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CRITICALITY OF SWITCHING CLASSES OF REVERSIBLE 2-STRUCTURES LABELED BY AN ABELIAN GROUP

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Abstract

Let V be a finite vertex set and let $(\mathbb{A}, +)$ be a finite abelian group. An $\mathbb{A}$-labeled and reversible 2-structure defined on V is a function $g : (V \times V) \setminus \{(v, v) : v \in V\} \longrightarrow \mathbb{A}$ such that for distinct $u, v \in V$, $g(u, v) = -g(v, u)$. The set of $\mathbb{A}$-labeled and reversible 2-structures defined on V is denoted by $\mathscr{L}(V, \mathbb{A})$. Given $g \in \mathscr{L}(V, \mathbb{A})$, a subset X of V is a *clan* of g if for any $x, y \in X$ and $v \in V \setminus X$, $g(x, v) = g(y, v)$. For example, $\emptyset$, V and $\{v\}$ (for $v \in V$) are clans of g, called *trivial*. An element g of $\mathscr{L}(V, \mathbb{A})$ is *primitive* if $|V| \geq 3$ and all the clans of g are trivial.

The set of the functions from V to $\mathbb{A}$ is denoted by $\mathscr{S}(V, \mathbb{A})$. Given $g \in \mathscr{L}(V, \mathbb{A})$, with each $s \in \mathscr{S}(V, \mathbb{A})$ is associated the *switch* g^s of g by s defined as follows: given distinct $x, y \in V$, $g^s(x, y) = s(x) + g(x, y) - s(y)$. The *switching class* of g is $\{g^s : s \in \mathscr{S}(V, \mathbb{A})\}$. Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ and $X \subseteq V$, $\{g_{\restriction (X \times X) \setminus \{(x,x) : x \in X\}} : g \in \mathfrak{S}\}$ is a switching class, denoted by $\mathfrak{S}[X]$.

Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, a subset X of V is a *clan* of $\mathfrak{S}$ if X is a clan of some $g \in \mathfrak{S}$. For instance, every $X \subseteq V$ such that $\min(|X|, |V \setminus X|) \leq 1$ is a clan of $\mathfrak{S}$, called *trivial*. A switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ is *primitive* if $|V| \geq 4$ and all the clans of $\mathfrak{S}$ are trivial. Given a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, $\mathfrak{S}$ is *critical* if for each $v \in V$, $\mathfrak{S} - v$ is not primitive. First, we translate the main results on the primitivity of $\mathbb{A}$-labeled and reversible 2-structures in terms of switching classes. For instance, we prove the following. For a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ such that $|V| \geq 8$, there exist $u, v \in V$ such that $u \neq v$ and $\mathfrak{S}[V \setminus \{u, v\}]$ is primitive. Second, we characterize the critical switching classes by using some of the critical digraphs described in [Y. Boudabous and P. Ille, *Indecomposability graph and critical vertices of an indecomposable graph*, Discrete Math. **309** (2009) 2839–2846].

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1. Introduction

Ehrenfeucht *et al.* [5, 6] generalized the decomposition of graphs in modules by introducing 2-structures and clans of 2-structures. They deduced a notion of primitivity for 2-structures [7] that leads to the notion of criticality for 2-structures [1, 11]. In another context, operations of switching were defined for graphs [12] and digraphs [3]. These operations generate switching classes. Ehrenfeucht *et al.* [5] generalized the switching to 2-structures by labeling them with a group.

Ehrenfeucht *et al.* [5] defined the family of clans of a switching class of (labeled) 2-structures as the collection of clans of the members of the class. A key difference is that the family of clans of a switching class is always closed under complementation whereas it is not the case for 2-structures. In particular, given a switching class and an element of the underlying set, one can construct a member of the class in which this element is isolated. This isolation process facilitates the study of the family of clans of a switching class.

Once one has the notion of clan, the notions of primitivity and criticality follow naturally. They have been used to good effect in a number of works on various structures. It is natural to seek analogous results for switching classes. Because the clans of a switching class are closed under complementation, some subtleties appear. The isolation process facilitates translating the main results on primitivity of 2-structures in terms of switching classes. However, the same does not apply as easily for criticality.

Schmerl and Trotter [11] and Bonizzoni [1] characterized independently the critical 2-structures. Ille [8] introduced the primitivity graph to study the primitive digraphs. Boudabbous and Ille [2] used this tool to bring out an elegant approach for describing critical digraphs. It follows from the characterization of Bonizzoni [1] that the critical 2-structures can be classify into four families by considering their primitivity graphs. The labeling of 2-structures by a group allows us to describe each of the four families as a unique algebraic formula of elementary labeled 2-structures that come from the critical digraphs of the corresponding family. In this paper, we adopt a similar approach for switching classes. Surprisingly, we only obtain three families. Indeed, two of the four families of critical 2-structures collapse into only one. In order to describe the switching classes of each of the three families, we still use the isolation process but technical issues arise that must be dealt with.

2. SWITCHING CLASSES

We only consider finite structures. Let $(\mathbb{A}, +)$ be an abelian group. The identity element of $(\mathbb{A}, +)$ is denoted by 0 and the inverse element of $a \in \mathbb{A}$ is denoted by $-a$. For each $a \in \mathbb{A}$, $o_{\mathbb{A}}(a)$ denotes the order of a in $\mathbb{A}$. Let V be a vertex set. An $\mathbb{A}$-labeled 2-structure defined on V is a function $g : (V \times V) \setminus \{(v, v) : v \in V\} \longrightarrow \mathbb{A}$. An $\mathbb{A}$-labeled 2-structure g defined on V is *reversible* if for distinct $x, y \in V$, we have $g(x, y) = -g(y, x)$. We denote the set of $\mathbb{A}$-labeled and reversible 2-structures defined on V by $\mathscr{L}(V, \mathbb{A})$. Given $g \in \mathscr{L}(V, \mathbb{A})$ and $X \subseteq V$, the element $g_{\restriction (X \times X) \setminus \{(x,x):x \in X\}}$ of $\mathscr{L}(X, \mathbb{A})$ is the 2-substructure of g induced by X, which is simply denoted by $g[X]$. Given $X \subseteq V$, $g[V \setminus X]$ is also denoted by $g - X$, and by $g - v$ when $X = \{v\}$.

Let G be a graph. Given $X \subseteq V(G)$, the *switch* G^X of G by $\{X, V(G) \setminus X\}$ [12] is the graph obtained from G by exchanging the edges and the non-edges between X and $V(G) \setminus X$. Analogously, given a tournament T and $X \subseteq V(T)$, the *switch* T^X of T by $\{X, V(G) \setminus X\}$ is the tournament obtained from T by reversing the arcs between X and $V(G) \setminus X$.

Ehrenfeucht *et al.* [5] generalized the switch to $\mathbb{A}$-labeled and reversible 2-structures as follows. Given a vertex set V, $\mathscr{S}(V, \mathbb{A})$ denotes the set of functions from V to $\mathbb{A}$ (such functions are called *selectors* in [5]). Now, with $g \in \mathscr{L}(V, \mathbb{A})$ and $s \in \mathscr{S}(V, \mathbb{A})$ associate the *switch* g^s of g by s defined as follows: given distinct $x, y \in V$,

$$g^s(x, y) = s(x) + g(x, y) - s(y).$$

For distinct $x, y \in V$, we have $g^s(y, x) = -g^s(x, y)$. Thus $g^s \in \mathscr{L}(V, \mathbb{A})$.

As for graphs and tournaments, Ehrenfeucht *et al.* [5] defined a binary relation R on $\mathscr{L}(V, \mathbb{A})$ as follows. Given $g, h \in \mathscr{L}(V, \mathbb{A})$, $R(g, h) = +$ if there exists

$s \in \mathscr{S}(V,\mathbb{A})$ such that $h = g^s$. Given $g \in \mathscr{L}(V,\mathbb{A})$, we have $g^{\mathcal{O}} = g$, where $\mathcal{O}$ is the element of $\mathscr{S}(V,\mathbb{A})$ defined by $\mathcal{O}(x) = 0$ for every $x \in V$. Hence R is reflexive. Furthermore, given $g \in \mathscr{L}(V,\mathbb{A})$ and $s,t \in \mathscr{S}(V,\mathbb{A})$, we have $(g^s)^t = g^{s+t}$, where $s+t$ is the element of $\mathscr{S}(V,\mathbb{A})$ defined by $(s+t)(x) = s(x)+t(x)$ for every $x \in V$. It follows that R is symmetric and transitive, so R is an equivalence relation. The switching classes in $\mathscr{L}(V,\mathbb{A})$ are the equivalence classes with respect to R. Let $g \in \mathscr{L}(V,\mathbb{A})$. The equivalence class of g is called the *switching class* of g, and is denoted by $\langle g \rangle$. Thus, for every $g \in \mathscr{L}(V,\mathbb{A})$, we have

$$\langle g \rangle = \{g^s : s \in \mathscr{S}(V,\mathbb{A})\}.$$

Given $g \in \mathscr{L}(V,\mathbb{A})$ and $X \subseteq V$, we have

$$\langle g[X] \rangle = \{h[X] : h \in \langle g \rangle\}.$$

Therefore, given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$, $\{g[X] : g \in \mathfrak{S}\}$ is a switching class in $\mathscr{L}(X,\mathbb{A})$, which is denoted by $\mathfrak{S}[X]$. Given $X \subseteq V$, $\mathfrak{S}[V \setminus X]$ is also denoted by $\mathfrak{S} - X$, and by $\mathfrak{S} - v$ when $X = \{v\}$.

3. Clans, Primitivity and Criticality

Given $g \in \mathscr{L}(V,\mathbb{A})$, a subset X of V is a *clan* [5] of g if for any $x,y \in X$ and $v \in V \setminus X$,

$$g(x,v) = g(y,v). \tag{1}$$

Observe that (1) implies that $g(v,x) = g(v,y)$ because g is reversible. Denote by $\mathrm{Cl}(g)$ the set of clans of g. The classic properties of clans follow.

Proposition 1 (Lemma 3.4 [5]). *Given $g \in \mathscr{L}(V,\mathbb{A})$, the following assertions hold.*

1. *$\emptyset, V \in \mathrm{Cl}(g)$, and for each $x \in V$, $\{x\} \in \mathrm{Cl}(g)$.*
2. *For any $X,Y \in \mathrm{Cl}(g)$, $X \cap Y \in \mathrm{Cl}(g)$.*
3. *For any $X,Y \in \mathrm{Cl}(g)$, if $X \cap Y \neq \emptyset$, then $X \cup Y \in \mathrm{Cl}(g)$.*
4. *For any $X,Y \in \mathrm{Cl}(g)$, if $X \setminus Y \neq \emptyset$, then $Y \setminus X \in \mathrm{Cl}(g)$.*

A subset X of V is a *clancut* [10] of g if $X, V \setminus X \in \mathrm{Cl}(g)$. Furthermore, given $a \in \mathbb{A}$, a subset X of V is an *a-clancut* of g if $g(x,y) = a$ for any $x \in X$ and $y \in V \setminus X$. A vertex x of g is *isolated* if $\{x\}$ is a clancut of g. Given $a \in \mathbb{A}$, a vertex x of g is *a-isolated* if $\{x\}$ is an a-clancut of g.

Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$, a subset X of V is a *clan* [5] of $\mathfrak{S}$ if there exists $g \in \mathfrak{S}$ such that X is a clan of g. We denote by $\mathrm{Cl}(\mathfrak{S})$ the set of clans of $\mathfrak{S}$. Thereby, we have

$$\mathrm{Cl}(\mathfrak{S}) = \bigcup_{g \in \mathfrak{S}} \mathrm{Cl}(g).$$

Proposition 2 (Lemma 13.4 [5]). *Consider $g \in \mathscr{L}(V, \mathbb{A})$ and $X \in \mathrm{Cl}(g)$. For each $a \in \mathbb{A}$, there is $s \in \mathscr{S}(V, \mathbb{A})$ such that X is an a-clancut of g^s.*

Two immediate consequences of Proposition 2 follow.

Corollary 3. *Let $g \in \mathscr{L}(V, \mathbb{A})$. For every $x \in V$ and for every $a \in \mathbb{A}$, there is $s \in \mathscr{S}(V, \mathbb{A})$ such that x is an a-isolated vertex of g^s.*

Corollary 4. *The set of clans of a switching class is closed under complementation.*

Let $g \in \mathscr{L}(V, \mathbb{A})$. By the first assertion of Proposition 1, $\emptyset$, V and $\{x\}$ (for $x \in V$) are clans of g, called *trivial clans*. We say that g is *indecomposable* if all the clans of g are trivial, otherwise g is said to be *decomposable*. Moreover, we say that g is *primitive* [5] if g is indecomposable and $|V| \geq 3$.

The following two results are important in the study of primitivity of $\mathbb{A}$-labeled and reversible 2-structures. The first one is established for binary relational structures in [11], and the second one is established for digraphs in [9]. In fact, both of them are directly transposable in terms of $\mathbb{A}$-labeled and reversible 2-structures.

Theorem 5 (Theorem 5.9 [11]). *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. If $|V| \geq 7$, then there exist $u, v \in V$ such that $u \neq v$ and $g - \{u, v\}$ is primitive.*

Theorem 6 (Theorem 1 [9]). *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. Consider $X \subseteq V$ such that $g[X]$ is primitive. If $|V \setminus X| \geq 6$, then there exist $u, v \in V \setminus X$ such that $u \neq v$ and $g - \{u, v\}$ is primitive.*

Let $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ be a switching class. It follows from Corollary 4 that

$$\{\emptyset, V\} \cup \{\{x\}, V \setminus \{x\} : x \in V\} \subseteq \mathrm{Cl}(\mathfrak{S}).$$

A clan C of $\mathfrak{S}$ is *trivial* if $\min(|C|, |V \setminus C|) \leq 1$. We say that $\mathfrak{S}$ is *indecomposable* if all the clans of $\mathfrak{S}$ are trivial, otherwise $\mathfrak{S}$ is said to be *decomposable*. Furthermore, $\mathfrak{S}$ is *primitive* if $\mathfrak{S}$ is indecomposable and $|V| \geq 4$.

Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. An element v of V is a *critical vertex* of g if $g - v$ is decomposable. We say that g is *critical* if all its vertices are critical. Analogously, given a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, an element v of V is a *critical vertex* of $\mathfrak{S}$ if $\mathfrak{S} - v$ is decomposable. We say that $\mathfrak{S}$ is *critical* if all its vertices are critical.

4. Main Results

In this section we list the main results, leaving the proofs of those which are less immediate to subsequent sections. The following result is basic in the study of primitive switching classes. It is proved in Section 9.

