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CHARACTERIZING ATOMS THAT RESULT FROM DECOMPOSITION BY CLIQUE SEPARATORS

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Abstract

A graph is defined to be an atom if no minimal vertex separator induces a complete subgraph; thus, atoms are the graphs that are immune to clique separator decomposition. Atoms are characterized here in two ways: first using generalized vertex elimination schemes, and then as generalizations of 2-connected unichord-free graphs (the graphs in which every minimal vertex separator induces an edgeless subgraph).

Keywords: clique separator, minimal separator, unichord-free graph.

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1. Introduction

If a graph G has $S \subset V(G)$, let $G - S$ denote the subgraph of G induced by $V(G) - S$, and if $v \in V(G)$, let $G - v = G - \{v\}$. A *component* of G is an inclusion-maximal connected subgraph. For nonadjacent vertices v and w in a connected graph G, a v,w-*separator* is a set $S \subseteq V(G) - \{v, w\}$ such that $V(G) - S$ induces a subgraph of G that has v and w in different components. A *minimal* v,w-*separator* is a v,w-separator that is not a proper subset of another v,w-separator, and a (*minimal*) *separator* of G is a (minimal) v,w-separator for some $v, w \in V(G)$. If S is a minimal separator of G, then $G - S$ has two components such that every vertex in S has neighbors in each of those components of $G - S$; see [3] for more on minimal separators.

A *clique* of a graph G is a nonempty set of pairwise-adjacent vertices of G, a *clique* (*minimal*) v,w-*separator* of G is a (minimal) v,w-separator that is a clique, and a *clique* (*minimal*) *separator* of G is a (minimal) separator that is a clique.

Define a graph G to be an *atom* if G is connected and has no clique separator (or, equivalently, G has no clique minimal separator). Therefore, a connected graph G is an atom if and only if G is not decomposable by *clique separator decomposition* as in [1, 9] (equivalently, G is not decomposable by *clique minimal separator decomposition* as in [1, 4]). Reference [1] surveys the significance, history, applications, and algorithmic aspects of clique (minimal) separator decomposition. In spite of this significance and the fundamental role that atom subgraphs play, these subgraphs do not seem to have been explicitly studied as actual graphs. The present paper will characterize such atom graphs.

A *chord* xy of a cycle C is an edge xy with $x, y \in V(C)$ and $xy \notin E(C)$. A graph G is *chordal* if every cycle of G with length 4 or more has a chord—in other words, if every cycle long enough to have a chord always does have a chord. See [2, 8] for many other characterizations (and applications) of chordal graphs, one of which is that every induced subgraph H of G has a *simplicial vertex,* meaning a vertex v whose neighborhood $N(v)$ is a clique of H (equivalently, whose closed neighborhood $N[v] = N(v) \cup \{v\}$ is a clique). Another characterization of being chordal is that every minimal separator is a clique minimal separator. As a result, being an atom graph can be viewed as, in a sense, the opposite of being a chordal graph.

Examples of atoms include all cycles, wheels, complete graphs, and complete bipartite graphs—but the complete tripartite graph $K_{1,1,2}$ (formed by deleting one edge from K_4) is not an atom, since the two degree-3 vertices form a clique separator. If G is connected but $G - v$ is not, then G is not an atom ($\{v\}$ is a minimal separator); thus, every atom is 2-connected or K_1 or K_2. Other examples of graphs that are not atoms include chordal graphs that are not complete (take $S = N(v)$ for any simplicial vertex v) and graphs that consist of a cycle augmented with a unique chord xy (take $S = \{x, y\}$). Figure 1 shows a more involved example of a graph G that is not an atom, with a clique minimal 1, 3-separator $S = \{2, 5, 6\}$. (The meaning of B and $B \cup C$ will be explained in Section 2.)

