since $\gamma_2(\pi K_s) = \gamma_2(K_s) = 1$ for every permutation π, the following result holds.

Corollary 3. *There exist disconnected graphs where every component is a universal γ_2-fixer while the graph itself is not a universal γ_2-fixer.*

Theorem 4. *For each positive integer k there exists a graph H_k and a permutation π of $V(H_k)$ such that $\gamma_2(H_k) = k+1$ and $\gamma_2(\pi H_k) = 2$.*

Proof. Let H_k be the graph with $V(H_k) = \{z, x_1, x_2, \ldots, x_{5k}, y_1, y_2, \ldots, y_{5k}\}$ for $k \geq 3$, and $E(H_k) = \{zx_i : 1 \leq i \leq 5k\} \cup \{zy_1\} \cup \{y_j y_{j+1} : 1 \leq j \leq 5k-1\}$ and let π the permutation given by $\pi(z) = z', \pi(x_i) = y'_i$ and $\pi(y_i) = x'_i$ for $1 \leq i \leq 5k$. The graph H_k satisfies $\gamma_2(H_k) = k+1$ and $\gamma_2(\pi H_k) = 2$, since $D = \{z, z'\}$ is a dominating γ_2-set of πH_k (see Figure 2). ■

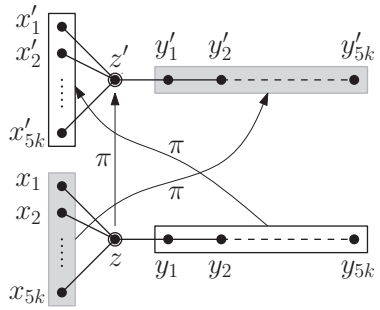

Figure 2. The graph πH_k. The set $\{z, z'\}$ is a distance 2-dominating set.

3. Paths and Cycles

This section is devoted to the charaterization of all paths and cycles that are universal γ_2-fixers. For any vertex u of a graph G, the 2-*neighborhood* of u, denoted by $N_G^2[u]$, is the set of vertices v of G for which $d_G(u,v) \leq 2$.

Observation 5. *Let G be a path or a cycle. Then $\gamma_2(G) \leq \gamma_2(\pi G)$ for all permutation π.*

Proof. If G has n vertices, then $\gamma_2(G) = \lceil \frac{n}{5} \rceil$. Moreover, if $v \in V(\pi G)$, then $|N_{\pi G}^2[v]| \leq 10$. Therefore, for all permutations π of $V(G)$, $\gamma_2(\pi G) \geq \lceil \frac{2n}{10} \rceil = \lceil \frac{n}{5} \rceil = \gamma_2(G)$. ■

Observation 6. *If P_n is a universal γ_2-fixer, then C_n is a universal γ_2-fixer.*

Proof. If a set of vertices of πP_n is a γ_2-set of πP_n, then the corresponding set of vertices of πC_n is a γ_2-set of πC_n. ■

Our main result is the following.

Theorem 7. *The path P_n is a universal γ_2-fixer if and only if $n \in \{1,2,3,6\}$. The cycle C_n is a universal γ_2-fixer if and only if $n \in \{3,6,7\}$.*

In what follows, if G is a path or a cycle of order n, we denote the vertices of two copies of G by $\{1,2,\ldots,n\}$ and by the first n letters of the alphabet, $\{a,b,c,\ldots\}$, respectively. For any permutation $\pi : \{1,\ldots,n\} \longrightarrow \{a,b,c,\ldots\}$, the prism πG has vertex set $\{1,2,\ldots,n\} \cup \{a,b,c,\ldots\}$. The set of edges is $E(\pi P_n) = \{\{i,i+1\} : i = 1,\ldots,n-1\} \cup \{\{i,\pi(i)\} : i = 1,\ldots,n\} \cup \{\{a,b\},\{b,c\},\{c,d\},\ldots\}$, when G is the path of order n, and $E(\pi C_n)$ is obtained from $E(\pi P_n)$ by adding the edges joining the end-vertices of the two copies of the path of order n. We denote by xy the edge $\{x,y\}$ if it is not misleading.

Theorem 7 is a consequence of the following propositions and corollaries.

Proposition 8. *If P_n or C_n is a universal γ_2-fixer, then $n \in \{1,2,3,4,6,7,8,11,12,16\}$.*

Proof. Let G be a path or a cycle with n vertices and let $\pi = I$ be the identity permutation. If $v \in V(IG)$, then $|N_{IG}^2[v]| \leq 8$ which implies that $8\gamma_2(IG) \geq 2n$. If G is a universal γ_2-fixer, then $\gamma_2(IG) = \gamma_2(G) = \lceil \frac{n}{5} \rceil$. Therefore, $8\lceil \frac{n}{5} \rceil \geq 2n$.

