Discussiones Mathematicae
Graph Theory 37 (2017) 623–632
doi:10.7151/dmgt.1941

ON THE MAXIMUM AND MINIMUM SIZES OF A GRAPH WITH GIVEN k-CONNECTIVITY

Yuefang Sun

Department of Mathematics
Shaoxing University
Zhejiang 312000, P.R. China

e-mail: yfsun2013@gmail.com

Abstract

The concept of k-connectivity $\kappa_k(G)$, introduced by Chartrand in 1984, is a generalization of the cut-version of the classical connectivity. For an integer $k \geq 2$, the k-connectivity of a connected graph G with order $n \geq k$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. In this paper, we get a sharp upper bound for the size of G with $\kappa_k(G) = t$, where $1 \leq t \leq n - k$ and $k \geq 3$; moreover, the unique extremal graph is given. Based on this result, we get the exact values for the maximum size, denoted by $g(n, k, t)$, of a connected graph G with order n and $\kappa_k(G) = t$. We also compute the exact values and bounds for another parameter $f(n, k, t)$ which is defined as the minimum size of a connected graph G with order n and $\kappa_k(G) = t$, where $1 \leq t \leq n - k$ and $k \geq 3$.

Keywords: k-connectivity, generalized connectivity, connectivity.

2010 Mathematics Subject Classification: 05C35, 05C40.

1. Introduction

We refer to book [1] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ be the set of vertices, the set of edges of G, respectively. For $X \subseteq V(G)$, we denote by $G \setminus X$ the subgraph obtained by deleting from G the vertices of X together with the edges incident with them. For a set S, we use $|S|$ to denote the size of it. We use P_n, C_m and K_ℓ to denote a path of order n, a cycle of order m and a complete graph of order ℓ, respectively.

Connectivity is one of the most basic concepts in graph theory, both in a combinatorial sense and in an algorithmic sense. In theoretical computer science, connectivity is a basic measure of reliability of networks. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set $X \subseteq V(G)$ such that $G \setminus X$ is disconnected or has only one vertex. This definition is called the *cut-version* definition of the connectivity. A well-known theorem of Menger provides an equivalent definition, which can be called the *path-version* definition of the connectivity. For any two distinct vertices x and y in G, the local connectivity $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y)|x, y \in V(G), x \neq y\}$ is defined to be the connectivity of G.

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings and so people tried to generalize this concept. The generalized k-connectivity of a graph G which was mentioned by Hager [6] in 1985 is a natural generalization of the path-version definition of the connectivity. There are many results on this type of generalized connectivity, see [3, 5–8, 10–12, 15–21] The reader is also referred to a recent survey [9] on the state-of-the-art of research on the generalized k-connectivity and their applications.

For the cut-version definition of the connectivity, the above minimum vertex set does not take into account the number of components of $G \setminus X$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n-1}$ and the path P_n ($n \geq 3$) are both trees of order n and therefore have connectivity 1, but the deletion of a cut-vertex from $K_{1,n-1}$ produces a graph with $n-1$ components while the deletion of a cut-vertex from P_n produces only two components. Chartrand *et al.* [2] generalized the cut-version definition of the connectivity as follows: For an integer $k \geq 2$ and a graph G of order $n \geq k$, the *k-connectivity* $\kappa_k(G)$ is the smallest number of vertices whose removal from G produces a graph with at least k components or a graph with fewer than k vertices. By definition, we clearly have $\kappa_2(G) = \kappa(G)$. Thus, the concept of k-connectivity could be seen as a generalization of the classical connectivity. For more details about this topic, we refer to [2, 4, 13, 14, 22].