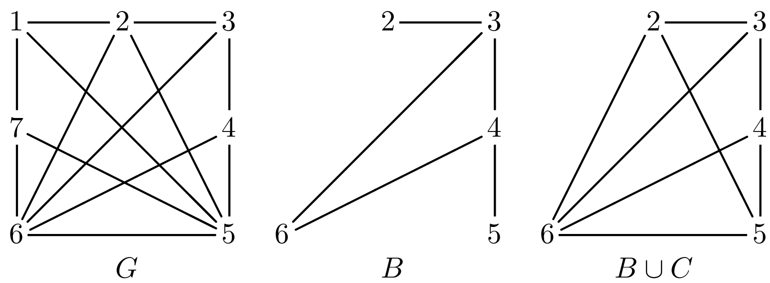

Figure 1. A non-atom G and a nonsingular bridge B of the triangle $C = 256$.

Sections 2 and 3 will characterize atoms using two standard graph-theoretic approaches. Section 2 involves strategically deleting vertices, which loosely evokes the familiar simplicial vertex elimination characterization of chordal graphs. Section 3 focuses on cycles and chords, partially generalizing a characterization of a previously-studied subclass of atoms—the "unichord-free graphs" (the graphs in which no cycle has a unique chord).

2. CHARACTERIZATIONS INVOLVING VERTEX DELETIONS

In Theorem 1, vertices u and v are called *twins* if u and v are adjacent and $N[u] = N[v]$ or if u and v are nonadjacent and $N(u) = N(v)$. Vertices $u_1, u_2, \ldots$ are *pairwise nonadjacent twins* if u_i and u_j are nonadjacent twins whenever $i \neq j$.

Theorem 1. *If a graph G has either adjacent twin vertices w and x or pairwise nonadjacent twin vertices x, y, and z, then G is an atom if and only if $G - x$ is an atom.*

Proof. First suppose w and x are adjacent twin vertices of a graph G and S is any clique minimal separator of G. If $w \in S$, then $x \in S$; otherwise, x would be in a component H_x of $G - S$ with $N(x) \subseteq V(H_x) \cup S$ while some $v \in N(w) - \{x\}$ would be in a different component of $G - S$ (contradicting $N[w] = N[x]$). If $w \notin S$, then the twins w and x are both in the component H_x of $G - S$. Hence, w and x are either both in S or both in the same component of $G - S$, and so each $S \subset V(G)$ will be a clique minimal separator of G if and only if $S - \{x\}$ is a clique minimal separator of $G - x$. Therefore, G is an atom if and only if $G - x$ is an atom.

Now suppose x, y, and z are pairwise nonadjacent twin vertices of G. If y is simplicial, then neither G nor $G - x$ can be an atom (because $N(y)$ cannot be a clique minimal y, z-separator). Hence, suppose y is not simplicial (so none of x, y, z is simplicial) and suppose S is any clique minimal separator of G. If any one of x, y, z is in S, then it will have neighbors u and v in different components of $G - S$, and so each of the nonadjacent twins x, y, z is in $N(u) \cap N(v) \subseteq S$, which would contradict S being a clique. Thus, none of x, y, z can be in S, so they will be, respectively, in (possibly identical) components H_x, H_y, H_z of $G - S$. If y has a neighbor $u \in H_y$, then $u \in N(x) \cap N(z)$; thus x, y, z are all in H_y, and so $H_x = H_y = H_z$. If, instead, y has no neighbor in H_y, then each of H_x, H_y, H_z is a singleton. Either way, $S \subset V(G)$ will be a clique minimal separator of G if and only if $S - \{x\}$ is a clique minimal separator of $G - x$. Therefore, G is an atom if and only if $G - x$ is an atom. ■

The remainder of this section will move towards the characterization in Theorem 5, showing that a graph G is an atom if and only if either G is a particular very simple type of graph—specifically, a chordless cycle or a complete graph—or G is recursively reducible to this simple type of graph by repeated vertex deletions (where each reduction step simultaneously deletes all the vertices of certain sorts of subgraphs).