If $n = 5m$, then $8\lceil \frac{n}{5} \rceil \geq 2n$ becomes $8m \geq 10m$ which implies $m = 0$. If $n = 5m + p$, with $1 \leq p \leq 4$, then $8\lceil \frac{n}{5} \rceil \geq 2n$ becomes $8(m+1) \geq 2(5m+p)$ which implies $m \leq 3$ for $p = 1$, $m \leq 2$ for $p = 2$, $m \leq 1$ for $p = 3$, and $m = 0$ for $p = 4$. Therefore $n \in \{1,2,3,4,6,7,8,11,12,16\}$. ■

Proposition 9. *The paths P_1, P_2, P_3 and P_6 are universal γ_2-fixers.*

Proof. The cases P_1, P_2 and P_3 are trivial.

If $n = 6$, then $V(P_6) = \{1, 2, 3, 4, 5, 6\}$. For any permutation π, we have $\{1, 2, 3, \pi(1), \pi(2), \pi(3)\} \subseteq N^2_{\pi P_6}[2]$ and $\{4, 5, 6, \pi(4), \pi(5), \pi(6)\} \subseteq N^2_{\pi P_6}[5]$. Therefore $D = \{2, 5\}$ is a γ_2-set of πP_6 and $\gamma_2(\pi P_6) = 2 = \gamma_2(P_6)$ for each permutation π of $V(P_6)$. Hence P_6 is a universal γ_2-fixer. ■

By Observation 6, we obtain the following.

Corollary 10. *C_3 and C_6 are universal γ_2-fixers.*

Proposition 11. *The cycle C_4 is not a universal γ_2-fixer.*

Proof. If π is the identity permutation I, then for any vertex $v \in V(IC_4)$ there exists a vertex $w \in V(IC_4)$ such that $d_{IC_4}(v, w) = 3$, therefore $\gamma_2(IC_4) \geq 2 > 1 = \gamma_2(C_4)$. Hence C_4 is not a universal γ_2-fixer. ■

Again by Observation 6, we obtain the following.

Corollary 12. *The path P_4 is not a universal γ_2-fixer.*

Proposition 13. *The path P_7 is not a universal γ_2-fixer.*

Proof. We will prove that $\gamma_2(\pi P_7) > 2 = \gamma_2(P_7)$ for some permutation π of $V(P_7)$. Let $V(P_7) = \{1, 2, \ldots, 7\}$, $V(P'_7) = \{a, b, \ldots, g\}$ and consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ f & b & c & d & e & a & g \end{pmatrix}$$

(see Figure 3).

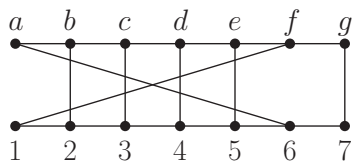

Figure 3. The graph πP_7.

Suppose $D = \{x, y\}$ is a γ_2-set of πP_7. By definition of distance 2-dominating set, $D \cap N^2_{\pi P_7}[7] \neq \emptyset$. Without loss of generality we assume $x \in N^2_{\pi P_7}[7] = \{5, 6, 7, a, f, g\}$. Furthermore, by the symmetry of πP_7, we can also assume $x \in \{a, 5, 6, 7\}$.

1. If $x = a$, then $S = V(\pi P_7) - N^2_{\pi P_7}[a] = \{1, 3, 4, d, e, f, g\}$ has to be 2-dominated by a vertex in D, say y, other than x. Therefore $y \in \bigcap_{v \in S} N^2_{\pi P_7}[v]$. But, $N^2_{\pi P_7}[3] \cap N^2_{\pi P_7}[4] \cap N^2_{\pi P_7}[g] = \emptyset$.

