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THE THICKNESS OF AMALGAMATIONS AND CARTESIAN PRODUCT OF GRAPHS

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Abstract

The thickness of a graph is the minimum number of planar spanning subgraphs into which the graph can be decomposed. It is a measurement of the closeness to the planarity of a graph, and it also has important applications to VLSI design, but it has been known for only few graphs. We obtain the thickness of vertex-amalgamation and bar-amalgamation of graphs, the lower and upper bounds for the thickness of edge-amalgamation and 2-vertex-amalgamation of graphs, respectively. We also study the thickness of Cartesian product of graphs, and by using operations on graphs, we derive the thickness of the Cartesian product $K_n \Box P_m$ for most values of m and n.

Keywords: thickness, amalgamation, Cartesian product, genus.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph is said to be *planar* if it can be drawn on the plane without edge crossings. Suppose G_1, $G_2, \ldots, G_k$ are spanning subgraphs of G; if $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) = E(G)$

and $E(G_i) \cap E(G_j) = \emptyset$ ($i \neq j$, $i, j = 1, 2, \ldots, k$), then $\{G_1, G_2, \ldots, G_k\}$ is a *decomposition* of G. Furthermore, if $G_1, G_2, \ldots, G_k$ are all planar graphs, then $\{G_1, G_2, \ldots, G_k\}$ is a *planar decomposition* of G. The minimum number of planar spanning subgraphs over all possible planar decompositions of G is called the *thickness* of G, denoted by $\theta(G)$.

The thickness of a graph was first defined by Tutte [21] in 1963. As a topological invariant of a graph, it is an important research object in topological graph theory, and it also has important applications to VLSI design [1]. But the results about thickness are few, compared with other topological invariants, e.g., genus and crossing number. The only types of graphs whose thickness have been obtained are complete graphs [3, 6, 22], complete bipartite graphs [7] and hypercubes [16]. Since determining the thickness of a graph is NP-hard [17], it is very difficult to get the exact number of thickness for arbitrary graphs, and people study the lower and upper bounds for the thickness of a graph [12, 14] and introduce heuristic algorithms to approximate it [11, 20]. Some relations between thickness and other topological invariants, such as genus, are also established [4]. The reader is referred to [18, 19] for more background and results about the thickness problems.

In this paper, the thickness of graphs that are formed from vertex-amalgamation and bar-amalgamation of any two graphs are given. The lower and upper bounds for the thickness of graphs that are obtained by edge-amalgamation and 2-vertex-amalgamation of any two graphs are also derived. Some results about the thickness of Cartesian product graph are also obtained, in particular, the thickness of the Cartesian product $K_n \Box P_m$ is obtained for most values of m and n (K_n is the complete graph with n vertices and P_m is the path with m vertices).

Graphs in this paper are simple. For undefined terminology, see [9].

2. Thickness of Graph Amalgamations

The *union* of graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *intersection* $G_1 \cap G_2$ of G_1 and G_2 is defined analogously.

Let G_1 and G_2 be subgraphs of a graph G. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$ (a vertex of G), then we say that G is the *vertex-amalgamation* of G_1 and G_2 at vertex v, denoted $G = G_1 \bigvee^1_{\{v\}} G_2$. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{u, v\}$ (two distinct vertices of G), then we say that G is the 2*-vertex-amalgamation* of G_1 and G_2 at vertices u and v, denoted $G = G_1 \bigvee^1_{\{u,v\}} G_2$. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{e\}$ (an edge of G), then we say that G is the *edge-amalgamation* of G_1 and G_2 on edge e, denoted $G = G_1 \bigvee^2_{\{e\}} G_2$.

Let G and H be two disjoint graphs. The *bar-amalgamation* of G and H is obtained by adding a new edge between a vertex of G and a vertex of H.

The four kinds of amalgamations defined above are important operations on graphs; by these amalgamations, one can create larger graphs (i.e., graphs with larger order) from small ones. It is a general method to study problems in graph theory by using operations on graphs. In the following, we list some results about genus of graph amalgamations which will be applied in our proof.

The *genus* of a graph G, denoted by $\gamma(G)$, is the minimum integer k such that G can be embedded on the orientable surface of genus k. A graph G is planar if and only if $\gamma(G) = 0$.