As in [11], define a *nonsingular bridge* B of a cycle C in a graph G to be a subgraph of G formed from a nonempty component H of $G - V(C)$ by appending all the edges xy that have $x \in V(H)$ and $y \in V(C)$; thus B is a (not necessarily induced) subgraph of $B \cup C$. Figure 1 shows an example with H just the edge 34. Call the vertices in $V(B) \cap V(C)$ the *vertices of attachment* of B, and let $B^\circ = V(B) - V(C) = V(H) \neq \emptyset$.

Lemma 2. *If an atom G has a chordless cycle with a nonsingular bridge B, then $G - B^\circ$ is an atom.*

Proof. Suppose B is a nonsingular bridge of a chordless cycle C in an atom G. Suppose $G' = G - B^\circ$ is not an atom, say because G' has a clique minimal separator S'. Since a chordless cycle of G' cannot contain vertices from more than one component of $G' - S'$, let H'_1 be the component of $G' - S'$ that contains $V(C) - S'$, with say $v_1 \in V(H'_1)$. Choose $v_2 \in V(G')$ in a component $H'_2 \neq H'_1$ of $G' - S'$. Since every $b \in B^\circ$ has $N_G[b] \subseteq V(B) \subseteq B^\circ \cup V(C) \subseteq B^\circ \cup V(H'_1) \cup S'$, there is a minimal v_1, v_2-separator of G contained in the clique S', contradicting that G is an atom. ■

Lemma 3. *If a graph G is not an atom and has a chordless cycle C with a nonsingular bridge B such that $B \cup C$ is an atom and if B has at least two nonadjacent vertices of attachment, then $G - B^\circ$ is not an atom.*

Proof. Suppose G is not an atom, say because G has a clique minimal separator S, and suppose B is a nonsingular bridge of a chordless cycle C in G such that $B \cup C$ is an atom and B has nonadjacent vertices of attachment $c_1, c_2 \in V(B) - B^\circ \subseteq V(C)$. Since the nonadjacent vertices c_1 and c_2 cannot both be in the clique S, say $c_1 \notin S$. Since C is chordless and S is a clique minimal separator of G, there is a component H_1 of $G - S$ that contains $V(C) - S$ (and so contains c_1). Choose $v_2 \in V(G)$ in some component $H_2 \neq H_1$ of $G - S$, so there is a clique minimal c_1, v_2-separator contained in S. Since $B \cup C$ is an atom, S does not contain a clique minimal separator of $B \cup C$, so $v_2 \notin B^\circ$, and so $S - B^\circ$ is a clique minimal c_1, v_2-separator of $G - B^\circ$. Therefore, $G - B^\circ$ is not an atom. ■

Figure 1 illustrates the necessity of the conditions on B and C in Lemma 3 for the non-atom graph G shown there. If C is the chordless cycle induced by $\{2, 5, 6\}$, then the nonsingular bridge B with $B^\circ = \{3, 4\}$ has the three vertices of attachment $2, 5, 6$, and the 4-spoke wheel $B \cup C$ is an atom; note that B does not have nonadjacent vertices of attachment and the 4-spoke wheel $G - B^\circ$ is an atom. Alternatively, if C is the chordless cycle induced by $\{2, 3, 4, 5\}$, then the nonsingular bridge B with $B^\circ = \{1, 6, 7\}$ has the four vertices of attachment $2, 3, 4, 5$, of which 2 and 4 are nonadjacent; in this case $B \cup C \cong G$ is not an atom and the length-4 cycle $G - B^\circ$ is an atom.