2. If $x = 5$, then $S = V(\pi P_7) - N^2_{\pi P_7}[5] = \{1, 2, b, c, g\}$ has to be 2-dominated by a vertex in D, say y, other than x. Therefore $y \in \bigcap_{v \in S} N^2_{\pi P_7}[v]$. But, $N^2_{\pi P_7}[b] \cap N^2_{\pi P_7}[c] \cap N^2_{\pi P_7}[g] = \emptyset$.
3. If $x = 6$, then $S = V(\pi P_7) - N^2_{\pi P_7}[6] = \{1, 2, 3, c, d, f\}$ has to be 2-dominated by a vertex in D, say y, other than x. Therefore $y \in \bigcap_{v \in S} N^2_{\pi P_7}[v]$. But, $N^2_{\pi P_7}[1] \cap N^2_{\pi P_7}[2] \cap N^2_{\pi P_7}[3] = \{1, 2, 3, b\}$ and $N^2_{\pi P_7}[c] \cap N^2_{\pi P_7}[d] \cap N^2_{\pi P_7}[f] = \{d, e\}$ which are disjoint sets.
4. If $x = 7$, then $S = V(\pi P_7) - N^2_{\pi P_7}[7] = \{1, 2, 3, 4, b, c, d, e\}$ has to be 2-dominated by a vertex in D, say y, other than x. Therefore $y \in \bigcap_{v \in S} N^2_{\pi P_7}[v]$. But, $N^2_{\pi P_7}[1] \cap N^2_{\pi P_7}[2] \cap N^2_{\pi P_7}[3] \cap N^2_{\pi P_7}[4] = \{2, 3\}$ and $N^2_{\pi P_7}[b] \cap N^2_{\pi P_7}[c] \cap N^2_{\pi P_7}[d] \cap N^2_{\pi P_7}[e] = \{c, d\}$ which are disjoint sets. ■

Proposition 14. *The cycle C_7 is a universal γ_2-fixer.*

Proof. Let $V(C_7) = \{1, 2, \ldots, 7\}$, $V(C'_7) = \{a, b, \ldots, g\}$ and $\pi : V(C_7) \to V(C'_7)$ be a permutation. We will prove $\gamma_2(\pi C_7) = 2$ by showing that, for all possible cases, there exists a γ_2-set of πC_7 of cardinality 2. By Observation 5, this implies $\gamma_2(\pi C_7) = \gamma_2(C_7) = 2$. By the symmetry of πC_7, we may assume that $\pi(1) = a$. The proposition is a consequence of the following claims.

Claim 1. *If $\{\pi(4), \pi(5)\} \cap \{d, e\} \neq \emptyset$, then $\gamma_2(\pi C_7) = 2$.*

Proof. Let $A = \{1, 2, 3, 6, 7, a, b, g\} \subseteq N^2_{\pi C_7}[1]$ and let $B = V(\pi C_7) - A = \{4, 5, c, d, e, f\}$. If $d \in \{\pi(4), \pi(5)\}$, then $B \subseteq N^2_{\pi C_7}[d]$ and $D = \{1, d\}$ is a γ_2-set of πC_7. Similarly, if $e \in \{\pi(4), \pi(5)\}$, then $B \subseteq N^2_{\pi C_7}[e]$ and $D = \{1, e\}$ is a γ_2-set of πC_7. □

Claim 2. *If $\pi(2) = b$ and $\pi(7) = g$, then $\gamma_2(\pi C_7) = 2$.*

Proof. If there exists an edge of the form $\{3c, 3d, 4c, 4d, 4e, 5d, 5e, 5f, 6e, 6f\}$, then Claim 1 can be applied by renaming the vertices in $V(\pi C_7)$. So, we only have to consider the case where $\pi(4) = f$ and $\pi(5) = c$ which, in turn, implies $\pi(3) = e$ and $\pi(6) = d$. Observe that $N^2_{\pi C_7}[2] = \{1, 2, 3, 4, 7, a, b, c, e\}$ and $N^2_{\pi C_7}[7] = \{1, 2, 5, 6, 7, a, d, f, g\}$, therefore $D = \{2, 7\}$ is γ_2-set of πC_7. □

Claim 3. *If $\pi(2) = b$, $\pi(3) \neq c$ and $\pi(7) \neq g$, then $\gamma_2(\pi C_7) = 2$.*

Proof. If there exists an edge of the form $\{4d, 4e, 5d, 5e, 5f, 6e, 6f\}$, then Claim 1 can be applied by renaming the vertices in $V(\pi C_7)$. So, we only have to consider the cases where $\pi(5) \in \{c, g\}$ and $\pi^{-1}(e) \in \{3, 7\}$. Without loss of generality we may assume $\pi(5) = g$ which implies $\pi(4) \in \{c, f\}$. This gives the following cases.