By definition, we know that the k-connectivity takes into account not only the number of vertices whose removal disconnects a graph, but also how many "pieces" it falls into. There is another graph parameter, the toughness of a graph, which also compares the number of vertices removed to the number of resulting pieces. In the following, we use $\omega(G)$ to denote the number of components of a graph G. Let h be a positive real number. A connected graph G is *h-tough* if $\omega(G \setminus X) \leq |X|/h$ for every vertex cut X of G. The largest value of h, denoted by $\tau(G)$, for which a graph is h-tough is called its *toughness*. By definition, we know that if G is not complete, then $\tau(G) = \min\{|X|/\omega(G \setminus X)\}$, where the minimum is taken over all vertex cuts in G. So there is a natural question: Is there any relationship between k-connectivity and toughness? Indeed, we can derive that $\kappa_k(G) \geq k\tau(G)$ under the condition that $\kappa_k(G) \leq n-k$. On one hand, if $\kappa_k(G) \leq n-k$, then there exists a subset $X_0 \subseteq V(G)$ with $|X_0| = \kappa_k(G)$ such that $\omega(G \setminus X_0) \geq k$ by the definition of $\kappa_k(G)$. On the other hand, we know that $\omega(G \setminus X_0) \leq |X_0|/\tau(G)$ by the definition of the toughness. Thus, $\kappa_k(G)/\tau(G) = |X_0|/\tau(G) \geq k$, that is, $\kappa_k(G) \geq k\tau(G)$. These bound is tight since we can consider the cycle C_n. Clearly, we have $\kappa_k(C_n) = k$ and $\tau(C_n) = 1$ and then $\kappa_k(C_n) = k\tau(C_n)$. There is another question: Does the equality $\kappa_k(G) = k\tau(G)$ always hold? In the following, we will give two examples which show that in some cases we have $\kappa_k(G) > k\tau(G)$. The wheel graph W_n is defined as the join of C_n and K_1, constructed by joining a new vertex to every vertex of C_n. It is not hard to show that $\kappa_k(W_n) = k+1$ for $n \geq 2k$ (this condition guarantees that $\kappa_k(W_n) \leq n+1-k$). By definition, we have $\tau(W_n) = \min\{|X|/\omega(W_n \setminus X)\}$, where the minimum is taken over all vertex cuts in W_n. Furthermore, let $n \geq 2k$ be even, and $\mathcal{X}_1$ be the set of vertex cuts X with $3 \leq |X| \leq n/2+1$ and $\mathcal{X}_2$ be the set of remaining vertex cuts of W_n. Clearly, each vertex cut X of W_n must contain the unique vertex of degree n, and so $\omega(W_n \setminus X) \leq |X|-1$. It is not hard to show that

$$\begin{aligned}\min\{|X|/\omega(W_n \setminus X)|X \in \mathcal{X}_1\} &= \min\{|X|/(|X|-1)|X \in \mathcal{X}_1\} \\ &= (n/2+1)/(n/2) = (n+2)/n\end{aligned}$$

and

$$\min\{|X|/\omega(W_n \setminus X)|X \in \mathcal{X}_2\} > (n/2+2)/(n/2) = (n+4)/n > (n+2)/n.$$

Thus, for an even integer $n \geq 2k$, we have $\tau(W_n) = (n+2)/n$, and so $\kappa_k(W_n) = k+1 > k(n+2)/n = k\tau(W_n)$. Another example is the star graph $K_{1,n}$, where $n \geq k+1$. We clearly have $\kappa_k(K_{1,n}) = 1$ and $\tau(K_{1,n}) = 1/n$, so $\kappa_k(K_{1,n}) = 1 > k\tau(K_{1,n}) = k/n$.

In [2] and [13], several sufficient conditions for $\kappa_k(G) \geq t$ were provided. In Section 2, we will study a necessary condition, and get a sharp upper bound for the size of a graph G with $\kappa_k(G) = t$ where $1 \leq t \leq n-k$; moreover, the unique extremal graph $G(n,k,t)$ will be given (Theorem 5). For three integers n,k,t with $1 \leq t \leq n-k+1$ and $k \geq 3$, the parameter $g(n,k,t)$ is defined as the maximum size of a connected graph G with order n and $\kappa_k(G) = t$. Based on Theorem 5, we will get the exact values of $g(n,k,t)$ (Theorem 6). We will also investigate another parameter $f(n,k,t)$ which is defined as the minimum size of a connected graph G with order n and $\kappa_k(G) = t$ where $1 \leq t \leq n-k$ and $k \geq 3$. In Section 3, we will compute the exact values and bounds for $f(n,k,t)$ (Theorem 13).