Lemma 4. *Every atom is either a chordless cycle or a complete graph or has a chordless cycle C with a nonsingular bridge B such that $B \cup C$ is an atom and B has at least two nonadjacent vertices of attachment.*

Proof. Suppose G is an n-vertex atom with $C_n \ncong G \ncong K_n$. Therefore, G is 2-connected and has a chordless cycle C of length $k \geq 4$ in G (otherwise G would be chordal, so $G \ncong K_k$ would have a simplicial vertex v with $N(v)$ a clique minimal separator of G, contradicting that G is an atom), and C has a nonsingular bridge B (since C is chordless and $G \ncong C_k$).

Suppose for the moment that $B \cup C$ is not an atom, say because $B \cup C$ has a clique minimal v, w-separator S. If $v, w \in V(C)$, then S cannot contain a unique vertex of C or exactly two consecutive vertices of C (since v and w would still be connected by the remaining subpath of $C - S$) or two or more nonconsecutive vertices of C (since C is chordless and S is a clique); hence C would have to be inside of one component of $B \cup C - S$ (contradicting that S is a v, w-separator of $B \cup C$). If, on the other hand, at least one of v, w is in $V(B) - V(C) = B^\circ$ (and so is nonadjacent with each vertex of $G - V(B)$), then the clique S is inside $V(B)$ with every v-to-w path in G containing a vertex in S; but then S would be a clique minimal v, w-separator of all of G (contradicting that G is an atom). Therefore, $B \cup C$ is an atom.

Now suppose that every two vertices of attachment of B are adjacent. Since C is chordless with length at least 4, B has at most two vertices of attachment and, if two, they are adjacent. Thus the vertices of attachment of B would include a clique minimal v, w-separator of G with, say, $v \in B^\circ$ and $w \in V(C) - V(B)$ (contradicting that G is an atom). Therefore, B has at least two nonadjacent vertices of attachment. ∎

Theorem 5. *A graph is an atom if and only if it can be reduced to a chordless cycle or a complete graph by repeatedly deleting the vertices in B° where B is a nonsingular bridge of a chordless cycle C such that $B \cup C$ is an atom and B has at least two nonadjacent vertices of attachment.*

Proof. The "only if" direction follows from (repeated use of) Lemmas 2 and 4. The "if" direction follows from cycles and complete graphs being atoms and from (repeated use of) Lemma 3. ∎

3. Characterizations Involving Cycles and Chords

The graphs in which no cycle has a unique chord were introduced independently in [6, 10] and characterized in [6] by every minimal separator inducing an edgeless subgraph (additional characterizations appear in [7, 10]). These graphs have become known as the *unichord-free graphs*; see [5] and its references. Thus, every 2-connected unichord-free graph is an atom. Since a graph is chordal if and only if every minimal separator is a clique, being unichord-free is the opposite of being chordal in a stronger sense than being an atom is.

Another characterization in [6] is that a graph is unichord-free if and only if, for every chord xy of every cycle C, there exist adjacent vertices $v, w \in V(C) - \{x, y\}$ such that the vertices x, v, y, w come in that order around C. (Such chords xy and vw are often called "crossing chords".) That characterization corresponds to the special case of Theorem 6 in which alternative (1) holds with π_1 having exactly one edge not in $E(C)$ (a chord of C).

Theorem 6. *A 2-connected graph G is an atom if and only if, for every two vertices v and w, every cycle C through v and w, and every chord xy of C such that the four vertices x, v, y, w come in that order around C, one of the following alternatives holds*:

(1) *$G - \{x, y\}$ contains a v-to-w path π_1 such that no internal vertex z_1 of π_1 forms a triangle xyz_1.*

(2) *$G - \{x, y\}$ contains two v-to-w paths π_1 and π_2 with, respectively, distinct nonadjacent internal vertices z_1 and z_2 that form triangles xyz_1 and xyz_2.*

Proof. For the "only if" direction, suppose an atom G contains vertices v and w, cycle C, and chord xy as described in the theorem. For any path π, let π° denote the set of internal vertices of π. Since G is an atom, the clique $\{x, y\}$ cannot be a minimal v, w-separator, and so $G - \{x, y\}$ contains a v-to-w path π_1.