Proposition 7. *For a switching class $\mathfrak{S}$, the following assertions are equivalent*

1. *$\mathfrak{S}$ is primitive,*
2. *there exists $g \in \mathfrak{S}$ which admits an isolated vertex x such that $g - x$ is primitive,*
3. *for each $g \in \mathfrak{S}$ which admits an isolated vertex x, $g - x$ is primitive.*

Let $\mathfrak{S}$ be a switching class. To study the primitivity of $\mathfrak{S}$ by using Proposition 7, we have to examine the primitivity of $g - x$, where $g \in \mathfrak{S}$ and x is an isolated vertex of g. Given $x \in V$, consider $g, h \in \mathfrak{S}$ such that x is an isolated vertex of g and h. There exists $s \in \mathscr{S}(V, \mathbb{A})$ such that $h = g^s$. For any $y, z \in V \setminus \{x\}$, we have $g(y,x) = g(z,x)$ and $h(y,x) = h(z,x)$. Since $h(y,x) = s(y) + g(y,x) - s(x)$ and $h(z,x) = s(z) + g(z,x) - s(x)$, we obtain $s(y) = s(z)$. It follows that $g - x = h - x$. For convenience, given $g \in \mathfrak{S}$ and an isolated vertex x of g, we denote $g - x$ by $\mathfrak{S}_x$.

Proposition 7 allows us to translate the most important results on primitivity of $\mathbb{A}$-labeled and reversible 2-structures in terms of switching classes. For instance, we obtain the following analogues of Theorems 5 and 6, which are proved in Section 9.

Theorem 8. *Consider a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$. If $|V| \geq 8$, then there exist $u, v \in V$ such that $u \neq v$ and $\mathfrak{S} - \{u, v\}$ is primitive.*

Theorem 9. *Consider a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$. Let $X \subseteq V$ such that $\mathfrak{S}[X]$ is primitive. If $|V \setminus X| \geq 6$, then there exist $u, v \in V \setminus X$ such that $u \neq v$ and $\mathfrak{S} - \{u, v\}$ is primitive.*

Now, concerning the critical switching classes, we obtain the following result that is an immediate consequence of Lemmas 58 and 52. This result is interesting because it is clearly false for critical and $\mathbb{A}$-labeled, reversible 2-structures.

Theorem 10. *Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ such that $|V| \geq 6$, $\mathfrak{S}$ is critical if and only if there exist distinct $x, y \in V$ such that $\mathfrak{S}_x$ and $\mathfrak{S}_y$ are critical.*

Theorem 5 led Ille [8] to introduce the following graph. With a primitive and $\mathbb{A}$-labeled, reversible 2-structure g, associate its *primitivity graph* $\Pi(g)$ defined on V as follows. Given distinct $u, v \in V$,

$$uv \in E(\Pi(g)) \text{ if } g - \{u, v\} \text{ is primitive.}$$

In terms of primitive switching classes, Theorem 8 leads us to a similar graph. With a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, we associate its *primitivity graph* $\Pi(\mathfrak{S})$ defined on V as follows. Given distinct $u, v \in V$,

$$uv \in E(\Pi(\mathfrak{S})) \text{ if } \mathfrak{S} - \{u, v\} \text{ is primitive.}$$

After studying the structural properties of the primitivity graph of a primitive switching class at the beginning of Section 10, we obtain the following corollary. We use the following graphs. For $n \geq 2$, P_n denotes the path $(\{0,\dots,n-1\},\{ij : |i-j| = 1\})$, and for $n \geq 3$, C_n denotes the cycle $(\{0,\dots,n-1\}, E(P_n)\cup\{0(n-1)\})$.

Corollary 11. *Given a critical switching class $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$ such that $|V| \geq 6$, one of the following holds*

- *$\Pi(\mathfrak{S})$ is isomorphic to $C_{|V|}$,*
- *$\Pi(\mathfrak{S})$ admits a unique isolated vertex x, $|V|$ is even and $\Pi(\mathfrak{S})-x$ is isomorphic to $C_{|V|-1}$.*

Lastly, we characterize the critical switching classes following their primitivity graphs. By Corollary 11, the primitivity graph of a critical switching class is an even cycle, an odd cycle or an odd cycle and an isolated vertex. To describe each of the three families, we use some of the critical digraphs obtained in [2]. We also need the following definition and notation. An *oriented graph* O is a digraph such that for distinct $u, v \in V(D)$, we cannot have both $uv \in A(O)$ and $vu \in A(O)$. Given an oriented graph O and $a \in \mathbb{A}$, O^a denotes the $\mathbb{A}$-labeled and reversible 2-structure defined on $V = V(O)$ by

$$\text{for distinct } x, y \text{ in } V, \quad O^a(x,y) = \begin{cases} a & \text{if } xy \in A(O), \\ -a & \text{if } yx \in A(O), \\ 0 & \text{if } xy \notin A(O) \text{ and } yx \notin A(O). \end{cases}$$

First, we consider the critical switching classes whose primitivity graph is an even cycle. We need the following digraphs. For $n \geq 2$, L_n denotes the usual linear order on $\{0,\dots,n-1\}$. For $n \geq 1$, O_{2n+1} denotes the partial order $(\{0,\dots,2n\},\{(2i)(2j) : 0 \leq i < j \leq n\})$. Furthermore, set

$$H_{2n+1} = (\{0,\dots,2n\}, A(L_{2n+1}) \setminus A(O_{2n+1})) \text{ (see Figure 1).}$$

The digraph H_{2n+1} is a critical digraph obtained in [2].

Theorem 12. *Let $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$ be a switching class and let $n \geq 3$. The following three assertions are equivalent*

- *$\mathfrak{S}$ is critical and $\Pi(\mathfrak{S}) \simeq C_{2n}$,*
- *there exists $x \in V$ such that $\mathfrak{S}_x \simeq (O_{2n-1})^a + (H_{2n-1})^b$, where $a, b \in \mathbb{A}$ satisfy $b + b \neq 0$, $a \neq b$ and $a + a = b + b$,*
- *there exist $a, b \in \mathbb{A}$ satisfying $b + b \neq 0$, $a \neq b$, $a + a = b + b$ and such that for every $x \in V$, $\mathfrak{S}_x \simeq (O_{2n-1})^a + (H_{2n-1})^b$ or $\mathfrak{S}_x \simeq (O_{2n-1})^b + (H_{2n-1})^a$.*

Second, we consider the critical switching classes whose primitivity graph is an odd cycle. We need the following partial orders. For $n \geq 1$, Q_{2n} denotes the

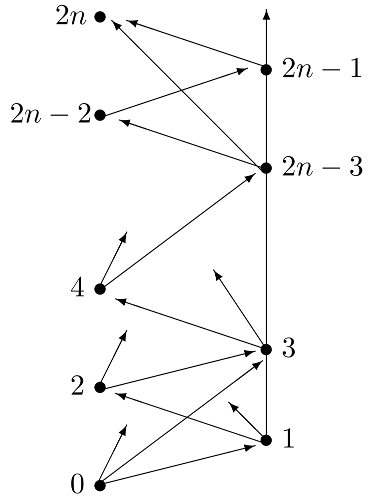

Figure 1. The digraph H_{2n+1}.

partial order $(\{0,\ldots,2n-1\},\{(2i)(2j+1) : 0 \leq i \leq j \leq n-1\})$ (see Figure 2), and R_{2n} denotes the partial order $(\{0,\ldots,2n-1\}, A(L_{2n}) \setminus A(Q_{2n}))$ (see Figure 3). The partial orders Q_{2n} and R_{2n} are critical digraphs obtained in [2].

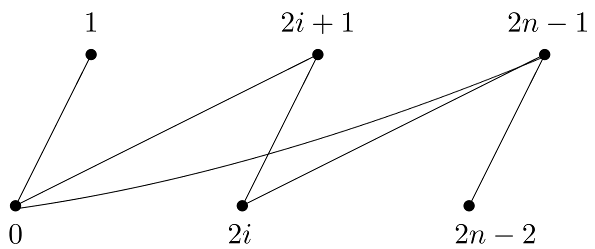

Figure 2. The Hasse diagram of Q_{2n}.

Theorem 13. *Let $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$ be a switching class and let $n \geq 3$. The following three assertions are equivalent*

- *$\mathfrak{S}$ is critical and $\Pi(\mathfrak{S}) \simeq C_{2n+1}$,*
- *there exists $x \in V$ such that $\mathfrak{S}_x \simeq (Q_{2n})^a + (R_{2n})^b$, where $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a,-a\}$,*
- *there exist $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a,-a\}$ such that for every $x \in V$, we have $\mathfrak{S}_x \simeq (Q_{2n})^a + (R_{2n})^b$.*

Third, we consider the critical switching classes whose primitivity graph consists of an odd cycle and an isolated vertex. We need the following tournaments.

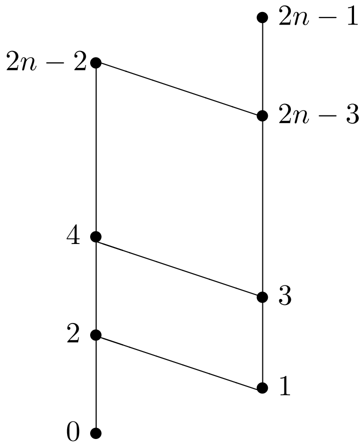

Figure 3. The Hasse diagram of R_{2n}.

For $n \geq 1$, T_{2n+1} denotes the tournament obtained from L_{2n+1} by reversing all the arcs between even and odd vertices (see Figure 4). For $n \geq 1$, W_{2n+1} denotes the tournament obtained from L_{2n+1} by reversing all the arcs between $2n$ and the even elements of $\{0, \ldots, 2n-1\}$ (see Figure 5). The tournaments T_{2n+1} and W_{2n+1} are critical digraphs obtained in [2].

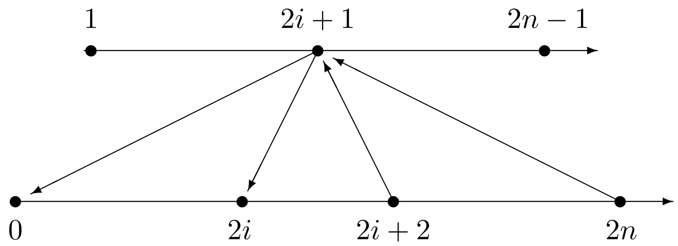

Figure 4. The tournament T_{2n+1}.

Theorem 14. *Let $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ be a switching class and let $n \geq 2$. Given $x \in V$, the following three assertions are equivalent*

1. *$\mathfrak{S}$ is critical, x is an isolated vertex of $\Pi(\mathfrak{S})$ and $\Pi(\mathfrak{S}) - x \simeq C_{2n+1}$,*
2. *there exists $a \in \mathbb{A}$, with $o_{\mathbb{A}}(a) = 4$, such that $\mathfrak{S}_x \simeq (T_{2n+1})^a$,*
3. *there exists $b \in \mathbb{A}$, with $o_{\mathbb{A}}(b) = 4$, such that for every $y \in V \setminus \{x\}$, there exists an isomorphism from $\mathfrak{S}_y$ onto $(W_{2n+1})^b$ that maps x on $2n$.*

Furthermore, if the last two assertions hold, then $a = b$.

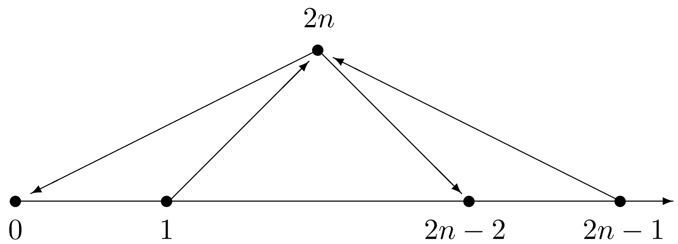

Figure 5. The tournament W_{2n+1}.

By using Corollary 11, we summarize Theorems 12, 13 and 14 as follows.

Theorem 15. *Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, where $|V| \geq 6$, $\mathfrak{S}$ is critical if and only if there exists $x \in V$ satisfying one of the following assertions for some $n \geq 3$*

- $\mathfrak{S}_x \simeq (O_{2n-1})^a + (H_{2n-1})^b$, *where $a, b \in \mathbb{A}$ such that $b + b \neq 0$, $a \neq b$, $a + a = b + b$,*
- $\mathfrak{S}_x \simeq (Q_{2n})^a + (R_{2n})^b$, *where $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a, -a\}$,*
- $\mathfrak{S}_x \simeq (T_{2n-1})^a$, *where $a \in \mathbb{A}$ such that $o_{\mathbb{A}}(a) = 4$.*

5. Graphs, Digraphs and 2-Structures

5.1. Switches

In this subsection, we compare the classic switch of graphs or tournaments with the switch of $\mathbb{A}$-labeled and reversible 2-structures. We use the following notion of a reversible 2-structure that is more general than the notion of an $\mathbb{A}$-labeled and reversible 2-structure. A 2-*structure* σ consists of a vertex set $V(\sigma)$ and an equivalence relation $\equiv_\sigma$ defined on $(V(\sigma) \times V(\sigma)) \setminus \{(v, v) : v \in V(\sigma)\}$, cf. [5]. A 2-structure σ is *reversible* if for any $u, v, x, y \in V(\sigma)$ such that $u \neq v$ and $x \neq y$, we have

$$(u, v) \equiv_\sigma (x, y) \implies (v, u) \equiv_\sigma (y, x).$$

The family of the equivalence classes of $\equiv_\sigma$ is denoted by $E(\sigma)$. Let σ be a reversible 2-structure. For each $e \in E(\sigma)$, $e^\star \in E(\sigma)$, where $e^\star = \{(u, v) : (v, u) \in e\}$, and we have either $e = e^\star$ or $e \cap e^\star = \emptyset$. In the first instance, e is said to be *symmetric*. It is called *asymmetric* in the second. The family of the asymmetric classes of $\equiv_\sigma$ is denoted by $E_a(\sigma)$ and that of the symmetric ones by $E_s(\sigma)$. A reversible 2-structure σ is *symmetric* when $E(\sigma) = E_s(\sigma)$, and it is *asymmetric* when $E(\sigma) = E_a(\sigma)$.

With each graph G, associate the 2-structure $\sigma(G)$ defined on $V(G)$ as follows. Given any $u, v, x, y \in V(\sigma)$ such that $u \neq v$ and $x \neq y$,

$$(u,v) \equiv_{\sigma(G)} (x,y) \text{ if } \begin{cases} uv, xy \in E(G), \text{ or} \\ uv, xy \notin E(G). \end{cases}$$

Clearly, $\sigma(G)$ is symmetric and $|E(\sigma(G))| = 1$ or 2. The *complement* of G is the graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. Note that $\sigma(G) = \sigma(\overline{G})$. In an analogous way, associate with each digraph D the 2-structure $\sigma(D)$ defined on $V(D)$ as follows. Given any $u, v, x, y \in V(\sigma)$ such that $u \neq v$ and $x \neq y$, $(u,v) \equiv_{\sigma(D)} (x,y)$ if the following two equivalences hold

- $uv \in A(D)$ if and only if $xy \in A(D)$,
- $vu \in A(D)$ if and only if $yx \in A(D)$.