Lemma 1 [5]. *If G is the vertex-amalgamation of G_1 and G_2, then*

$$\gamma(G) = \gamma(G_1) + \gamma(G_2).$$

Lemma 2 [10]. *If G is the bar-amalgamation of G_1 and G_2, then*

$$\gamma(G) = \gamma(G_1) + \gamma(G_2).$$

Lemma 3 [2]. *If G is the edge-amalgamation of G_1 and G_2, then*

$$\gamma(G) \le \gamma(G_1) + \gamma(G_2).$$

Lemma 4 [13]. *If G is the 2-vertex-amalgamation of G_1 and G_2, then*

$$\gamma(G_1) + \gamma(G_2) - 1 \le \gamma(G) \le \gamma(G_1) + \gamma(G_2) + 1.$$

In [4], a relation between genus and thickness of a graph was given as follows.

Lemma 5 [4]. *If G is a graph with genus 1, then the thickness of G is 2.*

In the following, some results about the thickness of vertex-amalgamation, bar-amalgamation, edge-amalgamation and 2-vertex-amalgamation of graphs are obtained.

Theorem 6. *If G is the vertex-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then*

$$\theta(G) = \max\{n_1, n_2\}.$$

Proof. Without loss of generality, one can assume that n_1 is not less than n_2 and $G_1 \cap G_2 = \{v\}$ (a vertex of G). Suppose that $\{G_{11}, G_{12}, \ldots, G_{1n_1}\}$ is a planar decomposition of G_1 and $\{G_{21}, G_{22}, \ldots, G_{2n_1}\}$ is a planar decomposition of G_2. From Lemma 1,

$$\gamma\left(G_{1i} \bigvee\nolimits_{\{v\}}^{1} G_{2i}\right) = \gamma(G_{1i}) + \gamma(G_{2i}) = 0, \qquad 1 \le i \le n_1.$$

Hence $\{G_{11} \bigvee_{\{v\}}^1 G_{21}, G_{12} \bigvee_{\{v\}}^1 G_{22}, \ldots, G_{1n_1} \bigvee_{\{v\}}^1 G_{2n_1}\}$ is a planar decomposition of G, which shows $\theta(G) \le n_1$. On the other hand, G_1 is a subgraph of G and $\theta(G_1) = n_1$, so we have $\theta(G) \ge n_1$. Summarizing the above, the thickness of G is n_1, and the theorem follows. ∎

Theorem 7. *If G is the bar-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then*

$$\theta(G) = \max\{n_1, n_2\}.$$

Proof. Suppose that $n_1 \geq n_2$ and edge e is the new edge between G_1 and G_2. Let $\{G_{11}, G_{12}, \ldots, G_{1n_1}\}$ be a planar decomposition of G_1 and $\{G_{21}, G_{22}, \ldots, G_{2n_1}\}$ be a planar decomposition of G_2. $G_{11} \cup G_{21} \cup e$ is the bar-amalgamation of G_{11} and G_{21}; from Lemma 2, the genus of $G_{11} \cup G_{21} \cup e$ is zero, that is to say, $G_{11} \cup G_{21} \cup e$ is a planar graph. Hence $\{G_{11} \cup G_{21} \cup e, G_{12} \cup G_{22}, \ldots, G_{1n_1} \cup G_{2n_1}\}$ is a planar decomposition of G, which shows $\theta(G) \leq n_1$. For $G = G_1 \cup G_2 \cup e$ and $\theta(G_1) = n_1$, we have $\theta(G) \geq n_1$. Summarizing the above, the thickness of G is n_1, and the theorem is obtained. ■

Theorem 8. *If G is the edge-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then*

$$\max\{n_1, n_2\} \leq \theta(G) \leq \max\{n_1, n_2\} + 1.$$

Proof. Suppose that n_1 is not less than n_2 and $G_1 \cap G_2 = \{e\}$ (an edge of G), the two end vertices of e are u and v. Let $\{G_{11}, G_{12}, \ldots, G_{1n_1}\}$ be a planar decomposition of G_1 and without loss of generality, we can assume $e \in E(G_{11})$. Let E_{uv} be the set of edges that are incident with u or v in G_2. It is easy to see that the graph $G_{11} \cup E_{uv}$ is a planar graph. Let $\{G_{21}, G_{22}, \ldots, G_{2n_2}\}$ be a planar decomposition of $G_2 - E_{uv}$.