2. Exact Values for $g(n,k,t)$

For two integers s,n with $1 \leq s \leq n-1$, the graph class $G(x,s)$ is defined as follows in [22]. For each graph $G \in G(x,s)$, there exists a cut vertex x such that $G \setminus \{x\}$ contains at least s components. By definition, for any $s_1 < s_2$, we have that $G(x,s_2)$ is a subclass of $G(x,s_1)$. We use $\alpha(G)$ to denote the independence number of a graph G. The following result will be useful in our later argument.

Proposition 1 [22]**.** *Let k,n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa_k(G) \leq n-k+1$. Moreover, $\kappa_k(G) = 1$ if and only if $k = n$ or $G \in G(x,k)$, and $\kappa_k(G) = n-k+1$ if and only if $\alpha(G) \leq k-1$.*

For example, we clearly have $\kappa_k(K_{1,n-1}) = 1$ and $\kappa_k(K_n) = n-k+1$. It is not hard to prove the following result.

Lemma 2. *Let G be a graph with order n and size m. If G contains at least k components, then $m \leq \binom{n-k+1}{2}$; the equality holds if and only if G has exactly k components such that $k-1$ of them are trivial, the remaining one is a clique of order $n-k+1$.*

For three integers n,k,t with $1 \leq t \leq n-k$, we define a class of graph $G(n,k,t)$ as follows: For each graph $G \in G(n,k,t)$, let $V(G) = A \cup B \cup \{v_{n-t-k+2}, \ldots, v_{n-t}\}$ with $A = \{u_i |\, 1 \leq i \leq t\}$ and $B = \{v_j |\, 1 \leq j \leq n-t-k+1\}$ such that both A and B are cliques of G, and $xy \in E(G)$ for each pair $(x,y) \in (A, V(G) \setminus A)$. Clearly, the size of G is $\binom{n-k+1}{2} + t(k-1)$. Furthermore, the following result holds.

Lemma 3. *$\kappa_k(G) = t$ for $G \in G(n,k,t)$.*

Proof. Let $G \in G(n,k,t)$. On one hand, for any set $X \subseteq V(G)$ with $|X| \leq t-1$, the subgraph $G \setminus X$ contains at least one vertex of A, then it is connected, so we have $\kappa_k(G) \geq t$ since $n \geq t+k$; on the other hand, since $G \setminus A$ contains t components, we have $\kappa_k(G) \leq t$. Thus, $\kappa_k(G) = t$. ■

By Proposition 1, Lemma 3 and the fact that $\kappa_k(K_n) = n-k+1$, the following result clearly holds.

Theorem 4. *For each triple (n,k,t) of three integers with $1 \leq t \leq n-k+1$ and $k \geq 3$, there exists a graph G of order n such that $\kappa_k(G) = t$.*

The following theorem concerns the case that $\kappa_k(G) \leq n-k$ and gives a necessary condition for $\kappa_k(G) = t$.

Theorem 5. *Let G be a connected graph of order n and size m. If $\kappa_k(G) = t$ where $1 \leq t \leq n-k$, then $m \leq \binom{n-k+1}{2} + t(k-1)$; moreover, the equality holds if and only if $G \in G(n,k,t)$.*

Proof. Since $\kappa_k(G) = t$ and $n \geq t + k$, by definition, there exists a subset $X \subseteq V(G)$ such that $G \setminus X$ contains at least k components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq k$. Without loss of generality, we can assume that $n(G_1) \leq n(G_2) \leq \cdots \leq n(G_\ell)$.