Clearly, $\sigma(D)$ is reversible and $1 \leq |E(\sigma(D))| \leq 4$. The *complement* of D is the digraph $\overline{D}$ defined by $V(\overline{D}) = V(D)$ and $A(\overline{D}) = \{uv : uv \notin A(D)\}$. Furthermore, the *dual* of D is the digraph $D^\star$ defined by $V(D^\star) = V(D)$ and $A(D^\star) = A(D)^\star$. Note that $\sigma(D) = \sigma(\overline{D}) = \sigma(D^\star)$.

Let σ be a reversible 2-structure. We can associate with σ an element $\sigma_{\mathbb{A}}$ of $\mathscr{L}(V(\sigma), \mathbb{A})$ under certain assumptions on $(\mathbb{A}, +)$. Indeed, for any $u, v, x, y \in V(\sigma)$ such that $u \neq v$ and $x \neq y$, we must have

$$(u,v) \not\equiv_\sigma (x,y) \iff \sigma_{\mathbb{A}}(u,v) \neq \sigma_{\mathbb{A}}(x,y).$$

Therefore, we have to suppose that

$$(2) \qquad \begin{cases} |E_a(\sigma)| \leq |\mathbb{A}| - |\mathbb{A}_{\leq 2}|, \text{ where } \mathbb{A}_{\leq 2} = \{\alpha \in \mathbb{A} : o_{\mathbb{A}}(\alpha) \leq 2\}, \text{ and} \\ |E_s(\sigma)| \leq |\mathbb{A}_{\leq 2}|. \end{cases}$$

Such an abelian group $\mathbb{A}$ always exists. Clearly, (2) holds if and only if there exists an injection $\varphi : E(\sigma) \longrightarrow \mathbb{A}$ satisfying

$$(3) \qquad \text{for each } e \in E(\sigma),\ \varphi(e^\star) = -\varphi(e).$$

When (3) holds, we associate with σ the element $\sigma_{\mathbb{A}}$ of $\mathscr{L}(V(\sigma), \mathbb{A})$ defined as follows. For distinct $u, v \in V(\sigma)$, let

$$\sigma_{\mathbb{A}}(u,v) = \varphi((u,v)_\sigma),$$

where $(u,v)_\sigma$ denotes the equivalence class of (u,v) modulo $\equiv_\sigma$. Given $g \in \mathscr{L}(V, \mathbb{A})$, we verify that there exists a reversible 2-structure $\sigma(g)$ defined on $V(\sigma(g)) = V$ such that

$$(4) \qquad \sigma(g)_{\mathbb{A}} = g.$$

Indeed, consider the reversible 2-structure $\sigma(g)$ defined on $V(\sigma(g)) = V$ as follows. For $u, v, x, y \in V$ such that $u \neq v$ and $x \neq y$, let

$$(u, v) \equiv_{\sigma(g)} (x, y) \text{ if } g(u, v) = g(x, y).$$

We also consider the injection

$$\begin{array}{rccl} \varphi: & E(\sigma(g)) & \longrightarrow & \mathbb{A} \\ & e & \longmapsto & g(u, v), \text{ where } (u, v) \in e. \end{array}$$

Since g is reversible, φ satisfies (3). For distinct $u, v \in V$,

$$\sigma(g)_{\mathbb{A}}(u, v) = \varphi((u, v)_{\sigma(g)}) = g(u, v).$$

Thus (4) holds.

Consider a graph G which is neither complete nor empty. We obtain that $\sigma(G)$ is symmetric and $|E(\sigma(G))| = 2$. Thus, (2) holds if and only if there exists $a \in \mathbb{A}$ such that $o_{\mathbb{A}}(a) = 2$. It follows that we can choose $\mathbb{Z}_2$ for $\mathbb{A}$. Moreover, we can choose φ in such a way that $\sigma(G)_{\mathbb{Z}_2}$ is defined as follows. For distinct $u, v \in V(G)$, let

$$\sigma(G)_{\mathbb{Z}_2}(u, v) = \begin{cases} 1 \text{ if } uv \in E(G), \\ 0 \text{ if } uv \notin E(G). \end{cases} \tag{5}$$

Now, consider a digraph D and suppose that there exist $u, v, w, x, y, z \in V(\sigma)$ satisfying $u \neq v$, $w \neq x$, $y \neq z$ and such that $uv, vu, wx \in A(D)$ and $xw, yz, zy \notin A(D)$. We obtain that $\sigma(D)$ is reversible, $|E_a(\sigma(D))| = 2$ and $|E_s(\sigma(D))| = 2$. Therefore, (2) holds if and only if $|\mathbb{A}| - |\mathbb{A}_{\leq 2}| \geq 2$ and $|\mathbb{A}_{\leq 2}| \geq 2$. It follows that we can choose $\mathbb{Z}_4$ for $\mathbb{A}$. Lastly, consider a tournament T. We obtain that $\sigma(T)$ is asymmetric and $|E(\sigma(T))| = 2$. Hence, (2) holds if and only if $|\mathbb{A}| - |\mathbb{A}_{\leq 2}| \geq 2$. It follows that we can choose $\mathbb{Z}_3$ for $\mathbb{A}$. Moreover, we can choose φ in such a way that $\sigma(T)_{\mathbb{Z}_3}$ is defined as follows. For distinct $u, v \in V(T)$, let

$$\sigma(T)_{\mathbb{Z}_3}(u, v) = \begin{cases} 1 \text{ if } uv \in A(T), \\ 2 \text{ if } uv \notin A(T). \end{cases} \tag{6}$$

For what follows, note that we can also choose $\mathbb{Z}_4$ for $\mathbb{A}$. In this case, we can choose φ in such a way that $\sigma(T)_{\mathbb{Z}_4}$ is defined as follows. For distinct $u, v \in V(T)$, let

$$\sigma(T)_{\mathbb{Z}_4}(u, v) = \begin{cases} 1 \text{ if } uv \in A(T), \\ 3 \text{ if } uv \notin A(T). \end{cases} \tag{7}$$

To complete this subsection, we compare the classic switch of graphs or tournaments with the switch of the associated $\mathbb{A}$-labeled and reversible 2-structures. Let G be a graph. Given $X \subseteq V(G)$, we consider the selector

$$\begin{array}{cccc} \mathbb{1}_X : & V(G) & \longrightarrow & \mathbb{Z}_2 \\ & v \in X & \longmapsto & 1 \\ & v \in V \setminus X & \longmapsto & 0. \end{array}$$

If $\sigma(G)_{\mathbb{Z}_2}$ and $(\sigma(G^X))_{\mathbb{Z}_2}$ are defined as in (5), then

$$(\sigma(G)_{\mathbb{Z}_2})^{\mathbb{1}_X} = (\sigma(G^X))_{\mathbb{Z}_2}.$$

Now, consider a tournament T. On the one hand, suppose that there exists $X \subseteq V(T)$ with $x, y \in X$ and $u, v \in V(T) \setminus X$ satisfying $xu, vy \in A(T)$. Suppose also that $\sigma(T)_{\mathbb{Z}_3}$ and $(\sigma(T^X))_{\mathbb{Z}_3}$ are defined as in (6). We verify that, whatever the selector $s : V(T) \longrightarrow \mathbb{Z}_3$ is, we obtain

$$(\sigma(T)_{\mathbb{Z}_3})^s \neq (\sigma(T^X))_{\mathbb{Z}_3}.$$

On the other hand, consider any $X \subseteq V(T)$, and suppose that $\sigma(T)_{\mathbb{Z}_4}$ and $(\sigma(T^X))_{\mathbb{Z}_4}$ are defined as in (7). By considering the selector

$$\begin{array}{cccc} \mathbb{1}'_X : & V(G) & \longrightarrow & \mathbb{Z}_4 \\ & v \in X & \longmapsto & 1 \\ & v \in V \setminus X & \longmapsto & 3, \end{array}$$

we obtain

$$(\sigma(T)_{\mathbb{Z}_4})^{\mathbb{1}'_X} = (\sigma(T^X))_{\mathbb{Z}_4}.$$

5.2. Primality and primitivity

In this subsection, we introduce the notion of clan, primitivity and criticality for reversible 2-structures. Afterwards, we compare each of these notions with its analogue for $\mathbb{A}$-labeled and reversible 2-structures. We conclude by considering the case of graphs and digraphs.

Let σ be a reversible 2-structure. A subset X of $V(\sigma)$ is a *clan* of σ if for any $x, y \in X$ and $v \in V \setminus X$, we have $(x, v) \equiv_\sigma (y, v)$, cf. [5]. As for $\mathbb{A}$-labeled and reversible 2-structures, the notions of primitivity and criticality for reversible 2-structures follow from the notion of clan. The primitivity graph associated with a primitive and reversible 2-structure is defined in the same way. Given a reversible 2-structure σ, to compare the clans of σ with those of an associated element of $\mathscr{L}(V(\sigma), \mathbb{A})$, consider an abelian group $\mathbb{A}$ such that (2) holds. The associated element $\sigma_{\mathbb{A}}$ of $\mathscr{L}(V(\sigma), \mathbb{A})$ is defined from an injection φ satisfying (3). Clearly, σ and $\sigma_{\mathbb{A}}$ share the same clans. It follows that σ is primitive if and only if $\sigma_{\mathbb{A}}$ is.

Thus, σ is critical if and only if $\sigma_{\mathbb{A}}$ is. In terms of primitivity graph, we obtain $\Pi(\sigma) = \Pi(\sigma_{\mathbb{A}})$.

Now, we introduce the classic notion of a module of a graph. Let G be a graph. A subset X of $V(G)$ is a *module* of G if for any $x, y \in X$ and $v \in V \setminus X$, $xv \in E(G)$ if and only if $yv \in E(G)$. Clearly, the set of the modules of G coincides with the set of the clans of $\sigma(G)$. Furthermore, a graph G is said to be *prime* if $|V(G)| \geq 4$, and $\emptyset$, $V(G)$ and $\{v\}$ $(v \in V(G))$ are its only modules. Hence, a graph G is prime if and only if $\sigma(G)$ is primitive. The same holds for criticality. By denoting by $\Pi(G)$ the primality graph of G, we obtain $\Pi(G) = \Pi(\sigma(G))$.

Lastly, we consider digraph D. A subset X of $V(D)$ is a *module* of D if for any $x, y \in X$ and $v \in V \setminus X$, we have

$$\begin{cases} xv \in A(D) \iff yv \in A(D), \text{ and} \\ vx \in A(D) \iff vy \in A(D). \end{cases}$$

The set of the modules of D coincides with the set of the clans of $\sigma(D)$. A digraph D is said to be *prime* if $|V(D)| \geq 3$, and $\emptyset$, $V(D)$ and $\{v\}$ $(v \in V(D))$ are its only modules. We end with the same observations as above for graphs.

6. Isomorphy Between Switching Classes

Let V and V' be vertex sets, and let $\mathbb{A}$ be an abelian group. Consider $g \in \mathscr{L}(V, \mathbb{A})$ and $h \in \mathscr{L}(V', \mathbb{A})$. A bijection β from V onto V' is an *isomorphism* from g onto h if for distinct $u, v \in V$, we have $g(u, v) = h(\beta(u), \beta(v))$. For convenience, given $g \in \mathscr{L}(V, \mathbb{A})$, $\beta(g)$ denotes the unique element of $\mathscr{L}(V', \mathbb{A})$ such that β is an isomorphism from g onto $\beta(g)$. Thereby, β may be identified with the function

$$\begin{array}{ccc} \mathscr{L}(V, \mathbb{A}) & \longrightarrow & \mathscr{L}(V', \mathbb{A}) \\ g & \longmapsto & \beta(g) \end{array}$$

that is a bijection from $\mathscr{L}(V, \mathbb{A})$ onto $\mathscr{L}(V', \mathbb{A})$.

Proposition 16. *Given $\mathscr{L}(V, \mathbb{A})$ and $\mathscr{L}(V', \mathbb{A})$, consider a bijection β from V onto V'. For every $g \in \mathscr{L}(V, \mathbb{A})$, we have*

$$\beta(\langle g \rangle) = \langle \beta(g) \rangle.$$

Proof. Given $s \in \mathscr{S}(V, \mathbb{A})$, we have $\beta(g^s) = (\beta(g))^{(s \circ \beta^{-1})}$. It follows that

$$\beta(\langle g \rangle) \subseteq \langle \beta(g) \rangle. \tag{8}$$

By exchanging V and V', β and β^{-1}, and g and $\beta(g)$, in (8), we obtain

$$\beta^{-1}(\langle \beta(g) \rangle) \subseteq \langle \beta^{-1}(\beta(g)) \rangle.$$

Thus $\beta^{-1}(\langle\beta(g)\rangle) \subseteq \langle g\rangle$ and hence $\langle\beta(g)\rangle \subseteq \beta(\langle g\rangle)$. It follows from (8) that

$$\beta(\langle g\rangle) = \langle\beta(g)\rangle.$$

■

Given switching classes $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ and $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$, a bijection β from V onto V' is an *isomorphism* from $\mathfrak{S}$ onto $\mathfrak{S}'$ if

$$\beta(\mathfrak{S}) = \mathfrak{S}'.$$

Remark 17. Given switching classes $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ and $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$, consider an isomorphism β from $\mathfrak{S}$ onto $\mathfrak{S}'$. For each $X \subseteq V$, $\beta_{\restriction X}$ is an isomorphism from $\mathfrak{S}[X]$ onto $\mathfrak{S}'[\beta(X)]$.

Theorem 18. *Consider switching classes $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ and $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$.*

- *Given a bijection β from V onto V', β is an isomorphism from $\mathfrak{S}$ onto $\mathfrak{S}'$ if and only if there exists $g \in \mathfrak{S}$ such that β is an isomorphism from g onto an element of $\mathfrak{S}'$.*
- *$\mathfrak{S}$ and $\mathfrak{S}'$ are isomorphic if and only if there exist $g \in \mathfrak{S}$ and $g' \in \mathfrak{S}'$ such that g and g' are isomorphic.*

Proof. The second assertion follows from the first one. For the first assertion, consider a bijection β from V onto V'. To begin, suppose that β is an isomorphism from $\mathfrak{S}$ onto $\mathfrak{S}'$, that is, $\beta(\mathfrak{S}) = \mathfrak{S}'$. Given $g \in \mathfrak{S}$, we obtain $\beta(g) \in \mathfrak{S}'$. Clearly, β is an isomorphism from g onto $\beta(g)$.

Conversely, suppose that β is an isomorphism from an element g of $\mathfrak{S}$ onto an element g' of $\mathfrak{S}'$. Necessarily $g' = \beta(g)$. It follows from Proposition 16 that

$$\beta(\mathfrak{S}) = \beta(\langle g\rangle) = \langle\beta(g)\rangle = \mathfrak{S}'.$$

■

7. BACKGROUND: CRITICALITY AND PRIMITIVITY OF $\mathbb{A}$-LABELED AND REVERSIBLE 2-STRUCTURES

7.1. Primitivity

In the sequel of this subsection, we list the main results on primitivity of $\mathbb{A}$-labeled and reversible 2-structures, other than Theorems 5 and 6. We begin with the existence of small primitive substructures.