(1) If $n_1 > n_2$, then $\{G_{11} \cup E_{uv}, G_{12} \cup G_{21}, \ldots, G_{1n_2+1} \cup G_{2n_2}, G_{1n_2+2}, \ldots, G_{1n_1}\}$ is a planar decomposition of G, which shows $\theta(G) \leq n_1$.
(2) If $n_1 = n_2$, then $\{G_{11} \cup E_{uv}, G_{12} \cup G_{21}, \ldots, G_{1n_1} \cup G_{2n_2-1}, G_{2n_2}\}$ is a planar decomposition of G, which shows $\theta(G) \leq n_1 + 1$.

For $G = G_1 \bigvee^2_{\{e\}} G_2$ and $\theta(G_1) = n_1$, we have $\theta(G) \geq n_1$. Summarizing the above, the theorem follows. ■

From the proof of Theorem 8, if G is the edge-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then $\theta(G) = \max\{n_1, n_2\}$, when $n_1 \neq n_2$; $\theta(G)$ is either $\max\{n_1, n_2\}$ or $\max\{n_1, n_2\} + 1$, when $n_1 = n_2$.

Theorem 9. *If G is the 2-vertex-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then*

$$\max\{n_1, n_2\} \leq \theta(G) \leq \max\{n_1, n_2\} + 1.$$

Proof. Suppose that $G_1 \cap G_2 = \{u, v\}$ (two distinct vertices of G), E_{1v} and E_{2v} are the sets of edges that are incident with v in G_1 and G_2, respectively. Then $G - E_{1v} - E_{2v}$ can be seen as the vertex-amalgamation of $G_1 - E_{1v}$ and

$G_2 - E_{2v}$ at the vertex u. From Theorem 6, there exists a planar decomposition of $G - E_{1v} - E_{2v}$ with $n = \max\{n_1, n_2\}$ planar subgraphs, and $\theta(G) \geq n$. Obviously, the subgraph induced by $E_{1v} \cup E_{2v}$ is a planar graph. So there is a planar decomposition of G with $n + 1$ planar subgraphs, which show $\theta(G) \leq n + 1$. Summarizing the above, the theorem follows. ∎

With a similar argument as in the proof of Theorem 9, one can obtain the following theorem about the q-vertex-amalgamation ($q \geq 3$) of two graphs.

Theorem 10. *If G is the q-vertex-amalgamation of G_1 and G_2, $\theta(G_1) = n_1$ and $\theta(G_2) = n_2$, then*

$$\max\{n_1, n_2\} \leq \theta(G) \leq \max\{n_1, n_2\} + q - 1.$$

3. Thickness of the Cartesian Product of Two Graphs

The *Cartesian product* of graphs G and H is the graph $G \square H$ with vertex set

$$V(G \square H) = V(G) \times V(H)$$

and edge set

$$E(G \square H) = \{(g, h)(g', h') \mid gg' \in E(G) \text{ and } h = h', \text{ or } hh' \in E(H) \text{ and } g = g'\}.$$

For any $h \in V(H)$, we denote by G^h the subgraph of $G \square H$ induced by $V(G) \times \{h\}$; it is isomorphic to G and called a *G-fiber*. The *H-fiber* is defined analogously.

3.1. Thickness of the Cartesian product of a t-minimal graph and an outerplanar graph

A graph G is said to be *t-minimal*, if every proper subgraphs of it have the thickness less than t. There are only two 2-minimal graphs, i.e., K_5 and $K_{3,3}$, up to homeomorphism. The only known t-minimal complete graph is K_9 for $t = 3$. A graph is an *outerplanar graph* if it can be embedded in the plane without crossings in such a way that all of the vertices belong to the unbounded region of the embedding.

Theorem 11 [8]**.** *Let G and H be connected graphs. Then the graph $G \square K_2$ is planar if and only if G is outerplanar.*

Theorem 12. *Let G be a t-minimal graph and H be an outerplanar graph. Then $\theta(G \square H) = \theta(G)$.*

Proof. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. Because G is t-minimal and the removal of a single edge from a graph cannot reduce the thickness of the graph by more than one, for any $e \in E(G)$, we have $\theta(G - e) = t - 1$. Without loss of generality, we suppose that $e = v_1v_2$ and the graph K_2 consists of the edge e. From the structure of $G\Box H$, we have $G\Box H = ((G - e)\Box H) \cup (K_2\Box H)$.