Let G' be a supergraph with vertex set $V(G') = V(G)$ such that $xy \in E(G')$ for each pair $(x, y) \in (X, V(G') \setminus X)$, both X and $V(G_i)$ induce cliques in G', where $1 \leq i \leq \ell$. By Lemma 2, we know that $m(G')$ reaches the maximum value if and only if $\ell = k$, $n(G'_k) = n - k + 1$ and $n(G'_i) = 1$ for $1 \leq i \leq k - 1$, that is, in this case we have that $G' \in G(n, k, t)$, and the sets X, $V(G'_k)$ correspond to A, B in $G(n, k, t)$, respectively. Thus, $m(G) \leq m(G') \leq \binom{n-k+1}{2} + t(k-1)$, and by Lemma 3, the conclusion holds. ∎

Note that in the previous theorem, we obtain a sharp upper bound for the size of G provided that $\kappa_k(G) = t$ where $1 \leq t \leq n - k$, and also get the unique extremal graph $G(n, k, t)$. By Proposition 1, Theorem 5 and the fact that $\kappa_k(K_n) = n - k + 1$, we can compute the exact value of $g(n, k, t)$.

Theorem 6. *For three integers n, k, t with $1 \leq t \leq n - k + 1$ and $k \geq 3$, we have*

$$g(n,k,t) = \begin{cases} \binom{n-k+1}{2} + t(k-1), & 1 \leq t \leq n-k, \\ \binom{n}{2}, & t = n-k+1. \end{cases}$$

3. Exact Values and Bounds for $f(n, k, t)$

In this section, we will study the parameter $f(n, k, t)$. Recall that $f(n, k, t)$ is defined as the minimum size of a connected graph G with order n and $\kappa_k(G) = t$ where $1 \leq t \leq n - k$ and $k \geq 3$. By definition, the following result is clear.

Observation 7. *$f(n, k, t) \geq n - 1$.*

For the case $k \geq 2t$, we will give the exact value of $f(n, k, t)$ in the following result.

Lemma 8. *If $k \geq 2t$, then $f(n, k, t) = n - 1$.*

Proof. As shown in Figure 1, we construct a tree T which contains $k + 1$ edge-disjoint paths, including $P : u_1, u_2, \ldots, u_t$ and P_i, where $1 \leq i \leq k$. For $1 \leq i \leq t - 1$, there is an edge between u_i and each of P_{2i-1} and P_{2i} such that $\{u_i\} \cup V(P_{2i-1})$ and $\{u_i\} \cup V(P_{2i})$ induce two edge-disjoint paths in T, say P'_{2i-1} and P'_{2i}. For $i = t$, there is an edge between u_t and each of $P_{2t-1}, \ldots, P_k$ such that $\{u_t\} \cup V(P_j)$ induces a path P'_j in T, where $2t - 1 \leq j \leq k$.

We choose $X = \{u_i | \, 1 \leq i \leq t\}$. Then $G \setminus X$ contains k components and each component is exactly the path P_i, where $1 \leq i \leq k$, so we have $\kappa_k(T) \leq t$.

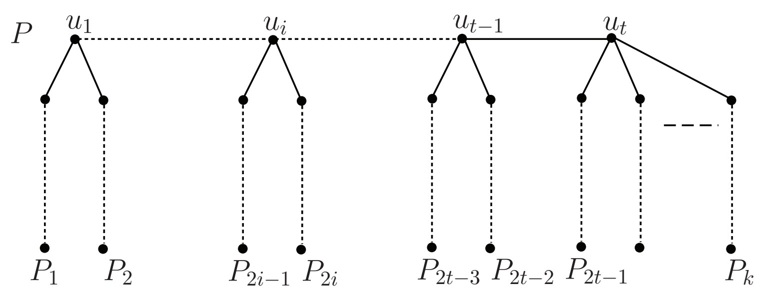

Figure 1. The tree T in Lemma 8.

Furthermore, it is not hard to show that for any set $X' \subseteq V(T)$ with $|X'| < t$, the graph $G \setminus X'$ contains at most $k-1$ components, so $\kappa_k(T) \geq t$. Hence, $\kappa_k(T) = t$ and then $f(n,k,t) \leq m(T) = n-1$. By Observation 7, we have $f(n,k,t) = n-1$. ■

For the case that $t \leq k \leq 2t-1$, we will get the following upper bound for $f(n,k,t)$.