1. The permutation π is given by $\pi(1) = a, \pi(2) = b, \pi(3) = e, \pi(4) = c, \pi(5) = g, \pi(6) = d, \pi(7) = f$.
   We have $N^2_{\pi C_7}[a] = \{1, 2, 5, 7, a, b, c, f, g\}$ and $N^2_{\pi C_7}[d] = \{3, 4, 5, 6, 7, b, c, d, e, f\}$, therefore $D = \{a, d\}$ is γ_2-set of πC_7.
2. The permutation π is given by $\pi(1) = a, \pi(2) = b, \pi(3) = e, \pi(4) = f, \pi(5) = g$ and $\{\pi(6), \pi(7)\} = \{c, d\}$.
   In this case we can apply Claim 2 by renaming the vertices in $V(\pi C_7)$.
3. The permutation π is given by $\pi(1) = a, \pi(2) = b, \pi(3) = f, \pi(4) = c, \pi(5) = g, \pi(6) = d, \pi(7) = e$.
   We have $N^2_{\pi C_7}[a] = \{1, 2, 5, 7, a, b, c, f, g\}$ and $N^2_{\pi C_7}[c] = \{2, 3, 4, 5, 6, a, b, c, d, e\}$, therefore $D = \{a, c\}$ is γ_2-set of πC_7.
4. The permutation π is given by $\pi(1) = a, \pi(2) = b, \pi(4) = f, \pi(5) = g, \pi(7) = e$, and $\{\pi(3), \pi(6)\} = \{c, d\}$.
   We have $N^2_{\pi C_7}[1] = \{1, 2, 3, 6, 7, a, b, e, g\}$ and $N^2_{\pi C_7}[3] = \{1, 2, 3, 4, 5, b, c, d, f\}$, therefore $D = \{1, 3\}$ is γ_2-set of πC_7. □

Claim 4. *If $\pi(6) = f$ and $\pi(7) \neq g$, then $\gamma_2(\pi C_7) = 2$.*

Proof. If there exists an edge of the form $\{2b, 2g, 5e, 5g, 7b, 7e\}$, then Claim 3 applies by renaming the vertices in $V(\pi C_7)$. Therefore $\pi(7) \in \{c, d\}$ and $\pi^{-1}(g) \in \{3, 4\}$. In any case, we have $\{1, 2, 5, 6, 7, a, c, d, f\} \subseteq N^2_{\pi C_7}[7]$ and $\{3, 4, 6, a, b, e, f, g\} \subseteq N^2_{\pi C_7}[g]$. Hence $D = \{7, g\}$ is a γ_2-set of πC_7. □

Claim 5. *For every permutation $\pi : V(C_7) \to V(C'_7)$ the graph $\pi(C_7)$ has $\gamma_2(\pi C_7) = 2$.*

Proof. By the symmetry of $\pi(C_7)$, we may assume that $\pi(1) = a$. The cases where $\pi(7) = e, \pi(7) = f, \pi(7) = g$ are symmetrical cases to $\pi(7) = d, \pi(7) = c, \pi(7) = b$, respectively. By Claim 3, if there is the edge $7g$, then $\gamma_2(\pi C_7) = 2$. So, we suppose $\pi(7) \in \{e, f\}$.

If $\pi(7) = f$ and there exists an edge of the form $\{3b, 3c, 4b, 4c, 4d, 4e, 5d, 5e\}$, then we can apply Claim 1 after renaming the vertices of πC_7. Therefore we can assume $\pi(4) = g$ and $\pi(3) \in \{d, e\}$.

1. If $\pi(3) = d$, $\pi(4) = g$ and $\pi(7) = f$, then $\{1, 2, 3, 6, 7, a, b, f, g\} \subseteq N^2_{\pi C_7}[1]$ and $\{1, 2, 3, 4, 5, c, d, e\} \subseteq N^2_{\pi C_7}[3]$. Hence $D = \{1, 3\}$ is γ_2-set of πC_7.
2. If $\pi(3) = e$, $\pi(4) = g, \pi(7) = f$ and $\pi(2) \in \{b, d\}$, then Claim 3 applies by renaming the vertices of πC_7. Therefore we may assume that $\pi(2) = c$ in which case $\{1, 2, 3, 6, 7, a, b, c, f, g\} \subseteq N^2_{\pi C_7}[1]$ and $\{1, 2, 3, 4, 5, d, e\} \subseteq N^2_{\pi C_7}[3]$. Hence $D = \{1, 3\}$ is γ_2-set of πC_7.

If $\pi(7) = e$ and there exists an edge of the form $\{2b, 3b, 4b, 4d, 4e, 5d, 5e, 6d, 6f\}$, then either Claim 1 or Claim 3 applies after renaming the vertices of πC_7.

Hence, we can assume $\pi^{-1}(d) \in \{2,3\}$. By the symmetry of $\pi(C_7)$, the case $\pi(2) = c$ is equivalent to the case $\pi(7) = f$, and $\pi(2) = g$ is equivalent to the case $\pi(2) = b$, so we may assume $\pi(2) \in \{d, f\}$.