If some such path π_1 has no internal vertex that forms a triangle with xy, then alternative (1) holds. Otherwise, suppose (1) fails and $z_1 \in \pi_1^\circ$ forms a triangle xyz_1.

Since G is an atom, the clique $\{x, y, z_1\}$ cannot be a minimal v, w-separator, and so $G - \{x, y, z_1\}$ contains a v-to-w path π_2 that can be assumed to contain a $z_2 \in \pi_2^\circ$ that forms a triangle xyz_2 (otherwise, π_2 and z_2 would satisfy alternative (1)). Moreover, $z_2 \neq z_1$ can be assumed to not be adjacent to z_1 (otherwise, all the internal vertices of all the v-to-w paths would form a clique with x and y that would be a clique minimal v, w-separator, contradicting that G is an atom), and so (2) holds.

Therefore, either (1) or (2) holds.

For the "if" direction, suppose a 2-connected graph G is not an atom, say because G has a clique minimal separator S. Since G is 2-connected, there exist adjacent $x, y \in S$, nonadjacent v and w from two different components of $G - S$, and a cycle C through v and w that has chord xy. Thus, the four vertices x, v, y, w come in that order around C.

If $|S| = 2$, then no v-to-w paths exist in $G - \{x, y\}$, and so (1) and (2) would both fail.

Hence, suppose $|S| \geq 3$, so a v-to-w path exists in $G - \{x, y\}$. Every v-to-w path π_1 in $G - \{x, y\}$ contains a $z_1 \in \pi_1^\circ \cap S$ that forms a triangle xyz_1, and so (1) fails. If $|S| = 3$, then no v-to-w path π_2 would exist in $G - \{x, y, z_1\}$, and so (2) would also fail.

Hence, suppose $|S| \geq 4$, so v-to-w paths π_1 in $G - \{x, y\}$ and π_2 in $G - \{x, y, z_1\}$ exist, each containing $z_i \in \pi_i^\circ \cap S$ that forms a triangle xyz_i. But $z_1, z_2 \in S$ implies that, for every such π_2 and z_2, the vertices z_1 and z_2 are adjacent, and so (2) also fails.

Therefore, in every case, both (1) and (2) would fail. ■

Corollary 7 will again characterize atoms (now without assuming 2-connectedness), but with Corollary 7 using an existential quantifier $(\exists C)$ where Theorem 6 has a universal quantifier $(\forall C)$.

Corollary 7. *A graph G with at least three vertices is an atom if and only if, for every two vertices v and w, there exists a cycle C through v and w such that, for every chord xy of C such that the four vertices x, v, y, w come in that order around C, alternative* (1) *or* (2) *of Theorem* 6 *holds.*

Proof. For the "only if" direction, suppose an atom G contains distinct vertices u, v, and w. Since K_1 and K_2 are the only atoms that are not 2-connected, G is 2-connected and so has a cycle C through v and w. If $|V(C)| = 3$, then C is chordless, which makes "for every chord xy of C ..." vacuously true in Theorem 6, and so alternative (1) or (2) holds. If $|V(C)| > 3$ and xy is a chord of C such that the four vertices x, v, y, w come in that order around C, then (1) or (2) holds by Theorem 6.

For the "if" direction, suppose G has $|V(G)| \geq 3$ and is not an atom. If G is not 2-connected, then there are vertices v and w that are are not in any cycle, which makes "there exists a cycle C through v and w ..." false. Hence, suppose the non-atom G is 2-connected with a clique minimal v, w-separator S. Thus there is a cycle C that contains v and w, and there are adjacent $x, y \in S$ that form a chord xy of C, with the four vertices x, v, y, w coming in that order around C. The remainder of the proof uses the same argument (word for word, with cases for $|S| = 2$, $|S| = 3$, and $|S| \geq 4$) as the proof of the "if" direction of Theorem 6. ■

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