Lemma 9. *If $t < k \leq 2t-1$, then $f(n,k,t) \leq n-1+2t-k$.*

Proof. Let $\ell = k-t$; we will construct a graph G in the following. The graph G consists of $k+1$ edge-disjoint paths, including $P : u_1, u_2, \ldots, u_t$ and P_i for $1 \leq i \leq k$. As shown in Figure 2, for $1 \leq i \leq \ell$, there is an edge between u_i and each of P_{2i-1} and P_{2i} such that $\{u_i\} \cup V(P_{2i-1})$ and $\{u_i\} \cup V(P_{2i})$ induce two edge-disjoint paths in G, say P'_{2i-1} and P'_{2i}; for $\ell+1 \leq i \leq t$, there is an edge between u_i and $P_{i+\ell}$ such that $\{u_i\} \cup V(P_{i+\ell})$ induces a path $P'_{i+\ell}$ in G. Let $V(G) = V(P) \cup \left(\bigcup_{i=1}^{k} V(P_i)\right)$ and $E(G) = E(P) \cup \left(\bigcup_{i=1}^{k} E(P'_i)\right) \cup M$, where $M = \{u_i v_{i+1} | \ell+1 \leq i \leq t\}$ with $v_{t+1} = v_{\ell+1}$.

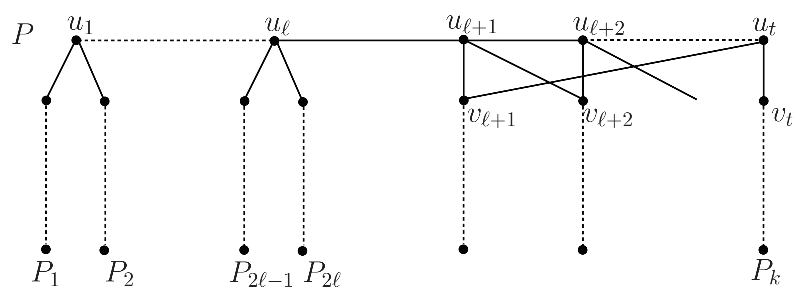

Figure 2. The graph G in Lemma 9.

Choose $X = \{u_i | 1 \leq i \leq t\}$. Then the graph $G \setminus X$ contains k components and each component is exactly the path P_i, where $1 \leq i \leq k$, so we have $\kappa_k(G) \leq t$. Furthermore, it is not hard to show that for any set $X' \subseteq V(G)$ with $|X'| < t$, the graph $G \setminus X'$ contains at most $k-1$ components. Thus $\kappa_k(G) \geq t$. Hence, $\kappa_k(G) = t$ and so we have $f(n,k,t) \leq m(G) = n-1+(t-\ell)$ $= n-1+2t-k$. ■

For the case that $k = t$, since $\kappa_k(C_n) = k$, we clearly have the following result.

Lemma 10. $f(n,k,k) = n$.

For the case $k \leq t-1$, the following upper bound of $f(n,k,t)$ holds.

Lemma 11. *If $k \leq t-1$, then $f(n,k,t) \leq n-1+(t-1)(k-1)$.*

Proof. Let G be a connected graph with vertex set $V(G) = A \cup B \cup C$ such that $A = \{u_i | 1 \leq i \leq t\}, B = \{v_j | 1 \leq j \leq k-1\}$, $C = \{v_j | k \leq j \leq n-t\}$, $G[A]$ and $G[B]$ are independent sets, $G[A, B \cup \{v_k\}]$ is a complete bipartite graph, and $G[C] = v_k, v_{k+1}, \ldots, v_{n-t}$ is a path.