Lemma 19 (Theorem 6.1 [5])**.** *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. There exists $X \subseteq V$ such that $|X| = 3$ or 4, and $g[X]$ is primitive.*

Lemma 19 was strengthened as follows.

Proposition 20 (Theorem 4 [4]). *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. For each $x \in V$, there exists $X \subseteq V$ such that $x \in X$, $|X| = 4$ or 5, and $g[X]$ is primitive.*

Given $g \in \mathscr{L}(V, \mathbb{A})$, consider $X \subseteq V$ such that $g[X]$ is primitive. Suppose that $|V \setminus X| \geq 2$. When g is primitive, to attempt to construct $Y \subseteq V$ such that $X \subsetneq Y$ and $g[Y]$ is primitive, we introduce the following subsets of $V \setminus X$:

- $\text{Ext}^g(X)$ is the set of $v \in V \setminus X$ such that $g[X \cup \{v\}]$ is primitive,
- $\widehat{X}^g$ is the set of $v \in V \setminus X$ such that X is a clan of $g[X \cup \{v\}]$,
- for $y \in X$, $X^g(y)$ is the set of $v \in V \setminus X$ such that $\{y, v\}$ is a clan of $g[X \cup \{v\}]$.

We consider the partition

$$p_{(g,X)} = \{\text{Ext}^g(X), \widehat{X}^g\} \cup \{X^g(y) : y \in X\} \tag{9}$$

of $V \setminus X$. The next result follows.

Proposition 21 (Theorem 6.5 [5]). *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. Consider $X \subseteq V$ such that $g[X]$ is primitive. If $|V \setminus X| \geq 2$, then there exist distinct $u, v \in V \setminus X$ such that $g[X \cup \{u, v\}]$ is primitive.*

An immediate consequence of Lemma 19 and Proposition 21 follows.

Corollary 22 (Theorem 6.4 [5]). *Let $g \in \mathscr{L}(V, \mathbb{A})$ be primitive. If $|V| \geq 5$, then there exist $u, v \in V$ such that $g - \{u, v\}$ is primitive.*

Hence, Theorem 5 strengthens Corollary 22.

7.2. Criticality

The main properties of the primitivity graph are described in the next three results. Given a primitive element g of $\mathscr{L}(V, \mathbb{A})$, the set of critical vertices of g is denoted by $\mathcal{C}(g)$.

Lemma 23 (Lemma 10 [2]). *Consider a primitive element g of $\mathscr{L}(V, \mathbb{A})$, where $|V| \geq 5$. For every $v \in \mathcal{C}(g)$, we have $d_{\Pi(g)}(v) \leq 2$. Moreover, for each $v \in \mathcal{C}(g)$, we have*

- *if $d_{\Pi(g)}(v) = 1$, then $V \setminus (N_{\Pi(g)}(v) \cup \{v\}) \in \text{Cl}(g - v)$,*
- *if $d_{\Pi(g)}(v) = 2$, then $N_{\Pi(g)}(v) \in \text{Cl}(g - v)$.*

The next result follows from Lemma 23.

Proposition 24 (Proposition 11 [2]). *Consider a primitive element g of $\mathscr{L}(V, \mathbb{A})$, where $|V| \geq 5$. If there exists a component C of $\Pi(g)$ such that $|V(C)| \geq 2$ and $V(C) \subseteq \mathcal{C}(g)$, then one of the following holds*

- *$\Pi(g)$ is isomorphic to C_{2n+1}, where $n \geq 2$,*

- *$\Pi(g)$ is isomorphic to P_{2n+1}, where $n \geq 2$,*
- *C is isomorphic to P_{2n}, where $n \geq 2$, and $|V \setminus V(C)| \leq 1$.*

The following result is an immediate consequence of Corollary 22 and Proposition 24.

Corollary 25. *Given a critical element g of $\mathscr{L}(V, \mathbb{A})$, where $|V| \geq 5$, one of the following holds*

- *$\Pi(g)$ is isomorphic to C_{2n+1}, where $n \geq 2$,*
- *$\Pi(g)$ is isomorphic to P_n, where $n \geq 5$,*
- *$\Pi(g)$ admits a unique isolated vertex x, and $\Pi(g) - x$ is isomorphic to P_{2n}, where $n \geq 2$.*

To complete this subsection, we provide a characterization of critical and $\mathbb{A}$-labeled, reversible 2-structures by using Corollary 25. An analogous characterization is given in [2] for critical digraphs. Since both approaches are similar, we omit the proofs for critical and $\mathbb{A}$-labeled, reversible 2-structures. We also use the characterization of critical and reversible 2-structures due to [1].

For $n \geq 1$, U_{2n+1} denotes the tournament obtained from L_{2n+1} by reversing all the arcs between even vertices (see Figure 6). Thus $A(U_{2n+1}) = A(H_{2n+1}) \cup A(O_{2n+1})^\star$.

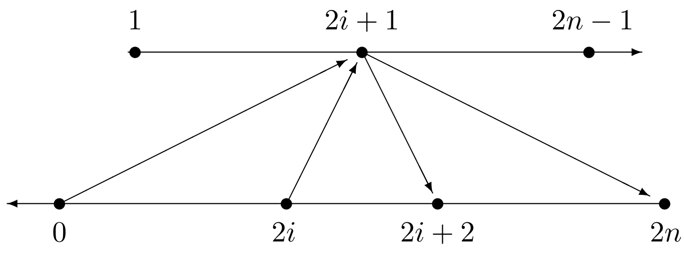

Figure 6. The tournament U_{2n+1}.

Boudabbous and Ille [2] obtained the following characterization of critical digraphs whose primitivity graph is a path of odd size.

Theorem 26 (Proposition 19 [2]). *Given a digraph D defined on $V(D) = \{0, \ldots, 2n\}$, where $n \geq 2$, the following two assertions are equivalent*

- *D is critical and $\Pi(D) = P_{2n+1}$,*
- *one of the digraphs D, $D^\star$, $\overline{D}$ or $\overline{D^\star}$ equals H_{2n+1} or U_{2n+1}.*

We have

$$\begin{cases} E(\sigma(H_{2n+1})) = \{A(H_{2n+1}), A(H_{2n+1})^\star, A(O_{2n+1}) \cup A(O_{2n+1})^\star\}, \text{ and} \\ E(\sigma(U_{2n+1})) = \{A(H_{2n+1}) \cup A(O_{2n+1})^\star, A(H_{2n+1})^\star \cup A(O_{2n+1})\}. \end{cases}$$

Let D be a digraph. As observed in Subsection 5.2, D is critical if and only if $\sigma(D)$ is. Moreover, we have $\Pi(D) = \Pi(\sigma(D))$. Therefore, we obtain the following corollary.

Corollary 27. *Given a reversible* 2*-structure* σ *defined on* $V(\sigma) = \{0, \ldots, 2n\}$*, where* $n \geq 2$*, if*

$$E(\sigma) = \begin{cases} \{A(H_{2n+1}) \cup A(O_{2n+1})^{\star}, A(H_{2n+1})^{\star} \cup A(O_{2n+1})\}, & or \\ \{A(H_{2n+1}), A(H_{2n+1})^{\star}, A(O_{2n+1}) \cup A(O_{2n+1})^{\star}\}, \end{cases}$$

then σ *is critical and* $\Pi(\sigma) = P_{2n+1}$.

The next result follows from [1, Proposition 6.3].

Theorem 28. *Given a reversible* 2*-structure* σ *defined on* $V(\sigma) = \{0, \ldots, 2n\}$*, where* $n \geq 2$*,* σ *is critical and* $\Pi(\sigma) = P_{2n+1}$ *if and only if*

$$E(\sigma) = \begin{cases} \{A(H_{2n+1}) \cup A(O_{2n+1})^{\star}, A(H_{2n+1})^{\star} \cup A(O_{2n+1})\}, \\ \{A(H_{2n+1}), A(H_{2n+1})^{\star}, A(O_{2n+1}) \cup A(O_{2n+1})^{\star}\}, & or \\ \{A(H_{2n+1}), A(H_{2n+1})^{\star}, A(O_{2n+1}), A(O_{2n+1})^{\star}\}. \end{cases}$$

Given a reversible 2-structure σ defined on $V(\sigma) = \{0, \ldots, 2n\}$, where $n \geq 2$, we have

- $E(\sigma) = \{A(H_{2n+1}) \cup A(O_{2n+1})^{\star}, A(H_{2n+1})^{\star} \cup A(O_{2n+1})\}$ if and only if $\sigma_{\mathbb{A}} = (O_{2n+1})^{-b} + (H_{2n+1})^{b}$, where $b \in \mathbb{A}$ such that $b + b \neq 0$,
- $E(\sigma) = \{A(H_{2n+1}), A(H_{2n+1})^{\star}, A(O_{2n+1}) \cup A(O_{2n+1})^{\star}\}$ if and only if $\sigma_{\mathbb{A}} = (O_{2n+1})^{a} + (H_{2n+1})^{b}$, where $a, b \in \mathbb{A}$ such that $a + a = 0$ and $b + b \neq 0$,
- $E(\sigma) = \{A(H_{2n+1}), A(H_{2n+1})^{\star}, A(O_{2n+1}), A(O_{2n+1})^{\star}\}$ if and only if $\sigma_{\mathbb{A}} = (O_{2n+1})^{a} + (H_{2n+1})^{b}$, where $a, b \in \mathbb{A}$ such that $a + a \neq 0$, $b + b \neq 0$ and $a \neq b$.

Let $g \in \mathscr{L}(\{0, \ldots, 2n\}, \mathbb{A})$. By (4), there exists a reversible 2-structure $\sigma(g)$ such that $\sigma(g)_{\mathbb{A}} = g$. As observed in Subsection 5.2, g is critical if and only if $\sigma(g)$ is. Moreover, we have $\Pi(g) = \Pi(\sigma(g))$. Therefore, the next theorem follows.

Theorem 29. *Given* $g \in \mathscr{L}(\{0, \ldots, 2n\}, \mathbb{A})$ *such that* $n \geq 2$*, the following two assertions are equivalent*

- g *is critical and* $\Pi(g) = P_{2n+1}$*,*
- $g = (O_{2n+1})^{a} + (H_{2n+1})^{b}$*, where* $a, b \in \mathbb{A}$ *such that* $a \neq b$ *and* $b + b \neq 0$.

For a partial order P, $\text{comp}(P) = (V(P), A(P) \cup A(P)^{\star})$ is the *comparability digraph* of P. Boudabbous and Ille [2] obtained the following characterization of critical digraphs whose primitivity graph is a path of even size.

Theorem 30 (Proposition 20 [2]). *Given a digraph D defined on $V(D) = \{0, \dots, 2n-1\}$, where $n \geq 3$, the following two assertions are equivalent*

- *D is critical and $\Pi(D) = P_{2n}$,*
- *one of the digraphs D, $D^\star$, $\overline{D}$ or $\overline{D^\star}$ equals $\mathrm{comp}(Q_{2n})$, Q_{2n} or R_{2n}.*

We have

$$\begin{cases} E(\sigma(\mathrm{comp}(Q_{2n}))) = \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \\ E(\sigma(Q_{2n})) = \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \text{ and} \\ E(\sigma(R_{2n})) = \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}. \end{cases}$$

We deduce the following corollary.

Corollary 31. *Given a reversible 2-structure σ defined on $V(\sigma) = \{0, \dots, 2n-1\}$, where $n \geq 3$, if*

$$E(\sigma) = \begin{cases} \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \\ \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \ \textit{or} \\ \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}, \end{cases}$$

then σ is critical and $\Pi(\sigma) = P_{2n}$.

The next result follows from [1, Proposition 6.3].

Theorem 32. *Given a reversible 2-structure σ defined on $V(\sigma) = \{0, \dots, 2n-1\}$, where $n \geq 3$, σ is critical and $\Pi(\sigma) = P_{2n}$ if and only if*

$$E(\sigma) = \begin{cases} \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \\ \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}, \\ \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}, \ \textit{or} \\ \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}. \end{cases}$$

Given a reversible 2-structure σ defined on $V(\sigma) = \{0, \dots, 2n-1\}$, where $n \geq 3$, we have

- $E(\sigma) = \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (Q_{2n})^a + (R_{2n})^b$, where $a, b \in \mathbb{A}$ such that $a \neq b$, $a + a = 0$ and $b + b = 0$,
- $E(\sigma) = \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}) \cup A(R_{2n})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (Q_{2n})^a + (R_{2n})^b$, where $a, b \in \mathbb{A}$ such that $a + a \neq 0$ and $b + b = 0$,
- $E(\sigma) = \{A(Q_{2n}) \cup A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (Q_{2n})^a + (R_{2n})^b$, where $a, b \in \mathbb{A}$ such that $a + a = 0$ and $b + b \neq 0$,
- $E(\sigma) = \{A(Q_{2n}), A(Q_{2n})^\star, A(R_{2n}), A(R_{2n})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (Q_{2n})^a + (R_{2n})^b$, where $a, b \in \mathbb{A}$ such that $a \neq b$, $a \neq -b$, $a + a \neq 0$ and $b + b \neq 0$.

The next theorem follows.

Theorem 33. *Given $g \in \mathscr{L}(\{0,\ldots,2n-1\},\mathbb{A})$ such that $n \geq 3$, the following two assertions are equivalent*

- *g is critical and $\Pi(g) = P_{2n}$,*
- *$g = (Q_{2n})^a + (R_{2n})^b$, where $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a,-a\}$.*

Boudabbous and Ille [2] obtained the following characterization of critical digraphs whose primitivity graph is an odd cycle.

Theorem 34 (Proposition 18 [2])**.** *Given a digraph D defined on $V(D) = \{0,\ldots,2n\}$, where $n \geq 2$, the following two assertions are equivalent*

- *D is critical and $\Pi(D) = C_{2n+1}$,*
- *D equals T_{2n+1} or $(T_{2n+1})^\star$.*

We have $E(\sigma(T_{2n+1})) = \{A(T_{2n+1}), A(T_{2n+1})^\star\}$. We obtain the following corollary.

Corollary 35. *Given a reversible 2-structure σ defined on $V(\sigma) = \{0,\ldots,2n\}$, where $n \geq 2$, if $E(\sigma) = \{A(T_{2n+1}), A(T_{2n+1})^\star\}$, then σ is critical and $\Pi(\sigma) = C_{2n+1}$.*

The next result follows from [1, Proposition 6.1].

Theorem 36. *Given a reversible 2-structure σ defined on $V(\sigma) = \{0,\ldots,2n\}$, where $n \geq 2$, σ is critical and $\Pi(\sigma) = C_{2n+1}$ if and only if*

$$E(\sigma) = \{A(T_{2n+1}), A(T_{2n+1})^\star\}.$$

Given a reversible 2-structure σ defined on $V(\sigma) = \{0,\ldots,2n\}$, where $n \geq 2$, we have: $E(\sigma) = \{A(T_{2n+1}), A(T_{2n+1})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (T_{2n+1})^a$, where $a \in \mathbb{A}$ such that $a + a \neq 0$. The next theorem follows.