1. If $\pi(2) = f$, $\pi(3) = d$ and $\pi(7) = e$, then $\{1,2,3,6,7,a,b,e,f,g\} \subseteq N^2_{\pi C_7}[1]$ and $\{1,2,3,4,5,c,d,e\} \subseteq N^2_{\pi C_7}[3]$. Hence $D = \{1,3\}$ is γ_2-set of πC_7.
2. If $\pi(2) = d$, $\pi(7) = e$ and there exists an edge of the form $\{5g, 6g\}$, then Claim 1 applies after renaming the vertices of πC_7. Likewise, if $3c$ is an edge of πC_7, then Claim 3 applies, and if there exist a edge of the form $\{4f, 5c\}$, then Claim 4 applies. Therefore $\pi^{-1}(4) \in \{c, g\}$ and $\pi^{-1}(5) \in \{b, f\}$.
   (a) If $\pi(4) = c, \pi(5) = b$ or $\pi(4) = g, \pi(5) = f$, then Claim 3 applies after renaming the vertices of πC_7.
   (b) If $\pi(2) = d$, $\pi(4) = c, \pi(5) = f, \pi(7) = e$. Then $\{1,2,3,6,7,a,b,e,d,g\} \subseteq N^2_{\pi C_7}[1]$ and $\{2,3,4,5,6,c,d,f\} \subseteq N^2_{\pi C_7}[4]$. Hence $D = \{1,4\}$ is γ_2-set of πC_7.
   (c) Suppose $\pi(2) = d$, $\pi(4) = g, \pi(5) = b, \pi(7) = e$. Then $D = \{1, a\}$ is γ_2-set of πC_7 because $\{1,2,3,6,7,a,b,e,d,g\} \subseteq N^2_{\pi C_7}[1]$ and $\{4,5,b,c,f,g\} \subseteq N^2_{\pi C_7}[a]$. □

By Claims 1–5, the proposition follows. ■

Proposition 15. *The cycle C_8 is not a universal γ_2-fixer.*

Proof. Since $\gamma_2(C_8) = 2$, it suffices to prove that there is a permutation $\pi \in S_8$ such that $\gamma_2(\pi C_8) > 2$. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a & b & f & e & d & c & g & h \end{pmatrix}.$$

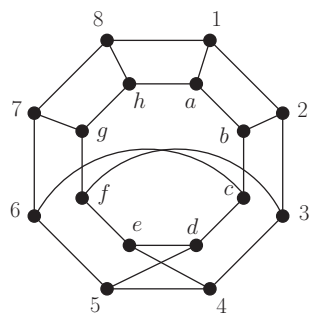

Figure 4. The graph πC_8.

Due to the symmetry of the graph πC_8, we may assume that D contains one of the vertices 1, 2, 3 or 4. If $1 \in D$, then $V(\pi C_8) \setminus N^2_{\pi C_8}[1] = \{4,5,6,c,d,e,f,g\}$.

However, no vertex of πC_8 2-dominates $\{4,5,6,c,d,e,f,g\}$. If $2 \in D$, then $V(\pi C_8) \setminus N^2_{\pi C_8}[2] = \{5,6,7,d,e,g,h\}$. However, no vertex of πC_8 2-dominates $\{5,6,7,d,e,g,h\}$. A similar argument shows that if $3 \in D$, then no vertex of πC_8 2-dominates $V(\pi C_8) \setminus N^2_{\pi C_8}[3]$, and if $4 \in D$, then no vertex of πC_8 2-dominates $V(\pi C_8) \setminus N^2_{\pi C_8}[4]$. ■

Proposition 16. *The cycle C_{11} is not a universal γ_2-fixer.*

Proof. Since $\gamma_2(C_{11}) = 3$, it suffices to prove that there is a permutation $\pi \in S_{11}$ such that $\gamma_2(\pi C_{11}) > 3$. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ j & d & 3 & b & g & f & e & i & h & a & k \end{pmatrix}.$$

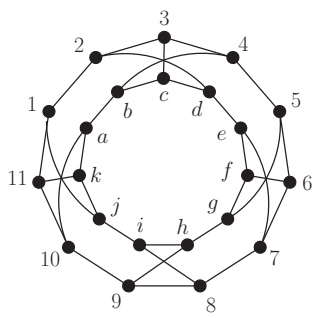

Figure 5. The graph πC_{11}.

Suppose that D is a 2-dominating set of πC_{11}. Since there is at least one vertex in D at distance 2 from vertex 3, D contains at least one vertex in $N^2_{\pi C_{11}}[3] = \{1,2,3,4,5,b,c,d\}$. Due to the symmetry of πC_{11}, we may assume that D contains one vertex in $S = \{1,2,3\}$.