From Theorem 11, $K_2\Box H$ is a planar graph. The H fibers $H^{v_3}, H^{v_4}, \ldots, H^{v_n}$ are also planar graphs. Furthermore, the graph $(K_2\Box H) \cup H^{v_3} \cup H^{v_4} \cup \cdots \cup H^{v_n}$ is a planar graph, since it is the union of $n-1$ disjoint planar graphs; denote it by G_t. The removal of the edges of the subgraph G_t from $G\Box H$ leaves $|V(H)|$ copies of disjoint graphs $G - e$, which can be decomposed into $t - 1$ subgraphs, because $\theta(G - e) = t - 1$. Summarizing the above, we can get a planar decomposition of $G\Box H$ with t subgraphs, i.e., $\theta(G\Box H) \leq t$.

On the other hand, since $G \subset G\Box H$, we have $\theta(G\Box H) \geq t$. The theorem follows. ∎

Corollary 13. *Let G be a t-minimal graph and C_m be a cycle graph. Then*

$$\theta(G\Box C_m) = \theta(G).$$

Corollary 14. *Let G be a t-minimal graph and P_n be a path graph. Then*

$$\theta(G\Box P_n) = \theta(G).$$

3.2. The thickness of $K_n\Box P_2$, $n \geq 2$

In the following, by using operations on graphs and some conclusions above, we obtain the thickness of $K_n\Box P_m$, for $n, m \geq 2$.

Lemma 15 [3, 6, 22]**.** *The thickness of the complete graph K_n is $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$, except that $\theta(K_9) = \theta(K_{10}) = 3$.*

Let K_n^1 be the complete graph with vertices $v_1, v_2, \ldots, v_n$. K_n^2 is a copy of K_n^1 and its vertices are labeled with $u_1, u_2, \ldots, u_n$, respectively. By joining the vertices v_i and u_i with an edge v_iu_i, $1 \leq i \leq n$, we get the graph $K_n\Box P_2$. Figure 1 illustrates $K_5\Box P_2$. From a planar decomposition of $K_n\Box P_2$, by contracting the edges from K_n^2 to a single vertex in all planar subgraphs, one can obtain a planar decomposition of K_{n+1}, so we have

$$\theta(K_n\Box P_2) \geq \theta(K_{n+1}). \tag{1}$$

By inserting a vertex w_i on edge v_iu_i, for $1 \leq i \leq n$, and merging these n 2-valent vertices $w_1, w_2, \ldots, w_n$ into one vertex w, one can get a new graph. This graph can also be seen as the vertex-amalgamation of K_{n+1} and K_{n+1} at w, denoted by $K_{n+1} \bigvee_{\{w\}}^1 K_{n+1}$. Figure 2 shows the graph $K_6 \bigvee_{\{w\}}^1 K_6$.

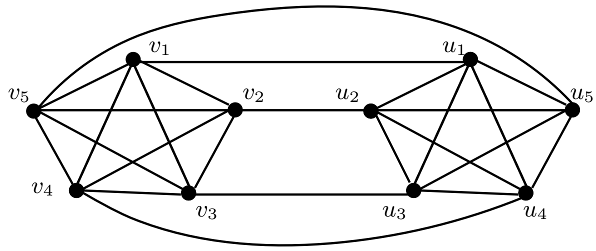

Figure 1. The graph $K_5 \square P_2$.

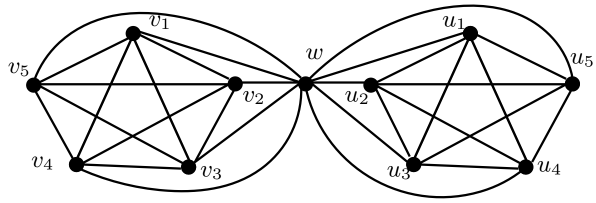

Figure 2. The graph $K_6 \bigvee^1_{\{w\}} K_6$.