Theorem 37. *Given $g \in \mathscr{L}(\{0,\ldots,2n\},\mathbb{A})$ such that $n \geq 2$, the following two assertions are equivalent*

- *g is critical and $\Pi(g) = C_{2n+1}$,*
- *$g = (T_{2n+1})^a$, where $a \in \mathbb{A}$ such that $a \neq -a$.*

Boudabbous and Ille [2] obtained the following characterization of critical digraphs whose primitivity graph is a path of even size with an isolated vertex.

Theorem 38 (Proposition 21 [2])**.** *Given a digraph D defined on $V(D) = \{0,\ldots,2n\}$, where $n \geq 2$, the following two assertions are equivalent*

- *D is critical, $2n$ is an isolated vertex of $\Pi(D)$ and $\Pi(D) - (2n) = P_{2n}$,*
- *D equals W_{2n+1} or $(W_{2n+1})^\star$.*

We have $E(\sigma(W_{2n+1})) = \{A(W_{2n+1}), A(W_{2n+1})^\star\}$. We obtain the following corollary.

Corollary 39. *Given a reversible* 2*-structure* σ *defined on* $V(\sigma) = \{0, \ldots, 2n\}$*, where* $n \geq 2$*, if* $E(\sigma) = \{A(W_{2n+1}), A(W_{2n+1})^\star\}$*, then* σ *is critical,* $2n$ *is an isolated vertex of* $\Pi(\sigma)$ *and* $\Pi(\sigma) - (2n) = P_{2n}$*.*

The next result follows from [1, Proposition 6.2].

Theorem 40. *Given a reversible* 2*-structure* σ *defined on* $V(\sigma) = \{0, \ldots, 2n\}$*, where* $n \geq 2$*,* σ *is critical,* $2n$ *is an isolated vertex of* $\Pi(\sigma)$ *and* $\Pi(\sigma) - (2n) = P_{2n}$ *if and only if*

$$E(\sigma) = \{A(W_{2n+1}), A(W_{2n+1})^\star\}.$$

Given a reversible 2-structure σ defined on $V(\sigma) = \{0, \ldots, 2n\}$, where $n \geq 2$, we have: $E(\sigma) = \{A(W_{2n+1}), A(W_{2n+1})^\star\}$ if and only if $\sigma_{\mathbb{A}} = (W_{2n+1})^a$, where $a \in \mathbb{A}$ such that $a + a \neq 0$. The next theorem follows.

Theorem 41. *Given* $g \in \mathscr{L}(\{0, \ldots, 2n\}, \mathbb{A})$ *such that* $n \geq 2$*, the following two assertions are equivalent*

- *g is critical, $2n$ is an isolated vertex of $\Pi(g)$ and $\Pi(g) - (2n) = P_{2n}$,*
- *$g = (W_{2n+1})^a$, where $a \in \mathbb{A}$ such that $a \neq -a$.*

8. CLANS OF A SWITCHING CLASS

We continue to describe the main properties of clans of a switching class, after Proposition 2 and Corollaries 3 and 4. The next result makes Corollary 4 clearer.

Proposition 42 (Theorem 13.5 [5])**.** *Let* $\mathfrak{S}$ *be a switching class. Consider an element* g *of* $\mathfrak{S}$ *admitting an isolated vertex* x*. For each* $X \in \mathrm{Cl}(\mathfrak{S})$*, we have*

$$\begin{array}{rlll} \textit{either} & x \notin X & \textit{and} & X \in \mathrm{Cl}(g) \\ \textit{or} & x \in X & \textit{and} & V \setminus X \in \mathrm{Cl}(g). \end{array}$$

Lemma 43. *Consider a switching class* $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$*. For* $W \subseteq V$ *and* $X \in \mathrm{Cl}(\mathfrak{S})$*, we have* $X \cap W \in \mathrm{Cl}(\mathfrak{S}[W])$*.*

Proof. There is $g \in \mathfrak{S}$ such that $X \in \mathrm{Cl}(g)$. Thus $X \cap W \in \mathrm{Cl}(g[W])$. Since $g[W] \in \mathfrak{S}[W]$, $X \cap W \in \mathrm{Cl}(\mathfrak{S}[W])$. ∎

Lemma 44 (Lemma 14.1 [5])**.** *Let* $\mathfrak{S}$ *be a switching class. Given* $X, Y \in \mathrm{Cl}(\mathfrak{S})$*, if* $X \cap Y \neq \emptyset$*, then* $X \cup Y \in \mathrm{Cl}(\mathfrak{S})$*.*

Remark 45. There exist $g \in \mathscr{L}(V, \mathbb{A})$ and $X, Y \in \mathrm{Cl}(\langle g\rangle)$ such that $X \cap Y \notin \mathrm{Cl}(\langle g\rangle)$ (compare with the second assertion of Proposition 1). Suppose that $|V| \geq 4$ and consider $x, y \in V$ such that $x \neq y$, and $u, v \in V \setminus \{x, y\}$ such that $u \neq v$. Let $g \in \mathscr{L}(V, \mathbb{A})$ such that $g(x, u) \neq g(x, v)$ and, for each $z \in V \setminus \{y\}$, $g(z, y) = 0$. Clearly $V \setminus \{y\} \in \mathrm{Cl}(g)$ and hence $V \setminus \{y\} \in \mathrm{Cl}(\langle g\rangle)$. Now consider the selector

$$\begin{array}{rccl} s: & V & \longrightarrow & \mathbb{A} \\ & x & \longmapsto & 0 \\ & z \in V \setminus \{x\} & \longmapsto & g(x, z). \end{array}$$

For every $z \in V \setminus \{x\}$, $g^s(x, z) = s(x) + g(x, z) - s(z) = 0$. We get $V \setminus \{x\} \in \mathrm{Cl}(g^s)$ and hence $V \setminus \{x\} \in \mathrm{Cl}(\langle g\rangle)$. Suppose that $(V \setminus \{x\}) \cap (V \setminus \{y\}) \in \mathrm{Cl}(\langle g\rangle)$. Since $y \notin (V \setminus \{x\}) \cap (V \setminus \{y\})$ and y is an isolated vertex of g, it follows from Proposition 42 that $(V \setminus \{x\}) \cap (V \setminus \{y\}) \in \mathrm{Cl}(g)$, which contradicts $g(x, u) \neq g(x, v)$. Consequently $(V \setminus \{x\}) \cap (V \setminus \{y\}) \notin \mathrm{Cl}(\langle g\rangle)$.

9. Primitive Switching Classes

We begin this section by proving Proposition 7.

Proof of Proposition 7. By Corollary 3, the third assertion implies the second. Now, we verify that the second assertion implies the first. Consider $g \in \mathfrak{S}$ that admits an isolated vertex x such that $g - x$ is primitive. Given $Y \in \mathrm{Cl}(\mathfrak{S})$, we have to prove that Y is trivial. By Corollary 4, we can suppose that $x \notin Y$. By Proposition 42, $Y \in \mathrm{Cl}(g)$ and hence $Y \in \mathrm{Cl}(g - x)$. Since $g - x$ is primitive, we obtain $Y = \emptyset, V \setminus \{x\}$ or $\{y\}$ where $y \in V \setminus \{x\}$. It follows that $\mathfrak{S}$ is primitive.

Lastly, we verify that the first assertion implies the third. Suppose that $\mathfrak{S}$ is primitive and consider $g \in \mathfrak{S}$ which admits an isolated vertex x. We verify that $g - x$ is primitive. Let $Y \in \mathrm{Cl}(g - x)$. As $V \setminus \{x\} \in \mathrm{Cl}(g)$, $Y \in \mathrm{Cl}(g)$ and hence $Y \in \mathrm{Cl}(\mathfrak{S})$. Since $\mathfrak{S}$ is primitive, $Y = \emptyset, V$ or $Y \in \{\{z\}, V \setminus \{z\} : z \in V\}$. As $x \notin Y$, we obtain $Y = \emptyset$ or $Y = \{z\}$, where $z \in V \setminus \{x\}$, or $Y = V \setminus \{x\}$. It follows that $g - x$ is primitive. ∎

Proposition 46. *Let $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ be a switching class. If $\mathfrak{S}$ is primitive, then there exists $g \in \mathfrak{S}$ such that g is primitive.*

Proof. Suppose that $\mathfrak{S}$ is primitive. Denote by $I(\mathfrak{S})$ the set of elements of $\mathfrak{S}$ that admit an isolated vertex. Given $(x, a) \in V \times \mathbb{A}$, we denote by $g_{(x,a)}$ the unique element of $\mathfrak{S}$ in which x is a-isolated. We have $I(\mathfrak{S}) = \{g_{(x,a)} : (x, a) \in V \times \mathbb{A}\}$. Consider the function

$$\begin{array}{rccc} \varphi: & V \times \mathbb{A} & \longrightarrow & I(\mathfrak{S}) \\ & (x, a) & \longmapsto & g_{(x,a)}. \end{array}$$

To verify that φ is injective, consider $(x,a),(y,b) \in V \times \mathbb{A}$ such that $g_{(x,a)} = g_{(y,b)}$. Thus x and y are isolated vertices of $g_{(x,a)}$. Therefore $V \setminus \{x\} \in \mathrm{Cl}(g_{(x,a)})$ and $V \setminus \{y\} \in \mathrm{Cl}(g_{(x,a)})$. It follows that $V \setminus \{x,y\} = (V \setminus \{x\}) \cap (V \setminus \{y\}) \in \mathrm{Cl}(g_{(x,a)})$ so that $V \setminus \{x,y\} \in \mathrm{Cl}(\mathfrak{S})$. Since $\mathfrak{S}$ is primitive, we obtain $x = y$. Clearly $x = y$ implies $a = b$. Consequently φ is injective. Hence $|V| \times |\mathbb{A}| \leq |I(\mathfrak{S})|$. Since $I(\mathfrak{S}) = \{g_{(x,a)} : (x,a) \in V \times \mathbb{A}\}$, we have $|I(\mathfrak{S})| \leq |V| \times |\mathbb{A}|$. It follows that

$$|I(\mathfrak{S})| = |V| \times |\mathbb{A}|.$$

We show that $I(\mathfrak{S}) \subsetneq \mathfrak{S}$. Since $|\mathfrak{S}| = |\mathbb{A}|^{|V|-1}$, it suffices to prove that $|\mathbb{A}|^{|V|-2} > |V|$. We distinguish the following two cases.

- Suppose that $|V| \geq 5$. We have $2^{|V|-2} > |V|$. Furthermore, $|\mathbb{A}| \geq 2$ because $\mathfrak{S}$ is primitive. Thus $|\mathbb{A}|^{|V|-2} \geq 2^{|V|-2}$, so $|\mathbb{A}|^{|V|-2} > |V|$.
- Suppose that $|V| = 4$. Since $\mathfrak{S}$ is primitive, $|\mathbb{A}| \geq 3$. We obtain $|\mathbb{A}|^{|V|-2} = |\mathbb{A}|^2 \geq 3^2 > 4 = |V|$.

Consequently $I(\mathfrak{S}) \subsetneq \mathfrak{S}$. Consider $g \in \mathfrak{S} \setminus I(\mathfrak{S})$. We have

$$\mathrm{Cl}(g) \subseteq \mathrm{Cl}(\mathfrak{S}) = \{\emptyset, V\} \cup \{\{x\}, V \setminus \{x\} : x \in V\}.$$

Since $g \notin I(\mathfrak{S})$, $V \setminus \{x\} \notin \mathrm{Cl}(g)$ for each $x \in V$. Therefore g is primitive. ■

Remark 47. The opposite direction in Proposition 46 does not hold. Given $n \geq 4$, consider the graph G defined on $\{x,y\} \cup \{0,\ldots,2n-1\}$ by $G - \{x,y\} = C_{2n}$, $N_G(x) = \{0,\ldots,n-1\} \cup \{y\}$ and $N_G(y) = \{n,\ldots,2n-1\} \cup \{x\}$. Set $X = \{0,\ldots,2n-1\}$. We have $G[X]$ is prime. We verify that G is prime too. Consider a module M of G such that $|M| \geq 2$. Suppose that $M \cap X = \emptyset$. Since $|M| \geq 2$, $M = \{x,y\}$, which contradicts $0 \in N_G(x) \setminus N_G(y)$. Thus $M \cap X \neq \emptyset$. Suppose that $M \cap X \subsetneq X$. Since $G[X]$ is prime, there is $i \in X$ such that $M \cap X = \{i\}$. As $|M| \geq 2$, we have $\{x,y\} \cap M \neq \emptyset$. Since $d_{G-y}(x) = d_{G-x}(y) = n$, we obtain $d_{G-\{x,y\}}(i) \geq n-1 > 2$, which contradicts $G - \{x,y\} = C_{2n}$. Therefore $M \supseteq X$. As $0 \in N_G(x) \setminus N_G(y)$ and $n \in N_G(y) \setminus N_G(x)$, we get $\{x,y\} \subseteq M$, so $M = V(G)$. It follows that G is prime. Hence $\sigma(G)_{\mathbb{Z}_2}$ (see (5)) is primitive. However, consider the selector s defined by $s^{-1}(\{1\}) = \{y\} \cup \{0,\ldots,n-1\}$ and $s^{-1}(\{0\}) = \{x\} \cup \{n,\ldots,2n-1\}$. We obtain that x and y are 0-isolated vertices of $(\sigma(G)_{\mathbb{Z}_2})^s$. Thus $\{x,y\} \in \mathrm{Cl}((\sigma(G)_{\mathbb{Z}_2})^s)$ and hence $\{x,y\} \in \mathrm{Cl}(\langle \sigma(G)_{\mathbb{Z}_2} \rangle)$. Consequently $\langle \sigma(G)_{\mathbb{Z}_2} \rangle$ is decomposable.

Proposition 48. *Given isomorphic switching classes $\mathfrak{S} \subseteq \mathscr{L}(V,\mathbb{A})$ and $\mathfrak{S}' \subseteq \mathscr{L}(V',\mathbb{A})$, $\mathfrak{S}$ is primitive if and only if $\mathfrak{S}'$ is primitive.*

Proof. We have

$$\mathrm{Cl}(\mathfrak{S}) = \bigcup_{g \in \mathfrak{S}} \mathrm{Cl}(g) \text{ and } \mathrm{Cl}(\mathfrak{S}') = \bigcup_{h \in \mathfrak{S}'} \mathrm{Cl}(h). \tag{10}$$

Since $\mathfrak{S}$ and $\mathfrak{S}'$ are isomorphic, there exists a bijection β from V onto V' such that $\mathfrak{S}' = \beta(\mathfrak{S})$. Thus

$$\mathrm{Cl}(\mathfrak{S}') = \bigcup_{g \in \mathfrak{S}} \mathrm{Cl}(\beta(g)).$$

Since β is an isomorphism from g onto $\beta(g)$ for every $g \in \mathscr{L}(V, \mathbb{A})$, we have

$$\mathrm{Cl}(\mathfrak{S}') = \bigcup_{g \in \mathfrak{S}} \beta(\mathrm{Cl}(g)).$$

It follows from (10) that $\{|C| : C \in \mathrm{Cl}(\mathfrak{S})\} = \{|C'| : C' \in \mathrm{Cl}(\mathfrak{S}')\}$. Consequently, $\mathfrak{S}$ is primitive if and only if $\mathfrak{S}'$ is. ■

We complete the section by translating results on primitive and $\mathbb{A}$-labeled, reversible 2-structures in terms of primitive switching classes. We begin with the analogue of Proposition 20 in terms of switching classes.