Choose $X = \{u_i | 1 \leq i \leq t\}$. Then $G \setminus X$ contains k components, so we have $\kappa_k(G) \leq t$. Furthermore, for any set $X' \subseteq V(G)$ with $|X'| < t$, the graph $G \setminus X'$ contains at least one vertex of A and so has at most $k-1$ components. Therefore $\kappa_k(G) \geq t$. Hence, $\kappa_k(G) = t$ and we have $f(n,k,t) \leq m(G) = tk + [(n-t)-(k-1)-1] = n-1+(t-1)(k-1)$. ■

The following result concerns a lower bound for $f(n,k,t)$.

Lemma 12. $f(n,k,t) \geq n+t-k$.

Proof. Consider a graph G with $\kappa_k(G) = t$. By definition, we know that there exists a set $X \subseteq V(G)$ with $|X| = t$ such that $G \setminus X$ contains ℓ components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq k$. By definition, we also have that for any set $X' \subseteq V(G)$ with $|X'| < t$, the subgraph $G \setminus X'$ contains less than k components, so for any set $X_1 \subseteq X$ with $|X_1| = t-1$, the subgraph $G \setminus X_1$ contains less than k components. Then for any vertex $v \in X$, v is adjacent to at least $\ell-(k-2)$ components of $G \setminus X$. Thus, the number of edges between X and $V(G) \setminus X$ is at least $[\ell-(k-2)]t$. Furthermore, we know that $\left|\bigcup_{i=1}^{\ell} E(G_i)\right| \geq n-t-\ell$. Hence, we have $|E(G)| \geq [\ell-(k-2)]t + (n-t-\ell) = \ell(t-1)-(k-2)t+n-t \geq k(t-1)-(k-2)t+n-t = n+t-k$, and our result holds. ■

By Observation 7, Lemmas 8, 9, 10, 11 and 12, we can get the following result.

Theorem 13. *The following assertions hold.*

(i) $f(n,k,t) = n-1$ *for* $k \geq 2t$.
(ii) *If* $t < k \leq 2t-1$, *then* $\max\{n+t-k, n-1\} \leq f(n,k,t) \leq n-1+2t-k$.
(iii) $f(n,k,k) = n$.
(iv) *If* $k \leq t-1$, *then* $n+t-k \leq f(n,k,t) \leq n-1+(t-1)(k-1)$.

From the above results, we can get the exact values of $f(n,k,t)$ for the case $t \in \{1,2\}$.

Proposition 14. $f(n,k,t) = n-1$ *for* $t \in \{1,2\}$.

Proof. For $t = 1$, by Lemma 8, we clearly have $f(n,k,t) = n-1$ since $k \geq 3$. For $t = 2$, if $k \geq 4$, then $f(n,k,t) = n-1$ by Lemma 8. If $k = 3$, consider the path P_n; we clearly have $\kappa_3(P_n) = 2$, then $f(n,3,2) = n-1$. Hence, the result holds. ■

4. Remarks

For a connected graph G with $\kappa_k(G) = t$, by definitions of $f(n,k,t)$ and $g(n,k,t)$, we have $f(n,k,t) \leq m(G) \leq g(n,k,t)$. In Section 2, we computed the exact value of $g(n,k,t)$ (Theorem 6), which was based on the necessary condition for $\kappa_k(G) = t$ in Theorem 5. In Section 3, we investigated the parameter $f(n,k,t)$ and computed the exact values and bounds for $f(n,k,t)$ (Theorem 13).

To the best of our knowledge, there is no complexity or algorithmic result on computing the k-connectivity. So the following question is natural: Is there a polynomial-time algorithm to compute the k-connectivity of a given graph G? Or, one may prove that the following question is NP-Complete: Given an input graph G, determine if $\kappa_k(G) \leq t$, where k, t are two positive integers with $k \geq 3$.

Acknowledgements

This research was supported by National Natural Science Foundation of China (No. 11401389) and China Scholarship Council (No. 201608330111). We would like to thank the anonymous referee for helpful comments and suggestions which indeed help us greatly to improve the quality of our paper.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, Berlin, 2008).

[2] G. Chartrand, S.F. Kappor, L. Lesniak and D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. **2** (1984) 1–6.