From Theorem 6, the thickness of $K_{n+1} \bigvee^1_{\{w\}} K_{n+1}$ is the same as the thickness of K_{n+1}. Let $\theta(K_{n+1}) = t$ and $\{G_1, G_2, \ldots, G_t\}$ be a planar decomposition of K_{n+1}. Then one can get a planar decomposition of $K_{n+1} \bigvee^1_{\{w\}} K_{n+1}$ as follows,

$$\left\{G_1 \bigvee^1_{\{w\}} G_1, G_2 \bigvee^1_{\{w\}} G_2, \ldots, G_t \bigvee^1_{\{w\}} G_t\right\}$$

in which $G_i \bigvee^1_{\{w\}} G_i$, $1 \le i \le t$ are planar graphs. A planar decomposition of $K_6 \bigvee^1_{\{w\}} K_6$ is shown in Figure 3.

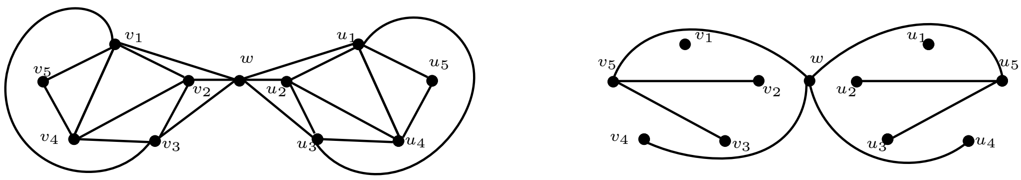

Figure 3. A planar decomposition of $K_6 \bigvee^1_{\{w\}} K_6$.

From the construction of $G_i \bigvee^1_{\{w\}} G_i$, if the edge $v_q w \in G_i \bigvee^1_{\{w\}} G_i$, then $u_q w \in G_i \bigvee^1_{\{w\}} G_i$, $1 \le q \le n$. For each graph $G_i \bigvee^1_{\{w\}} G_i$, $1 \le i \le t$, if $v_q w$, $u_q w \in G_i \bigvee^1_{\{w\}} G_i$, then we replace them by a new edge $v_q u_q$, for $1 \le q \le n$, and delete the vertex w. In this way, we obtain a new planar decomposition, which is exactly a planar decomposition of $K_n \square P_2$. Figure 4 illustrates a planar decomposition of $K_5 \square P_2$ by using this way.

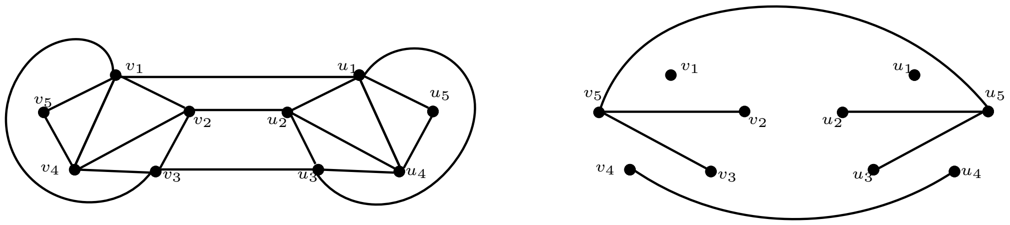

Figure 4. A planar decomposition of $K_5\Box P_2$.

From the argument and construction above, one can get a planar decomposition of $K_n\Box P_2$ from that of $K_{n+1}\bigvee^1_{\{w\}} K_{n+1}$, so we have

$$\theta(K_n\Box P_2)\le\theta\left(K_{n+1}\bigvee\nolimits^1_{\{w\}}K_{n+1}\right)=\theta(K_{n+1}). \tag{2}$$

Theorem 16. *The thickness of the Cartesian product $K_n\Box P_2$ $(n\ge 2)$ is*

$$\theta(K_n\Box P_2)=\left\lfloor\frac{n+8}{6}\right\rfloor,$$

except that $\theta(K_8\Box P_2)=\theta(K_9\Box P_2)=3$.