Lemma 49. *Let $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ be a primitive switching class. For each $x \in V$, there exists $X \subseteq V$ such that $|X| = 4$ or 5, $X \ni x$ and $\mathfrak{S}[X]$ is primitive.*

Proof. Let $x \in V$. By Corollary 3, there is $g \in \mathfrak{S}$ such that x is an isolated vertex of g. Since $\mathfrak{S}$ is primitive, it follows from Proposition 7 that $g - x$ is primitive. Since $|V \setminus \{x\}| \geq 3$, it follows from Lemma 19 that there exists $Y \subseteq V \setminus \{x\}$ such that $|Y| = 3$ or 4, and $(g - x)[Y]$ is primitive. We have $g[Y \cup \{x\}] \in \mathfrak{S}[Y \cup \{x\}]$. Moreover, x is an isolated vertex of $g[Y \cup \{x\}]$ as well. It follows from Proposition 7 applied to $\mathfrak{S}[Y \cup \{x\}]$ that $\mathfrak{S}[Y \cup \{x\}]$ is primitive. ■

The next result is the analogue of Proposition 21 for switching classes.

Proposition 50 (Theorem 6.5 [5])**.** *Consider a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$. Let $X \subseteq V$ such that $\mathfrak{S}[X]$ is primitive. If $|V \setminus X| \geq 2$, then there exist distinct $u, v \in V \setminus X$ such that $\mathfrak{S}[X \cup \{u, v\}]$ is primitive.*

Proof. Let $x \in X$. By Corollary 3, there is $g \in \mathfrak{S}$ such that x is an isolated vertex of g. Since $\mathfrak{S}$ is primitive, it follows from Proposition 7 that $g - x$ is primitive. Clearly x is an isolated vertex of $g[X]$ also. Since $g[X] \in \mathfrak{S}[X]$ and $\mathfrak{S}[X]$ is primitive, it follows from Proposition 7 that $g[X] - x$ is primitive. As $|(V \setminus \{x\}) \setminus (X \setminus \{x\})| = |V \setminus X| \geq 2$, it follows from Proposition 21 that there exist distinct $u, v \in (V \setminus \{x\}) \setminus (X \setminus \{x\})$ such that $g[(X \setminus \{x\}) \cup \{u, v\}]$ is primitive. Once again, x is an isolated vertex of $g[X \cup \{u, v\}]$. Since $g[X \cup \{u, v\}] \in \mathfrak{S}[X \cup \{u, v\}]$, it follows from Proposition 7 applied to $\mathfrak{S}[X \cup \{u, v\}]$ that $\mathfrak{S}[X \cup \{u, v\}]$ is primitive. ■

Now, we describe the analogue of the partition $p_{(g,X)}$ for switching classes. Consider a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, and $X \subsetneq V$ such that $\mathfrak{S}[X]$ is primitive.

We still denote by $\mathrm{Ext}^{\mathfrak{S}}(X)$ the set of $u \in V \setminus X$ such that $\mathfrak{S}[X \cup \{u\}]$ is primitive. Let $u \in V \setminus X$ such that $\mathfrak{S}[X \cup \{u\}]$ is decomposable. Consider a nontrivial clan C of $\mathfrak{S}[X \cup \{u\}]$. By interchanging C and $(X \cup \{u\}) \setminus C$, assume that $u \in C$. By Lemma 43, $C \setminus \{u\} \in \mathrm{Cl}(\mathfrak{S}[X])$. Since $\mathfrak{S}[X]$ is primitive, $C \setminus \{u\}$ is trivial, that is, $\min(|C \setminus \{u\}|, |X \setminus (C \setminus \{u\})|) \leq 1$. If $|X \setminus (C \setminus \{u\})| \leq 1$, then C is a trivial clan of $\mathfrak{S}[X \cup \{u\}]$, which is a contradiction. Thus $|C \setminus \{u\}| \leq 1$. Since $|C| \geq 2$, there is $y \in X$ such that $C = \{u, y\}$. Given $y \in X$, $X^{\mathfrak{S}}(y)$ denotes the set of $u \in V \setminus X$ such that $\{y, u\}$ is a clan of $\mathfrak{S}[X \cup \{u\}]$. It follows from Lemmas 43 and 44 that the family

$$p_{(\mathfrak{S},X)} = \{\mathrm{Ext}^{\mathfrak{S}}(X)\} \cup \{X^{\mathfrak{S}}(y) : y \in X\}$$

is a partition of $V \setminus X$.

An immediate consequence of Lemma 49 and Proposition 50 follows. It is the analogue of Corollary 22 for switching classes.

Corollary 51. *Consider a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$. If $|V| \geq 6$, then there exist $u, v \in V$ such that $\mathfrak{S} - \{u, v\}$ is primitive.*

***Proof of Theorem* 8.** Let $x \in V$. By Corollary 3, there is $g \in \mathfrak{S}$ such that x is an isolated vertex of g. Since $\mathfrak{S}$ is primitive, it follows from Proposition 7 that $g - x$ is primitive. Since $|V \setminus \{x\}| \geq 7$, it follows from Theorem 5 that there exist distinct $u, v \in (V \setminus \{x\})$ such that $g[(V \setminus \{x\}) \setminus \{u, v\}]$ is primitive. Once again, x is an isolated vertex of $g - \{u, v\}$. Since $g - \{u, v\} \in \mathfrak{S} - \{u, v\}$, it follows from Proposition 7 applied to $\mathfrak{S} - \{u, v\}$ that $\mathfrak{S} - \{u, v\}$ is primitive. ■

***Proof of Theorem* 9.** Let $x \in X$. By Corollary 3, there is $g \in \mathfrak{S}$ such that x is an isolated vertex of g. Since $\mathfrak{S}$ is primitive, it follows from Proposition 7 that $g - x$ is primitive. Clearly x is an isolated vertex of $g[X]$ also. Since $g[X] \in \mathfrak{S}[X]$ and $\mathfrak{S}[X]$ is primitive, it follows from Proposition 7 that $g[X] - x$ is primitive. Since $|(V \setminus \{x\}) \setminus (X \setminus \{x\})| = |V \setminus X| \geq 6$, it follows from Theorem 6 that there exist distinct $u, v \in (V \setminus \{x\}) \setminus (X \setminus \{x\})$ such that $(g - x) - \{u, v\}$ is primitive. Once again, x is an isolated vertex of $g - \{u, v\}$. Since $g - \{u, v\} \in \mathfrak{S} - \{u, v\}$, it follows from Proposition 7 applied to $\mathfrak{S} - \{u, v\}$ that $\mathfrak{S} - \{u, v\}$ is primitive. ■

10. CRITICAL SWITCHING CLASSES

Given a switching class $\mathfrak{S}$, the set of critical vertices of $\mathfrak{S}$ is denoted by $\mathcal{C}(\mathfrak{S})$. The next lemma follows from Proposition 7.

Lemma 52. *Let $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ be a switching class such that $|V| \geq 6$. Given $x \in V$, consider $g \in \mathfrak{S}$ such that x is an isolated vertex of g. If $g - x$ is critical, then $\mathfrak{S}$ is primitive and $V \setminus \{x\} \subseteq \mathcal{C}(\mathfrak{S})$.*

Proof. Since $g-x$ is primitive, it follows from Proposition 7 that $\mathfrak{S}$ is primitive. Let $y \in V \setminus \{x\}$. Since $g-x$ is critical, $(g-x)-y$ is decomposable. Since x is an isolated vertex of $g-y$ and $(g-x)-y$, that is, $(g-y)-x$ is decomposable, $\mathfrak{S}-y$ is decomposable by Proposition 7 applied to $\mathfrak{S}-y$. Therefore $V \setminus \{x\} \subseteq \mathcal{C}(\mathfrak{S})$. ■

The following three results are immediate consequences of Remark 17 and Proposition 48.

Corollary 53. *Given primitive switching classes* $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ *and* $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$*, if* β *is an isomorphism from* $\mathfrak{S}$ *onto* $\mathfrak{S}'$*, then* $\mathcal{C}(\mathfrak{S}') = \beta(\mathcal{C}(\mathfrak{S}))$.

Corollary 54. *Given isomorphic switching classes* $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ *and* $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$*,* $\mathfrak{S}$ *is critical if and only if* $\mathfrak{S}'$ *is critical.*

Corollary 55. *Given primitive switching classes* $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ *and* $\mathfrak{S}' \subseteq \mathscr{L}(V', \mathbb{A})$*, every isomorphism from* $\mathfrak{S}$ *onto* $\mathfrak{S}'$ *is an isomorphism from* $\Pi(\mathfrak{S})$ *onto* $\Pi(\mathfrak{S}')$.

The next result is the analogue of Lemma 23 for switching classes.

Lemma 56. *Consider a primitive switching class* $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ *such that* $|V| \geq 6$*. For every* $v \in \mathcal{C}(\mathfrak{S})$*,* $d_{\Pi(\mathfrak{S})}(v) = 0$ *or* 2*. Moreover, for each* $v \in \mathcal{C}(\mathfrak{S})$*, if* $d_{\Pi(\mathfrak{S})}(v) = 2$*, then* $N_{\Pi(\mathfrak{S})}(v) \in \mathrm{Cl}(\mathfrak{S} - v)$.

Proof. Let $v \in \mathcal{C}(\mathfrak{S})$ such that $d_{\Pi(\mathfrak{S})}(v) > 0$. Consider $x \in N_{\Pi(\mathfrak{S})}(v)$. We have $\mathfrak{S}[X]$ is primitive, where $X = V \setminus \{x, v\}$. Since $v \in \mathcal{C}(\mathfrak{S})$, $\mathfrak{S} - v = \mathfrak{S}[X \cup \{x\}]$ is decomposable. Thus there exists $y \in X$ such that $x \in X^{\mathfrak{S}}(y)$, that is, $\{x, y\} \in \mathrm{Cl}(\mathfrak{S}[X \cup \{x\}])$.

We prove that $y \in N_{\Pi(\mathfrak{S})}(v)$. Let $w \in X \setminus \{y\}$. By Corollary 3, there is $g \in \mathfrak{S}$ such that w is an isolated vertex of g. Clearly w is an isolated vertex of $g[X \cup \{x\}]$. Since $\{x, y\} \in \mathrm{Cl}(\mathfrak{S}[X \cup \{x\}])$, it follows from Proposition 42 that $\{x, y\} \in \mathrm{Cl}(g[X \cup \{x\}])$. Therefore, the function

$$\begin{array}{rccl} \varphi : & X & \longrightarrow & (X \setminus \{y\}) \cup \{x\} \\ & z \in X \setminus \{y\} & \longmapsto & z \\ & y & \longmapsto & x \end{array}$$

is an isomorphism from $g[X]$ onto $g[(X \setminus \{y\}) \cup \{x\}]$. Clearly w is an isolated vertex of $g[X]$. Since $\mathfrak{S}[X]$ is primitive, it follows from Proposition 7 that $g[X]-w$ is primitive. Thus $g[\varphi(X \setminus \{w\})] = g[((X \setminus \{y\}) \cup \{x\}) \setminus \{w\}]$ is primitive. Since w is an isolated vertex of $g[(X \setminus \{y\}) \cup \{x\}]$, it follows from Proposition 7 that $\mathfrak{S}[(X \setminus \{y\}) \cup \{x\}]$ is primitive. Since $(X \setminus \{y\}) \cup \{x\} = V \setminus \{v, y\}$, $y \in N_{\Pi(\mathfrak{S})}(v)$.

Lastly, we show that $N_{\Pi(\mathfrak{S})}(v) = \{x, y\}$. Let $z \in V \setminus \{v, x, y\}$. Since $\{x, y\} \in \mathrm{Cl}(\mathfrak{S}[X \cup \{x\}])$, that is, $\{x, y\} \in \mathrm{Cl}(\mathfrak{S} - v)$, we have $\{x, y\} \in \mathrm{Cl}(\mathfrak{S} - \{v, z\})$ by Lemma 43. Hence $\mathfrak{S} - \{v, z\}$ is decomposable, so $z \notin N_{\Pi(\mathfrak{S})}(v)$. Consequently, $N_{\Pi(\mathfrak{S})}(v) = \{x, y\}$ and $N_{\Pi(\mathfrak{S})}(v) \in \mathrm{Cl}(\mathfrak{S} - v)$. ■

The next result is the analogue of Proposition 24 for switching classes.

Proposition 57. *Consider a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ such that $|V| \geq 6$. If there exists a component D of $\Pi(\mathfrak{S})$ such that $|V(D)| \geq 2$ and $V(D) \subseteq \mathcal{C}(\mathfrak{S})$, then one of the following holds*

- $\Pi(\mathfrak{S})$ *is isomorphic to* $C_{|V|}$,
- $\Pi(\mathfrak{S})$ *admits a unique isolated vertex* x, $|V|$ *is even and* $\Pi(\mathfrak{S}) - x$ *is isomorphic to* $C_{|V|-1}$.

Proof. By Lemma 56, D is isomorphic to C_m, where $m \geq 3$. The vertices of D can be indexed as $d_0, \ldots, d_{m-1}$ so that for $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, m-1\}$, we have: $d_i d_j \in E(D)$ if and only if $|i - j| = 1$ or $m - 1$.

Suppose that there exists $x \in V \setminus V(D)$. We have to show that m is odd, and $V = V(D) \cup \{x\}$. Suppose to the contrary that $m = 2n$, where $n \geq 2$. It follows from Lemma 56 that $\{d_{2i-1}, d_{2i+1}\} \in \mathrm{Cl}(\mathfrak{S} - d_{2i})$ for $1 \leq i \leq n-1$. Consider $g \in \mathfrak{S}$ such that x is an isolated vertex of g. Let $i \in \{1, \ldots, n-1\}$. Since x is an isolated vertex of $g - d_{2i}$, we obtain $\{d_{2i-1}, d_{2i+1}\} \in \mathrm{Cl}(g - d_{2i})$ by Proposition 42. Thus $g(d_0, d_{2i-1}) = g(d_0, d_{2i+1})$. It follows that

$$g(d_0, d_1) = g(d_0, d_{2n-1}). \tag{11}$$

By Lemma 56, $\{d_1, d_{2n-1}\} \in \mathrm{Cl}(\mathfrak{S} - d_0)$. Since x is an isolated vertex of $g - d_0$, we obtain $\{d_1, d_{2n-1}\} \in \mathrm{Cl}(g - d_0)$ by Proposition 42. By (11), $\{d_1, d_{2n-1}\} \in \mathrm{Cl}(g)$. Therefore $\{d_1, d_{2n-1}\} \in \mathrm{Cl}(\mathfrak{S})$, which contradicts the primitivity of $\mathfrak{S}$. It follows that m is odd. Let $n \geq 1$ such that $m = 2n + 1$.