Likewise, the set D must contain a vertex in $N^2_{\pi C_{11}}[9] = \{7,8,9,10,11,a,g,h,i\}$. We will see that no set D, with cardinality 3, containing a vertex in S and a vertex in $N^2_{\pi C_{11}}[9]$ can 2-dominate the graph πC_{11}. To prove this, we will consider the 27 cases that arise combining one vertex of S with a vertex of $N^2_{\pi C_{11}}[9]$.

If there is a 2-dominating set of cardinality 3, for one of the 27 cases there must be a vertex that 2-dominates all the vertices not dominated by at least one of the two vertices considered in the corresponding case. Therefore, for at least one of the cases considered, the intersection of all the 2-neighborhoods of the vertices not dominated by at least one of the two vertices considered should be non-empty. For this purpose, for each $i \in \{1,2,\ldots,27\}$ let

$$x_i = \begin{cases} 1, & \text{if } 1 \le i \le 9; \\ 2, & \text{if } 10 \le i \le 18; \\ 3, & \text{if } 19 \le i \le 27, \end{cases}$$

and let

$$y_i = \begin{cases} a, & \text{if } i \in \{1, 10, 19\}; \\ g, & \text{if } i \in \{2, 11, 20\}; \\ h, & \text{if } i \in \{3, 12, 21\}; \\ i, & \text{if } i \in \{4, 13, 22\}; \\ 7, & \text{if } i \in \{5, 14, 23\}; \\ 8, & \text{if } i \in \{6, 15, 24\}; \\ 9, & \text{if } i \in \{7, 16, 25\}; \\ 10, & \text{if } i \in \{8, 17, 26\}; \\ 11, & \text{if } i \in \{9, 18, 27\}. \end{cases}$$

For each i, $1 \leq i \leq 27$, we calculate the set S_i of vertices not 2-dominated by the two vertices (x_i, y_i) and show that there is no vertex contained in the intersection of all the 2-neighborhoods of vertices in S_i.

The sets $N^2_{\pi C_{11}}[x]$ for $x \in V(\pi C_{11})$ are shown in Table 1 and the results obtained in each case are shown in Table 2. ■

x	$N^2_{\pi C_{11}}[x]$	x	$N^2_{\pi C_{11}}[x]$
1	$\{1, 2, 3, 10, 11, d, i, j, k\}$	a	$\{4, 9, 10, 11, a, b, c, j, k\}$
2	$\{1, 2, 3, 4, 11, c, d, e, j\}$	b	$\{3, 4, 5, 10, k, a, b, c, d\}$
3	$\{1, 2, 3, 4, 5, b, c, d\}$	c	$\{2, 3, 4, a, b, c, d, e\}$
4	$\{2, 3, 4, 5, 6, a, b, c, g\}$	d	$\{1, 2, 3, 7, b, c, d, e, f\}$
5	$\{3, 4, 5, 6, 7, b, f, g, h\}$	e	$\{2, 6, 7, 8, c, d, e, f, g\}$
6	$\{4, 5, 6, 7, 8, e, f, g\}$	f	$\{5, 6, 7, d, e, f, g, h\}$
7	$\{5, 6, 7, 8, 9, d, e, f, i\}$	g	$\{4, 5, 6, 9, e, f, g, h, i\}$
8	$\{6, 7, 8, 9, 10, e, h, i, j\}$	h	$\{5, 8, 9, 10, f, g, h, i, j\}$
9	$\{7, 8, 9, 10, 11, a, g, h, i\}$	i	$\{1, 7, 8, 9, g, h, i, j, k\}$
10	$\{1, 8, 9, 10, 11, a, b, h, k\}$	j	$\{1, 2, 8, 11, a, h, i, j, k\}$
11	$\{1, 2, 9, 10, 11, a, j, k\}$	k	$\{1, 10, 11, a, b, i, j, k\}$

Table 1. $N^2_{\pi C_{11}}[x]$ for $x \in V(\pi C_{11})$.

Proposition 17. *The cycle C_{12} is not a universal γ_2-fixer.*

Proof. Since $\gamma_2(C_{12}) = 3$, we only need to prove that $\gamma_2(IC_{12}) > 3$, where

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ a & b & c & d & e & f & g & h & i & j & k & l \end{pmatrix}.$$

Observe that each vertex belonging to $\{1, 2, \ldots, 12\}$ 2-dominates 5 vertices in $\{1, 2, \ldots, 12\}$ and 3 vertices in $\{a, b, \ldots, l\}$, and that each vertex in $\{a, b, \ldots, l\}$