[3] G. Chartrand, F. Okamoto and P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks **55** (2010) 360–367.

[4] D.P. Day, O.R. Oellermann and H.C. Swart, *The ℓ-connectivity function of trees and complete multipartite graphs*, J. Combin. Math. Combin. Comput. **10** (1991) 183–192.

[5] R. Gu, X. Li and Y. Shi, *The generalized* 3*-connectivity of random graphs*, Acta Math. Sinica **57** (2014) 321–330, in Chinese.

[6] M. Hager, *Pendant tree-connectivity*, J. Combin. Theory Ser. B **38** (1985) 179–189. doi:10.1016/0095-8956(85)90083-8

[7] H. Li, X. Li, Y. Mao and Y. Sun, *Note on the generalized connectivity*, Ars Combin. **114** (2014) 193–202.

[8] H. Li, X. Li and Y. Sun, *The generalized* 3*-connectivity of Cartesian product graphs*, Discrete Math. Theor. Comput. Sci. **14** (2012) 43–54.
doi:10.1016/j.commatsci.2011.09.003

[9] X. Li and Y. Mao, *A survey on the generalized connectivity of graphs.*
arXiv:1207.1838[math.CO]

[10] X. Li and Y. Mao, *On extremal graphs with at most ℓ internally disjoint Steiner trees connecting any $n-1$ vertices*, Graphs Combin. **31** (2015) 2231–2259.
doi:10.1007/s00373-014-1500-7

[11] X. Li and Y. Mao, *The generalized* 3*-connectivity of lexicographic product graphs*, Discrete Math. Theor. Comput. Sci. **16** (2014) 339–354.
doi:10.1007/978-3-319-12691-3_31

[12] X. Li, Y. Mao and Y. Sun, *On the generalized* (*edge*)*-connectivity of graphs*, Australas. J. Combin. **58** (2014) 304–319.

[13] O.R. Oellermann, *On the ℓ-connectivity of a graph*, Graphs Combin. **3** (1987) 285–299.
doi:10.1007/BF01788551

[14] O.R. Oellermann, *A note on the ℓ-connectivity function of a graph*, Congr. Numer. **60** (1987) 181–188.

[15] Y. Sun, *Generalized* 3*-*(*edge*)*-connectivity for undirected double-loop networks*, J. Discrete Math. Sci. Cryptogr. **17** (2014) 19–28.
doi:10.1080/09720529.2013.867672

[16] Y. Sun, *Generalized* 3*-edge-connectivity of Cartesian product graphs*, Czechoslovak Math. J. **65** (2015) 107–117.
doi:10.1007/s10587-015-0162-9

[17] Y. Sun, *Generalized* 3*-connectivity and* 3*-edge-connectivity for the Cartesian products of some graph classes*, J. Combin. Math. Combin. Comput. **94** (2015) 215–225.

[18] Y. Sun, *Maximum generalized local connectivities of cubic Cayley graphs on Abelian groups*, J. Combin. Math. Combin. Comput. **94** (2015) 227–236.

[19] Y. Sun, *Sharp upper bounds for generalized edge-connectivity of product graphs*, Discuss. Math. Graph Theory **36** (2016) 833–843.
doi:10.7151/dmgt.1924

[20] Y. Sun, *A sharp lower bound for the generalized* 3*-edge-connectivity of strong product graphs*, Discuss. Math. Graph Theory, accepted.

[21] Y. Sun, F. Li and Z. Jin, *On two generalized connectivities of graphs*, Discuss. Math. Graph Theory, accepted.

[22] Y. Sun and X. Li, *On the difference of two generalized connectivities of a graph*, J. Comb. Optim. (2015), in press.
doi:10.1007/s10878-015-9956-9

[23] Y. Sun and S. Zhou, *Tree connectivities of Cayley graphs on Abelian groups with small degrees*, Bull. Malays. Math. Sci. Soc. **39** (2016) 1673–1685.
doi:10.1007/s40840-015-0147-8

Received 19 January 2016
Revised 27 May 2016
Accepted 1 June 2016