We prove that $V(D) \in \mathrm{Cl}(g)$. Let $z \in V \setminus V(D)$. Now, consider $g \in \mathfrak{S}$ such that z is an isolated vertex of g. As previously, we have $\{d_{2i-1}, d_{2i+1}\} \in \mathrm{Cl}(g - d_{2i})$ for $i \in \{1, \ldots, n-1\}$. Thus

$$g(z, d_1) = \cdots = g(z, d_{2n-1}). \tag{12}$$

Similarly, we have $\{d_{2i}, d_{2i+2}\} \in \mathrm{Cl}(g - d_{2i+1})$ for $i \in \{0, \ldots, n-1\}$. Hence

$$g(z, d_0) = \cdots = g(z, d_{2n}). \tag{13}$$

We also have $\{d_1, d_{2n}\} \in \mathrm{Cl}(g - d_0)$. Thus $g(z, d_1) = g(z, d_{2n})$. It follows from (12) and (13) that $g(z, d_0) = g(z, d_i)$ for every $i \in \{0, \ldots, 2n\}$. Consequently, $V(D) \in \mathrm{Cl}(g)$ and hence $V(D) \in \mathrm{Cl}(\mathfrak{S})$. Since $\mathfrak{S}$ is primitive, we obtain $V = V(D) \cup \{x\}$. ∎

Corollary 11 is an easy consequence of Proposition 57 and Corollary 51. It is the analogue of Corollary 25 for switching classes.

***Proof of Corollary* 11.** By Corollary 51, there exist $u, v \in V$ such that $\mathfrak{S} - \{u, v\}$ is primitive. Since $\mathfrak{S}$ is critical, we have $u \neq v$. Hence $uv \in E(\Pi(\mathfrak{S}))$. Let D be the component of $\Pi(\mathfrak{S})$ such that $uv \in E(D)$. It is sufficient to apply Proposition 57 with D. ■

The next lemma is important in what follows.

Lemma 58. *Consider a critical switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ such that $|V| \geq 6$. For each $x \in V$, we have $\mathfrak{S}_x$ is critical and $\Pi(\mathfrak{S}_x) = \Pi(\mathfrak{S}) - x$.*

Proof. Let $x \in V$. Consider $g \in \mathfrak{S}$ such that x is an isolated vertex of g. We have $\mathfrak{S}_x = g - x$. By Proposition 7, $g - x$ is primitive. Let $y \in V \setminus \{x\}$. We have $g - y \in \mathfrak{S} - y$. Since $\mathfrak{S} - y$ is decomposable and x is an isolated vertex of $g - y$, it follows from Proposition 7 that $(g - y) - x$, that is, $(g - x) - y$ is decomposable. Thus $g - x$ is critical.

Now, we show that $\Pi(g - x) = \Pi(\mathfrak{S}) - x$. Let u, v be distinct elements of $V \setminus \{x\}$. We have

$$uv \in E(\Pi(\mathfrak{S}) - x) \iff uv \in E(\Pi(\mathfrak{S})).$$

By definition of $\Pi(\mathfrak{S})$,

$$uv \in E(\Pi(\mathfrak{S})) \iff \mathfrak{S} - \{u, v\} \text{ is primitive.}$$

Since x is an isolated vertex of $g - \{u, v\}$, it follows from Proposition 7 that

$$\mathfrak{S} - \{u, v\} \text{ is primitive} \iff (g - \{u, v\}) - x \text{ is primitive.}$$

Since $(g - \{u, v\}) - x = (g - x) - \{u, v\}$, it follows from the definition of $\Pi(g - x)$ that

$$(g - \{u, v\}) - x \text{ is primitive} \iff uv \in E(\Pi(g - x)).$$ ■

Theorem 10 is an immediate consequence of Lemmas 52 and 58. To conclude, we prove Theorems 12, 13 and 14. We need the next three lemmas to show Theorem 12.

Lemma 59. *Let $n \geq 3$ and let $a, b \in \mathbb{A}$ such that $b + b \neq 0$, and $a \neq b$. Consider the selector*

$$\begin{array}{rrcl} s: & \{0, \ldots, 2n-2\} & \longrightarrow & \mathbb{A} \\ & 2n-2 & \longmapsto & b + b \\ & p \in \{0, \ldots, 2n-3\} & \longmapsto & 0. \end{array} \tag{14}$$

We obtain that $2n - 3$ is an isolated vertex of $((O_{2n-1})^a + (H_{2n-1})^b)^s$. Moreover, the following assertions are equivalent

- $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$ *is decomposable,*
- $a + a = b + b$,
- $\{0, 2n-2\}$ *is a clan of* $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$.

Proof. Set $h = ((O_{2n-1})^a + (H_{2n-1})^b)^s$. Clearly $2n-3$ is a $(-b)$-isolated vertex of h. Set $Y = \{0, \ldots, 2n-4\}$. We verify that $((O_{2n-1})^a + (H_{2n-1})^b)[Y]$, that is, $(O_{2n-3})^a + (H_{2n-3})^b$ is primitive. If $n = 3$, then it is easy to verify that $((O_3)^a + (H_3)^b)$ is primitive because $b + b \neq 0$ and $a \neq b$. If $n \geq 4$, then it follows from Theorem 29 that $(O_{2n-3})^a + (H_{2n-3})^b$ is critical and hence primitive. Therefore, $((O_{2n-1})^a + (H_{2n-1})^b)[Y]$ is primitive for every $n \geq 3$. Since $s(p) = 0$ for each $p \in Y$, $((O_{2n-1})^a + (H_{2n-1})^b)[Y] = h[Y]$. Thus $h[Y]$ is primitive. Moreover, since $b + b \neq 0$ and $a \neq b$, we have

$$2n - 2 \notin \widehat{Y}^h \cup \left(\bigcup_{p=1}^{2n-4} Y^h(p) \right) \quad \text{(see (9))}.$$

Thus $2n-2 \in \mathrm{Ext}^h(Y) \cup Y^h(0)$. Consequently, $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$, that is, $h[Y \cup \{2n-2\}]$ is decomposable if and only if $\{0, 2n-2\}$ is a clan of $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$. Furthermore, it is easy to verify that $\{0, 2n-2\}$ is a clan of $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$ if and only if $a + a = b + b$. ■

Lemma 60. *Consider* $g \in \mathscr{L}(\{0, \ldots, 2n-1\}, \mathbb{A})$*, where* $n \geq 3$*, such that* $2n-1$ *is an isolated vertex of* g*. If* $\langle g \rangle$ *is critical and* $\Pi(\langle g \rangle) = C_{2n}$*, then* $g - (2n-1) = (O_{2n-1})^a + (H_{2n-1})^b$*, where* $a, b \in \mathbb{A}$ *such that* $b + b \neq 0$*,* $a \neq b$*, and* $a + a = b + b$*.*

Proof. By Lemma 58, $g - (2n-1)$ is critical and $\Pi(g - (2n-1)) = \Pi(\mathfrak{S}) - (2n-1) = P_{2n-1}$. It follows from Theorem 29 that $g - (2n-1) = (O_{2n-1})^a + (H_{2n-1})^b$, where $a, b \in \mathbb{A}$ such that $b + b \neq 0$ and $a \neq b$. Let s be the selector defined in (14). We have $((O_{2n-1})^a + (H_{2n-1})^b)^s \in \langle g \rangle - (2n-1)$ and $\langle g \rangle - (2n-1)$ is decomposable because $\langle g \rangle$ is critical. By Lemma 59, $2n-3$ is an isolated vertex of $((O_{2n-1})^a + (H_{2n-1})^b)^s$. It follows from Proposition 7 that $((O_{2n-1})^a + (H_{2n-1})^b)^s - (2n-3)$ is decomposable. By Lemma 59, $a + a = b + b$. ■

Lemma 61. *Given* $n \geq 3$*, consider* $g \in \mathscr{L}(\{0, \ldots, 2n-1\}, \mathbb{A})$ *such that* $2n-1$ *is an isolated vertex of* g*. If* $g - (2n-1) = (O_{2n-1})^a + (H_{2n-1})^b$*, where* $a, b \in \mathbb{A}$ *satisfying* $b + b \neq 0$*,* $a \neq b$ *and* $a + a = b + b$*, then* $\langle g \rangle$ *is critical and* $\Pi(\langle g \rangle) = C_{2n}$*.*

Proof. It follows from Theorem 29 applied to $g - (2n-1)$ that $g - (2n-1)$ is critical and $\Pi(g - (2n-1)) = P_{2n-1}$. Furthermore, since $g - (2n-1)$ is critical, it follows from Lemma 52 that $\langle g \rangle$ is primitive and

$$\{0, \ldots, 2n-2\} \subseteq \mathcal{C}(\langle g \rangle). \tag{15}$$

Let s be the selector defined in (14). Since $a+a=b+b$, it follows from Lemma 59 that $((O_{2n-1})^a+(H_{2n-1})^b)^s-(2n-3)$, that is, $(g-(2n-1))^s-(2n-3)$ is decomposable. Moreover, $2n-3$ is an isolated vertex of $(g-(2n-1))^s$ by Lemma 59. It follows from Proposition 7 that $\langle (g-(2n-1))^s\rangle$, that is, $\langle g\rangle-(2n-1)$ is decomposable. Hence $2n-1\in\mathcal{C}(\langle g\rangle)$. By (15), $\langle g\rangle$ is critical. Since $\Pi(g-(2n-1))=P_{2n-1}$, it follows from Lemma 58 and Corollary 11 that $\Pi(\langle g\rangle)=C_{2n}$. ■

***Proof of Theorem* 12.** To begin, we prove that the first assertion implies the second one. Consider a switching class $\mathfrak{S}\subseteq\mathscr{L}(V,\mathbb{A})$ such that $|V|\geq 6$ and suppose that $\mathfrak{S}$ is critical and $\Pi(\mathfrak{S})\simeq C_{2n}$. Up to isomorphy, we may assume that $V=\{0,\ldots,2n-1\}$ and $\Pi(\mathfrak{S})=C_{2n}$. Consider $g\in\mathfrak{S}$ such that $2n-1$ is an isolated vertex of g. It follows from Lemma 60 that $g-(2n-1)=(O_{2n-1})^a+(H_{2n-1})^b$, where $a,b\in\mathbb{A}$ such that $b+b\neq 0$, $a\neq b$, and $a+a=b+b$.

Now, we prove that the second assertion implies the first one. Suppose that there exists $x\in V$ and $g\in\mathfrak{S}$ such that x is an isolated vertex of g, and $g-x\simeq(O_{2n-1})^a+(H_{2n-1})^b$, where $n\geq 3$ and $a,b\in\mathbb{A}$ such that $b+b\neq 0$, $a\neq b$ and $a+a=b+b$. Up to isomorphy, we may assume that $V=\{0,\ldots,2n-1\}$, $x=2n-1$ and $g-(2n-1)=(O_{2n-1})^a+(H_{2n-1})^b$. It follows from Lemma 61 that $\langle g\rangle$ is critical and $\Pi(\langle g\rangle)=C_{2n}$.

Consequently, the first two assertions are equivalent. Clearly, the third assertion implies the second one. We complete the proof by proving that the second assertion implies the last one. Suppose that there exist $x\in V$ and $g\in\mathfrak{S}$ such that x is an isolated vertex of g and $g-x\simeq(O_{2n-1})^a+(H_{2n-1})^b$, where $n\geq 3$ and $a,b\in\mathbb{A}$ such that $b+b\neq 0$, $a\neq b$, and $a+a=b+b$. Up to isomorphy, we may assume that $V=\{0,\ldots,2n-1\}$, $x=2n-1$ and $g-(2n-1)=(O_{2n-1})^a+(H_{2n-1})^b$. It follows from Lemma 61 that $\langle g\rangle$ is critical and $\Pi(\langle g\rangle)=C_{2n}$. Let $m\in\{0,\ldots,2n-2\}$. Consider the selector

$$\begin{array}{llcl} t: & \{0,\ldots,2n-1\} & \longrightarrow & \mathbb{A}\\ & m & \longmapsto & 0\\ & p\in\{0,\ldots,2n-1\}\setminus\{m\} & \longmapsto & g(m,p). \end{array}$$

Clearly, m is an isolated vertex of g^t. Consider also the permutation γ of $\{0,\ldots,2n-1\}$ defined by

$$\begin{array}{llcl} \gamma: & \{0,\ldots,2n-1\} & \longrightarrow & \{0,\ldots,2n-1\}\\ & p & \longmapsto & p-1-m \pmod{2n}. \end{array}$$

Since γ is an isomorphism from g^t onto $\gamma(g^t)$ and m is an isolated vertex of g^t, $\gamma(m)$ is an isolated vertex of $\gamma(g^t)$. We have $\gamma(m)=-1=2n-1 \mod 2n$. Hence $2n-1$ is an isolated vertex of $\gamma(g^t)$. Clearly γ is an isomorphism from $\langle g^t\rangle$ onto $\langle\gamma(g^t)\rangle$. Since $\langle g\rangle$ is critical, it follows from Corollary 54 that $\langle\gamma(g^t)\rangle$ is critical. Moreover, by Corollary 55, γ is an isomorphism from $\Pi(\langle g\rangle)$ onto

$\Pi(\langle\gamma(g^t)\rangle)$. Since $\Pi(\langle g\rangle) = C_{2n}$ and γ is an automorphism of C_{2n}, we obtain that $\Pi(\langle\gamma(g^t)\rangle) = C_{2n}$. Consequently, $2n-1$ is an isolated vertex of $\gamma(g^t)$, $\langle\gamma(g^t)\rangle$ is critical and $\Pi(\langle\gamma(g^t)\rangle) = C_{2n}$. It follows from Lemma 60 that there exist $c, d \in \mathbb{A}$ such that $d + d \neq 0$, $c \neq d$, $c + c = d + d$, and

$$\gamma(g^t) - (2n-1) = (O_{2n-1})^c + (H_{2n-1})^d.$$

Clearly $(c, d) = (\gamma(g^t)(0, 2), \gamma(g^t)(0, 1))$. Given distinct $i, j \in \{0, \dots, 2n-2\}$, we have $\gamma(g^t)(i, j) = g(m, i+1+m) + g(i+1+m, j+1+m) - g(m, j+1+m)$, where $i+1+m$ and $j+1+m$ are considered modulo $2n$. It follows that

$$(c, d) = \begin{cases} (a, b) \text{ if } m \text{ is odd, and} \\ (b, a) \text{ if } m \text{ is even.} \end{cases}$$

Since γ is an isomorphism from g^t onto $\gamma(g^t)$ such that $\gamma(m) = 2n-1$ and $\gamma(g^t) - (2n-1) = (O_{2n-1})^c + (H_{2n-1})^d$, we obtain that $g^t - m$ is isomorphic to $(O_{2n-1})^a + (H_{2n-1})^b$ or $(O_{2n-1})^b + (H_{2n-1})^a$. ∎

***Proof of Theorem* 13.** To begin, we prove that the first assertion implies the second one. Suppose that the first assertion holds. Up to isomorphy, we may assume that $V = \{0, \dots, 2n\}$ and $\Pi(\mathfrak{S}) = C_{2n+1}$. By Lemma 58, $\mathfrak{S}_{2n}$ is critical and $\Pi(\mathfrak{S}_{2n}) = \Pi(\mathfrak{S}) - (2n) = P_{2n}$. It follows from Theorem 33 that $\mathfrak{S}_{2n} = (Q_{2n})^a + (R_{2n})^b$, where $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a, -a\}$.