Case	(x_i, y_i)	S_i	T_i
1	$(1, a)$	$\{5, 6, 7, 8, e, f, g, h\}$	$\{8, e, f, h\}$
2	$(1, g)$	$\{7, 8, a, b, c\}$	$\{7, a, c\}$
3	$(1, h)$	$\{4, 6, 7, a, b, c, e\}$	$\{7, a, c\}$
4	$(1, i)$	$\{4, 5, 6, a, b, c, e, f\}$	$\{a, f\}$
5	$(1, 7)$	$\{4, a, b, c, g, h\}$	$\{c, h\}$
6	$(1, 8)$	$\{4, 5, a, b, c, f, g\}$	$\{a, f\}$
7	$(1, 9)$	$\{4, 5, 6, b, c, e, f\}$	$\{6, b, e\}$
8	$(1, 10)$	$\{4, 5, 6, 7, c, e, f, g\}$	$\{5, c, f\}$
9	$(1, 11)$	$\{4, 5, 6, 7, 8, b, c, e, f, g, h\}$	$\{c, h\}$
10	$(2, a)$	$\{5, 6, 7, 8, f, g, h, i\}$	$\{5, 7, h, i\}$
11	$(2, g)$	$\{7, 8, 10, a, b, k\}$	$\{7, b, k\}$
12	$(2, h)$	$\{6, 7, a, b, k\}$	$\{6, k\}$
13	$(2, i)$	$\{5, 6, 10, a, b, f\}$	$\{5, 6, 10\}$
14	$(2, 7)$	$\{10, a, b, g, h, k\}$	$\{a, g, k\}$
15	$(2, 8)$	$\{5, a, b, f, g, k\}$	$\{a, g\}$
16	$(2, 9)$	$\{5, 6, b, f, k\}$	$\{6, k\}$
17	$(2, 10)$	$\{5, 6, 7, f, g, i\}$	$\{6, 7, g, i\}$
18	$(2, 11)$	$\{5, 6, 7, 8, b, f, g, h, i\}$	$\{b, h, i\}$
19	$(3, a)$	$\{6, 7, 8, e, f, g, h, i\}$	$\{6, 7, g, i\}$
20	$(3, g)$	$\{7, 8, 10, 11, a, j, k\}$	$\{7, a, k\}$
21	$(3, h)$	$\{6, 7, 11, a, e, k\}$	$\{6, k\}$
22	$(3, i)$	$\{6, 10, 11, a, e, f\}$	$\{a, f, \}$
23	$(3, 7)$	$\{10, 11, a, g, h, j, k, \}$	$\{a, g, k\}$
24	$(3, 8)$	$\{11, a, f, g, k\}$	$\{a, f\}$
25	$(3, 9)$	$\{6, e, f, j, k\}$	$\{6, k\}$
26	$(3, 10)$	$\{6, 7, e, f, g, i, j\}$	$\{6, 7, g, i\}$
27	$(3, 11)$	$\{6, 7, 8, e, f, g, h, i\}$	$\{6, 7, g, i\}$

Table 2. In each case the set $S_i = V \setminus (N^2_{\pi C_{11}}[x_i] \cup N^2_{\pi C_{11}}[y_i])$ and a subset T_i of S_i satisfying $\bigcap_{v \in T_i} N^2_{\pi C_{11}}[v] = \emptyset$ are given.

2-dominates 5 vertices in $\{a, b, \ldots, l\}$ and 3 vertices in $\{1, 2, \ldots, 12\}$. Suppose on the contrary that there exists a 2-dominating set D with 3 vertices, r of them in $\{1, 2, \ldots, 12\}$ and the remaining s in $\{a, b, \ldots, l\}$. Since each vertex of IC_{12} must be 2-dominated, then r, s must be integers satisfying:

$$5r + 3s = 12,$$
$$3r + 5s = 12,$$
$$r + s = 3.$$

From the previous equations we derive that r, s are integers satisfying $r = s$ and $r + s = 3$, which is a contradiction. ■

Proposition 18. *The cycle C_{16} is not a universal γ_2-fixer.*

Proof. Since $\gamma_2(C_{16}) = 4$, it suffices to prove that there is a permutation $\pi \in S_{16}$ such that $\gamma_2(\pi C_{16}) > 4$. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ n & b & c & d & e & f & g & h & i & j & k & l & m & a & p & o \end{pmatrix}.$$

We remark that each vertex in $\{1, 2, \ldots, 16\}$ 2-dominates exactly 5 vertices in $\{1, 2, \ldots, 16\}$ and at most 4 vertices in $\{a, b, \ldots, p\}$. Moreover, vertices in $\{3, 4, \ldots, 12, 15, 16\}$ and vertices in $\{c, d, \ldots, l, o, p\}$ 2-dominate exacty 3 vertices in $\{a, b, \ldots, o\}$ and in $\{1, 2, \ldots, 16\}$, respectively. Likewise, each vertex in $\{a, b, \ldots, o\}$ 2-dominates exactly 5 vertices in $\{a, b, \ldots, o\}$ and at most 4 vertices in $\{1, 2, \ldots, 16\}$ (see Figure 6).