Proof. From (1) and (2), we obtain that $\theta(K_n\Box P_2)=\theta(K_{n+1})$. By Lemma 15, the theorem follows. ■

3.3. The thickness of $K_n\Box P_m$, $n\ge 2, m\ge 3$

We use the method similar to that in Section 3.2. Firstly, we insert a 2-valent vertex into each "path edge" (the edges come from P_m). Secondly, we merge these $(m-1)n$ 2-valent vertices into $m-1$ vertices, each of which joint two adjacent K_n; then we get a new graph $\tilde{G}$. The graph $\tilde{G}$ can be seen as a vertex-amalgamation of m graphs, in which the first and the mth graphs are K_{n+1}, the others are $K_{n+2}-e$. From Theorem 6, one can get $\theta(\tilde{G})=\theta(K_{n+2}-e)$. In the following, we will construct a planar decomposition of $K_n\Box P_m$ $(m\ge 3)$ from a planar decomposition of $\tilde{G}$, which shows that

$$\theta(K_n\Box P_m)\le\theta(K_{n+2}-e)\le\theta(K_{n+2}). \tag{3}$$

Suppose that $\{G_1,G_2,\ldots,G_j\}$ is a planar decomposition of $K_{n+2}-e$, in which the vertices of K_{n+2} are labeled with $v_1,v_2,\ldots,v_{n+2}$, respectively and $e=v_{n+1}v_{n+2}$. For each $1\le i\le j$, we do a vertex-amalgamation of m graphs G_i as follows

$$G_i\bigvee\nolimits^1_{\{v_{n+1}\}}G_i\bigvee\nolimits^1_{\{v_{n+2}\}}G_i\bigvee\nolimits^1_{\{v_{n+1}\}}G_i\cdots\bigvee\nolimits^1_{\{v_p\}}G_i$$

in which $p = v_{n+2}$ when m is odd, and $p = v_{n+1}$ when m is even. Denote the resulting graph by $\widehat{G_i}$. For each $\widehat{G_i}$ $(1 \leq i \leq j)$, we delete the vertex v_{n+2} and the edges incident with it in the first G_i, delete the vertex v_{n+1} or v_{n+2} and the edges incident with it in the mth G_i according to m is odd or even. Denote the resulting graph by $\tilde{G}_i$. Then $\{\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_j\}$ is a planar decomposition of $\tilde{G}$. Finally, we delete $m-1$ vertices v_{n+1} and v_{n+2} in $\tilde{G}_i$, $1 \leq i \leq j$, and replace them by "path edge" as in Section 3.2, and denote the obtained graph by $\overline{G_i}$, $1 \leq i \leq j$. Clearly, $\{\overline{G_1}, \overline{G_2}, \ldots, \overline{G_j}\}$ is a planar decomposition of $K_n \square P_m$, $m \geq 3$. Figure 5 shows a planar decomposition of a vertex-amalgamation of four graphs $K_7 - e$ and a planar decomposition of $K_5 \square P_4$ from it is illustrated in Figure 6.

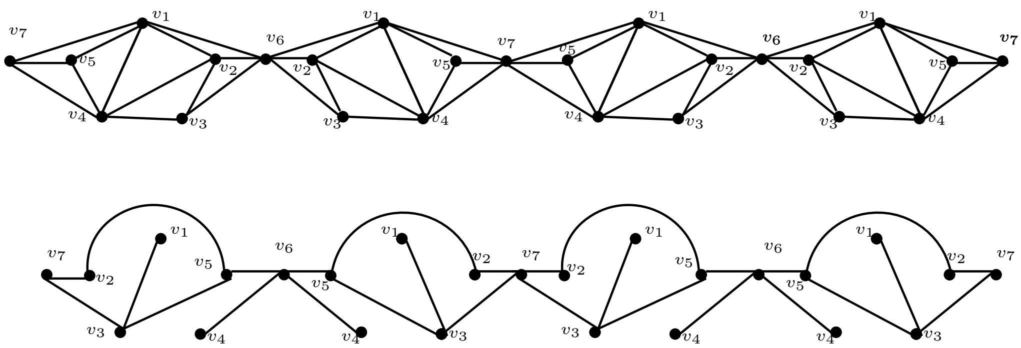

Figure 5. A planar decomposition of a vertex-amalgamation of 4 graphs $K_7 - e$.