Now, we prove that the second assertion implies the first one. Suppose that the second assertion holds. Up to isomorphy, we may assume that $V = \{0, \dots, 2n\}$, $x = 2n$ and $\mathfrak{S}_{2n} = (Q_{2n})^a + (R_{2n})^b$. It follows from Theorem 33 that $\mathfrak{S}_{2n}$ is critical and $\Pi(\mathfrak{S}_{2n}) = P_{2n}$. By Lemma 52, $\mathfrak{S}$ is primitive and

$$\{0, \dots, 2n-1\} \subseteq \mathcal{C}(\mathfrak{S}). \tag{16}$$

Consider the selector

$$\begin{array}{rrcl} s: & \{0, \dots, 2n-1\} & \longrightarrow & \mathbb{A} \\ & 2n-1 & \longmapsto & a+b \\ & p \in \{0, \dots, 2n-2\} & \longmapsto & 0. \end{array}$$

We obtain, for $0 \leq m \leq n-2$, $(\mathfrak{S}_{2n})^s(2m+1, 2n-1) = -a$, and for $0 \leq m \leq n-1$, $(\mathfrak{S}_{2n})^s(2m, 2n-1) = -b$. Thus $2n-2$ is a $(-b)$-isolated vertex of $(\mathfrak{S}_{2n})^s$. Furthermore, $\{0, 2n-1\}$ is a clan of $(\mathfrak{S}_{2n})^s$. Therefore $\{0, 2n-1\}$ is a clan of $\mathfrak{S} - (2n)$. Hence $2n \in \mathcal{C}(\mathfrak{S})$ and $\mathfrak{S}$ is critical by (16). Lastly, since $|V|$ is odd, it follows from Corollary 11 that $\Pi(\mathfrak{S})$ is an odd cycle.

Consequently, the first two assertions are equivalent. As the third assertion implies the second one, we complete the proof by showing that the first two assertions imply the third one. Suppose that the first two assertions hold. We may assume that $V = \{0, \dots, 2n\}$, $\mathfrak{S}$ is critical and $\Pi(\mathfrak{S}) = C_{2n+1}$. Moreover, we may assume that $x = 2n$ in the second assertion. Thus, there exists $g \in \mathfrak{S}$ such that $2n$ is an isolated vertex of g, and $g - 2n = (Q_{2n})^a + (R_{2n})^b$, where $a \in \mathbb{A}$ and $b \in \mathbb{A} \setminus \{a, -a\}$. Let $m \in \{0, \dots, 2n-1\}$. Consider the selector

$$\begin{array}{llcl} t: & \{0, \dots, 2n\} & \longrightarrow & \mathbb{A} \\ & m & \longmapsto & 0 \\ & p \in V \setminus \{m\} & \longmapsto & g(m,p). \end{array}$$

We obtain that m is an isolated vertex of g^t. The permutation

$$\begin{array}{llcl} \delta: & \{0, \dots, 2n\} & \longrightarrow & \{0, \dots, 2n\} \\ & p & \longmapsto & p + (2n - m) \pmod{2n+1} \end{array}$$

is an isomorphism from g^t onto $\delta(g^t)$ that maps m to $2n$. Thus $2n$ is an isolated vertex of $\delta(g^t)$. Furthermore, δ is an isomorphism from $\langle g^t \rangle$, that is, $\langle g \rangle = \mathfrak{S}$ onto $\langle \delta(g^t) \rangle$. Since $\mathfrak{S}$ is critical, $\langle \delta(g^t) \rangle$ is as well by Corollary 54. Moreover, by Corollary 55, δ is an isomorphism from $\Pi(\mathfrak{S})$ onto $\Pi(\langle \delta(g^t) \rangle)$. Since $\Pi(\mathfrak{S}) = C_{2n+1}$ and δ is an automorphism of C_{2n+1}, we obtain $\Pi(\langle \delta(g^t) \rangle) = C_{2n+1}$, so $\Pi(\langle \delta(g^t) \rangle) - 2n = P_{2n}$. It follows from Lemma 58 that $\delta(g^t) - 2n$ is critical and $\Pi(\delta(g^t) - 2n) = P_{2n}$. By Theorem 33, there exist $c \in \mathbb{A}$ and $d \in \mathbb{A} \setminus \{c, -c\}$ such that $\delta(g^t) - 2n = (Q_{2n})^c + (R_{2n})^d$. For distinct elements p and q of $\{0, \dots, 2n-1\}$, we have $(\delta(g^t))(p,q) = g(m, p+m+1) + g(p+m+1, q+m+1) - g(m, q+m+1)$, where $p+m+1$ and $q+m+1$ are considered modulo $2n+1$. Since $c = (\delta(g^t))(0,1)$ and $d = (\delta(g^t))(0,2)$, we obtain $c = g(m, m+1) + g(m+1, m+2) - g(m, m+2)$ and $d = g(m, m+1) + g(m+1, m+3) - g(m, m+3)$, where, $m+1$, $m+2$ and $m+3$ are considered modulo $2n+1$. It follows that $c = a$ and $d = b$. Thus $\delta(g^t) - 2n = (Q_{2n})^a + (R_{2n})^b$. Since δ is an isomorphism from g^t onto $\delta(g^t)$ such that $\delta(m) = 2n$, we obtain $g^t - m \simeq (Q_{2n})^a + (R_{2n})^b$. ∎

We need the next lemma to show Theorem 14.

Lemma 62. *Given $n \geq 2$, consider $g \in \mathscr{L}(\{0, \dots, 2n+1\}, \mathbb{A})$. Let φ be the transposition of $\{0, \dots, 2n+1\}$ that exchanges $2n$ and $2n+1$. Given $a \in \mathbb{A}$, consider the selector*

$$\begin{array}{llcl} s: & \{0, \dots, 2n+1\} & \longrightarrow & \mathbb{A} \\ & \{2p : 0 \leq p \leq n\} & \longmapsto & 0 \\ & \{2p+1 : 0 \leq p \leq n-1\} & \longmapsto & a + a \\ & 2n+1 & \longmapsto & a. \end{array} \tag{17}$$

1. *We have $2n+1$ is a 0-isolated vertex of g, and $g-(2n+1) = (T_{2n+1})^a$ if and only if the following four assertions hold*

$$(18) \qquad \begin{cases} 2n \text{ is a } (-a)\text{-isolated vertex of } g^s, \\ \text{for } 0 \leq p \leq n-1,\ g^s(2p, 2n+1) = -a, \\ \text{for } 0 \leq p \leq n-1,\ g^s(2p+1, 2n+1) = a, \\ g^s - \{2n, 2n+1\} = (Q_{2n})^{-(a+a+a)} + (R_{2n})^a. \end{cases}$$

2. *Suppose that* (18) *holds. If $o_{\mathbb{A}}(a) = 4$, then $\varphi_{\restriction \{0,\ldots,2n-1\}\cup\{2n+1\}}$ is an isomorphism from $g^s - (2n)$ onto $(W_{2n+1})^a$.*
3. *Suppose that* (18) *holds. If $a + a \neq 0$ and $\langle g \rangle - (2n+1)$ is decomposable, then $o_{\mathbb{A}}(a) = 4$.*

Proof. We easily verify that the first two assertions hold. To prove the last one, suppose that $\langle g \rangle - (2n+1)$ is decomposable. Since $2n$ is a $(-a)$-isolated vertex of $g^s - (2n+1)$ and $\langle g \rangle - (2n+1) = \langle g^s \rangle - (2n+1)$, it follows from Proposition 7 that $g^s - \{2n, 2n+1\}$, that is, $(Q_{2n})^{-(a+a+a)} + (R_{2n})^a$ is decomposable. We show that

$$\text{if } (a+a+a) \in \mathbb{A} \setminus \{-a, a\}, \text{then } (Q_{2n})^{-(a+a+a)} + (R_{2n})^a \text{ is primitive.}$$

If $n \geq 3$, then it suffices to apply Theorem 33. If $n = 2$, then we verify directly that $(Q_4)^{-(a+a+a)} + (R_4)^a$ is primitive. Since $(Q_{2n})^{-(a+a+a)} + (R_{2n})^a$ is decomposable, we obtain $a+a+a = a$ or $a+a+a = -a$. The first instance is not possible because $a + a \neq 0$. Thus, the second instance holds, so $o_{\mathbb{A}}(a) = 4$. ∎

***Proof of Theorem* 14.** To begin, we prove that the first assertion implies the second one. Suppose that the first assertion holds and consider $g \in \mathfrak{S}$ such that x is a 0-isolated vertex of g. We may assume that $V = \{0, \ldots, 2n+1\}$, $x = 2n+1$ and $\Pi(\mathfrak{S}) - (2n+1) = C_{2n+1}$. By Lemma 58, $g - (2n+1)$ is critical and $\Pi(g-(2n+1)) = \Pi(\mathfrak{S}) - (2n+1) = C_{2n+1}$. It follows from Theorem 37 that

$$g - (2n+1) = (T_{2n+1})^a,$$

where $a \in \mathbb{A}$ such that $a \neq -a$. By the first assertion of Lemma 62, (18) holds. Since $\langle g \rangle - (2n+1)$ is decomposable, it follows from the last assertion of Lemma 62 that $o_{\mathbb{A}}(a) = 4$.

Now, we prove that the second assertion implies the third one. Suppose that the second assertion holds. Up to isomorphy, we may assume that $V = \{0, \ldots, 2n+1\}$, $x = 2n+1$ and $g - (2n+1) = (T_{2n+1})^a$, where $g \in \mathfrak{S}$ such that x is a 0-isolated vertex of g, and $a \in \mathbb{A}$ such that $o_{\mathbb{A}}(a) = 4$. By the first assertion of Lemma 62, (18) holds. Let $y \in \{0, \ldots, 2n\}$. Since the permutation of $\{0, \ldots, 2n+1\}$ defined by

$$\begin{array}{rcl} \{0,\ldots,2n+1\} & \longrightarrow & \{0,\ldots,2n+1\} \\ p \in \{0,\ldots,2n\} & \longmapsto & p+(2n-y) \pmod{2n+1} \\ 2n+1 & \longmapsto & 2n+1, \end{array}$$

is an automorphism of g that maps y to $2n$, it suffices to prove the third assertion holds for $y = 2n$. Let s be the selector defined in (17), and let φ be the transposition of $\{0,\ldots,2n+1\}$ that exchanges $2n$ and $2n+1$. Since (18) holds and $o_{\mathbb{A}}(a) = 4$, it follows from the second assertion of Lemma 62 that $\varphi_{\restriction\{0,\ldots,2n-1\}\cup\{2n+1\}}$ is an isomorphism from $g^s - (2n)$ onto $(W_{2n+1})^a$. Consequently, the third assertion holds with $b = a$. Hence, observe that if the last two assertions hold, then $a = b$.

Lastly, we prove that the third assertion implies the first one. Suppose that the third assertion holds. Up to isomorphy, we may assume that $V = \{0,\ldots, 2n+1\}$, $x = 2n$ and there exists $h \in \mathfrak{S}$ such that $2n+1$ is a $(-b)$-isolated vertex of h and $h-(2n+1) = (W_{2n+1})^b$, where $b \in \mathbb{A}$ such that $o_{\mathbb{A}}(b) = 4$. By Theorem 41, $h-(2n+1)$ is critical. It follows from Lemma 52 that $\mathfrak{S}$ is primitive and

$$\{0,\ldots,2n\} \subseteq \mathcal{C}(\mathfrak{S}). \tag{19}$$

To prove that $2n+1 \in \mathcal{C}(\mathfrak{S})$, we proceed at follows. Let φ be the transposition of $\{0,\ldots,2n+1\}$ that exchanges $2n$ and $2n+1$. We obtain that $2n$ is a $(-b)$-isolated vertex of $\varphi(h)$ and $\varphi(h) - \{2n, 2n+1\} = (L_{2n})^b$. Furthermore, for $0 \leq p \leq n-1$, $(\varphi(h))(2p, 2n+1) = -b$ and $(\varphi(h))(2p+1, 2n+1) = b$. Since $\varphi(h) - \{2n, 2n+1\} = (L_{2n})^b$ and $o_{\mathbb{A}}(b) = 4$, we obtain

$$\varphi(h) - \{2n, 2n+1\} = (Q_{2n})^{-(b+b+b)} + (R_{2n})^b.$$

Let s be the selector defined in (17). After replacing b by a, and $\varphi(h)$ by g^s, we obtain that (18) holds. Consider the selector t obtained from s by replacing a by b. It follows from the first assertion of Lemma 62 that $2n+1$ is a 0-isolated vertex of $(\varphi(h))^{-t}$ and $(\varphi(h))^{-t} - (2n+1) = (T_{2n+1})^b$. Clearly, $\langle(\varphi(h))^{-t}\rangle = \langle\varphi(h)\rangle$. By Proposition 16, $\langle\varphi(h)\rangle = \varphi(\langle h\rangle)$. Therefore, $\langle(\varphi(h))^{-t}\rangle = \varphi(\mathfrak{S})$. Since $(\varphi(h))^{-t} - (2n+1) = (T_{2n+1})^b$ and $b+b \neq 0$, $(\varphi(h))^{-t} - (2n+1) = (T_{2n+1})^b$ is critical by Theorem 37. It follows from Lemma 52 that $2n \in \mathcal{C}(\varphi(\mathfrak{S}))$. Since $\mathcal{C}(\varphi(\mathfrak{S})) = \varphi(\mathcal{C}(\mathfrak{S}))$ by Corollary 53, we obtain that $2n+1 \in \mathcal{C}(\mathfrak{S})$. By (19), $\mathfrak{S}$ is critical. We have $\Pi(h-(2n+1)) = \Pi((W_{2n+1})^b)$. By Theorem 41, $2n$ is an isolated vertex of $\Pi(h-(2n+1))$. Since $\Pi(h-(2n+1)) = \Pi(\mathfrak{S}) - (2n+1)$ by Lemma 58, it follows from Lemma 56 that $2n$ is an isolated vertex of $\Pi(\mathfrak{S})$. By Corollary 11, $\Pi(\mathfrak{S}) - (2n) \simeq C_{2n+1}$. To conclude, it suffices to recall that $2n = x$. ∎

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