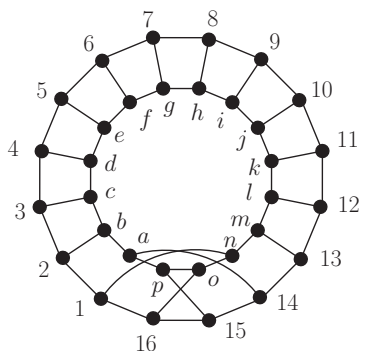

Figure 6. The graph πC_{16}.

We will prove that it is not possible to 2-dominate πC_{16} with 4 vertices. Suppose on the contrary that there exists a 2-dominating set D with 4 vertices.

Let $S = \{1, 2, 13, 14, a, b, m, n\}$. We claim that if D is a 2-dominating set of πC_{16} with 4 vertices, then one of the following cases holds:

(i) all vertices are 2-dominated by exactly one vertex in D,

(ii) all vertices are 2-dominated by exactly one vertex in D, except at most two vertices that are 2-dominated by two vertices in S,

(iii) all vertices are 2-dominated by exactly one vertex in D, except one vertex that is 2-dominated by three vertices in S.

To see this, observe that there are 12 vertices not lying in the union of the 2-neighborhoods of the vertices in S, hence D contains at most two vertices of S. Since πC_{16} has 32 vertices, vertices of πC_{16} are 2-dominated exactly by one vertex of D, except two vertices that are 2-dominated both of them by two vertices of D or except one vertex that is 2-dominated by three vertices of D.

There must be a vertex that 2-dominates vertex 7, that is a vertex from $N^2_{\pi C_{16}}[7] = \{5, 6, 7, 8, 9, f, g, h\}$. Due to the symmetry of the graph πC_{16}, we may assume that D contains one of the vertices 5, 6 or 7.

If $5 \in D$, then observe that $N^2_{\pi C_{16}}[5] = \{3, 4, 5, 6, 7, d, e, f\}$ and $g \notin N^2_{\pi C_{16}}[5]$. By our claim, we may assume that $i \in D$. Now, also by our claim, we may assume $13 \in D$ because $11 \notin N^2_{\pi C_{16}}[5] \cup N^2_{\pi C_{16}}[i]$. The set of vertices not in $N^2_{\pi C_{16}}[5] \cup N^2_{\pi C_{16}}[i] \cup N^2_{\pi C_{16}}[13]$ is $S_5 = \{1, 2, 16, b, c, o, p\}$. By the above remark, only vertex a can 2-dominate all vertices in S_5, but $1 \notin N^2_{\pi C_{16}}[a]$.

For the case where $6 \in D$, observe that $N^2_{\pi C_{16}}[6] = \{4, 5, 6, 7, 8, e, f, g\}$ and $h \notin N^2_{\pi C_{16}}[6]$. By our claim, we may assume that $j \in D$. Now, also by our claim, we may assume $14 \in D$ because $12 \notin N^2_{\pi C_{16}}[6] \cup N^2_{\pi C_{16}}[j]$. The set of vertices not in $N^2_{\pi C_{16}}[6] \cup N^2_{\pi C_{16}}[j] \cup N^2_{\pi C_{16}}[14]$ is $S_6 = \{1, 2, 3, c, d, n, o\}$. By the above remark, no vertex can 2-dominate all vertices in S_6.

Finally, if $5 \in D$, observe that $N^2_{\pi C_{16}}[7] = \{5, 6, 7, 8, 9, f, g, h\}$ and $e, i \notin N^2_{\pi C_{16}}[5]$. By our claim, we may assume that $c, k \in D$. The set of vertices not in $N^2_{\pi C_{16}}[7] \cup N^2_{\pi C_{16}}[c] \cup N^2_{\pi C_{16}}[k]$ is $S_7 = \{1, 13, 14, 15, 16, n, o, p\}$. By the above remark, only vertex 15 can 2-dominate all vertices in S_7, but $n \notin N^2_{\pi C_{16}}[15]$. ■

By Observation 6, we have the following:

Corollary 19. *The paths P_8, P_{11}, P_{12} and P_{16} are not universal γ_2-fixers.*

4. Final Comment

A natural problem, unsolved here, is that of determining all graphs which are γ_2-fixers. A more modest problem is that of characterizing those graphs G for which $\gamma_2(G) \leq \gamma_2(\pi G)$ for all permutations π.

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