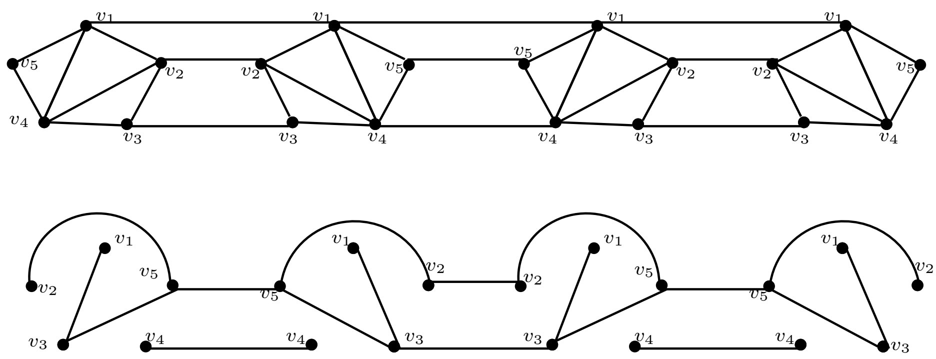

Figure 6. A planar decomposition of $K_5 \square P_4$.

On the other hand, $K_n \square P_2$ is a subgraph of $K_n \square P_m$ $(m \geq 3)$, and combing it with (1), we have

$$\theta(K_n \square P_m) \geq \theta(K_n \square P_2) \geq \theta(K_{n+1}). \tag{4}$$

Theorem 17. *The thickness of the Cartesian product $K_n \square P_m$ $(n \geq 2, m \geq 3)$ is*

$$\theta(K_n \square P_m) = \left\lfloor \frac{n+9}{6} \right\rfloor,$$

except that $\theta(K_3 \square P_m) = 1$, $\theta(K_8 \square P_m) = 3$ and possibly when $n = 6p+3$ $(p \geq 2)$.

Proof. When $n \neq 7$, from (3), (4) and Lemma 15, we obtain $\theta(K_n \square P_m) = \theta(K_{n+2})$, except possibly when $n = 6p + 3$ (p is a nonnegative integer).

When $n = 3$, because $\theta(K_4) \leq \theta(K_3 \square P_m) \leq \theta(K_5 - e)$ and both K_4 and $K_5 - e$ are planar graphs, we have $\theta(K_3 \square P_m) = 1$.

When $n = 9$, because $\theta(K_{10}) \leq \theta(K_9 \square P_m) \leq \theta(K_{11})$ and $\theta(K_{10}) = \theta(K_{11}) = 3$, we have $\theta(K_9 \square P_m) = 3$.

When $n = 8$, we have $\theta(K_8 \square P_m) = \theta(K_{10}) = 3$. When $n = 7$, we have $2 \leq \theta(K_7 \square P_m) \leq \theta(K_9 - e)$. We give a planar decomposition of $K_9 - e$ as shown in Figure 7, and $K_9 - e$ is a non-planar graph, which shows $\theta(K_9 - e) = 2$. So we have $\theta(K_7 \square P_m) = 2$.

Summarizing the above, the theorem is obtained. ■

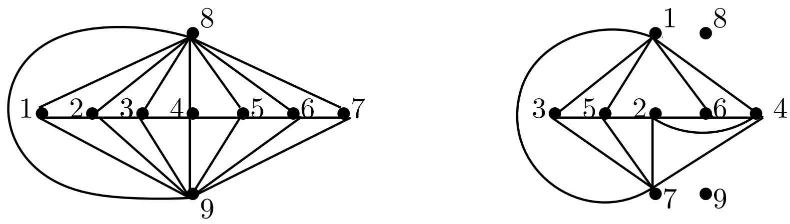

Figure 7. A planar decomposition of $K_9 - e$.

From Theorems 16 and 17, the only unsolved case for the thickness of the Cartesian product $K_n \square P_m$ is when is $n = 6p + 3$ $(p \geq 2)$ and $m \geq 3$. For this case, $\theta(K_n \square P_m) = \theta(K_{n+1})$ or $\theta(K_{n+2} - e)$. What is the exact number for this case is still open. It was conjectured in [15] that K_{6t-7} is t-minimal for $t \geq 5$. If this conjecture is true, then $\theta(K_n \square P_m) = \theta(K_{n+1}) = \theta(K_{n+2} - e) = \left\lfloor \frac{n+8}{6} \right\rfloor$, for $n = 6p + 3$ $(p \geq 3)$ and $m \geq 3$.

The method of the current paper is not strong enough to determine the thickness of the Cartesian product of the complete graph K_n and the cycle graph C_m. We pose the following problem for possible consideration.

Problem 18. Find an explicit formula for the thickness of the Cartesian product of the complete graph K_n and the cycle graph C_m for $n, m \geq 3$.

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